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# Series and Exponentially-Fitted Two-Point Hybrid Method for General Second Order Ordinary Differential Equations

**Sunday Jacob Kayode, Friday Oghenerukevwe Obarhua*, Oluwatoyin Christiana Osuntope**

Department of Mathematical Sciences, The Federal University of Technology, Akure, Nigeria
Email: *obarhuafo@futa.edu.ng

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## Abstract

This article considered the development of a two-point hybrid method for the numerical solution of initial value problems of second order ordinary differential Equations (ODEs) using power series and exponentially-fitted basis function. Interpolation and collocation techniques were used to derive the method. The method was implemented in predictor-corrector mode. In order to increase the accuracy of the results of the method, the predictor was designed to have same order of accuracy as the corrector. The method is symmetric, consistent, zero-stable and has small error constant and has better accuracy over other methods in the reviewed literature when tested with some numerical examples.

## Subject Areas

Numerical Mathematics, Ordinary Differential Equation

## Keywords

Exponentially-Fitted, Zero Stability, Hybrid Method, Symmetric, Error Constant, Basis Functions

## 1. Introduction

The pursuit of finding solutions to empirical problems in real life situations has given rise to mathematical models. These models most often resulted to differential equations of various types and order. In this work, we consider one of such important differential equations—second order ordinary differential Equations (ODEs) expressed in the form

$$y'' = f(x, y, y'),\ y(\mu) = \omega_0,\ y'(\mu) = \omega_1 \tag{1}$$

Equation (1) appears in almost all physical or biological processes in relation with a wide range of issues that arise in various facets of daily life. In the past, it was a convention to solve this type of equations by breaking it down into a set of first order ODEs and solve the resulting equations using analytical or numerical methods. Many authors, including [1]-[6], have thoroughly explored the direct numerical solution of equations of type (1) without reducing to system of first order.

Many studies have also been done on the application of variants of numerical methods to obtain solutions to (1), which includes block mode. Due to its self-starting nature, which mitigates its inherent disadvantages compared to predictor-corrector mode of implementation, where the predictor is of lower order, the block mode implementation has been reported in various papers to be superior [7] [8] [9]. In their study, Kayode and Adeyeye [10] examined a hybrid predictor-corrector method for direct solution of second order ODEs. While the introduction of hybrid points increases the order of accuracy of the method, the order of the major predictor in the work is equivalent to the order of the method itself to overcome the setback of predictor-corrector mode of implementation. Kayode and Obarhua [5] presented a 3-step y-function for second order ordinary differential equations to overcome the cost of functions evaluation. The accuracy of this method is much higher than that of existing block methods.

It has been found in literature that a number of methods have been developed using various basis functions as approximation solutions, which include power series, exponential function, legendry, trigonometric polynomial and Chebyshev polynomial, among others. Although the type of problems to be solved are sometimes influenced by the choice of the approximate solution. Alabi *et al.* [11] found that most of these methods do not have good stability features, which causes them to fail when the problem is stiff or oscillatory.

Given the aforementioned, the motivation for this research is the need to derive a predictor-corrector mode method that requires fewer function evaluations and a combination of power series with exponential functions.

## 2. Derivation of the Method

To derive this method, two off-grid points is introduced. The two off-grid points are $x_{n+\frac{1}{2}}$ and $x_{n+\frac{3}{2}}$. These points are carefully selected to guarantee symmetry and zero stability conditions. The basis function adopted as approximate solution to Equation (1) is a combination of power series with exponential function given as:

$$y(x) = \sum_{j=0}^{p} \alpha_j x^j + a_j \sum_{j=0}^{p} \frac{x^j}{j!} \tag{2}$$

where $x \in (a, b)$, $a's$ are real unknown parameters to be determined and $c + i$ is the sum of the number of collocation and interpolation points of a basis function with a single variable $x$, where $x \in [a,b]$, $a_j's$ are real unknown parameters to be determined and $c+i$ is the sum of the numbers of collocation and interpolation points.

The second derivative of (2) is

$$y''(x) = \sum_{j=2}^{p} j(j-1)\alpha_j x^{j-2} + a_j \sum_{j=0}^{p} \frac{x^{j-2}}{(j-2)!} \tag{3}$$

Combination of (3) and (1) generates the differential system of the form:

$$\sum_{j=2}^{p} j(j-1)\alpha_j x^{j-2} + a_j \sum_{j=2}^{p} \frac{x^{j-2}}{(j-2)!} = f(x,y,y') \tag{4}$$

Equations (4) and (2) are respectively collocated and interpolated at $x_{n+i}, i = 0\left(\frac{1}{2}\right)2$ and $x_{n+i}, i = 0,1$ to give rise to the following

$$f_n = 3a_2 + 7a_3x_n + \frac{25}{2}a_4x_n^2 + \frac{121}{6}a_5x_n^3 + \frac{721}{24}a_6^4$$

$$f_{n+\frac{1}{2}} = 3a_2 + 7a_3x_{n+\frac{1}{2}} + \frac{25}{2}a_4x_{n+\frac{1}{2}}^2 + \frac{121}{6}a_5x_{n+\frac{1}{2}}^3 + \frac{721}{24}a_6x_{n+\frac{1}{2}}^4$$

$$f_{n+1} = 3a_2 + 7a_3x_{n+1} + \frac{25}{2}a_4x_{n+1}^2 + \frac{121}{6}a_5x_{n+1}^3 + \frac{721}{24}a_6x_{n+1}^4$$

$$f_{n+\frac{3}{2}} = 3a_2 + 7a_3x_{n+\frac{3}{2}} + \frac{25}{2}a_4x_{n+\frac{3}{2}}^2 + \frac{121}{6}a_5x_{n+\frac{3}{2}}^3 + \frac{721}{24}a_6x_{n+\frac{3}{2}}^4$$

$$f_{n+2} = 3a_2 + 7a_3x_{n+2} + \frac{25}{2}a_4x_{n+2}^2 + \frac{121}{6}a_5x_{n+2}^3 + \frac{721}{24}a_6x_{n+2}^4$$

$$y_n = 2a_0 + 2a_1x_n + \frac{3}{2}a_2x_n^2 + \frac{7}{6}a_3x_n^3 + \frac{25}{24}a_4x_n^4 + \frac{121}{120}a_5x_n^5 + \frac{721}{720}a_6x_n^6$$

$$y_{n+1} = 2a_0 + 2a_1x_{n+1} + \frac{3}{2}a_2x_{n+1}^2 + \frac{7}{6}a_3x_{n+1}^3 + \frac{25}{24}a_4x_{n+1}^4 + \frac{121}{120}a_5x_{n+1}^5 + \frac{721}{720}a_6x_{n+1}^6 \tag{5}$$

The system of linear Equations (5) is solved to have the values of the unknown parameters $a_j's$ to give

$$a_0 = \frac{1}{2}\left[y_n - 2a_1x_n - \frac{3}{2}a_2x_n^2 - \frac{7}{6}a_3x_n^3 - \frac{25}{24}a_4x_n^4 - \frac{121}{120}a_5x_n^5 - \frac{721}{720}a_6x_n^6\right]$$

$$\begin{aligned} a_1 = \frac{-1}{720h^4}\Big[ & 360h^3(y_{n+1}+y_n) + \left(-3h^5 + 90x_n^2h^3 + 220x_n^3h^2 + 180x_n^4h + 48x_n^5\right)f_{n+2} \\ & + \left(16h^5 - 480x_n^2h^3 - 1120x_n^3h^2 - 840x_n^4h - 192x_n^5\right)f_{n+\frac{3}{2}} \\ & + \left(-30h^5 + 1080x_n^2h^3 + 2280x_n^3h^2 + 1920x_n^4h + 288x_n^5\right)f_{n+1} \\ & + \left(114h^5 - 1440x_n^2h^3 - 2080x_n^3h^2 - 1080x_n^4h - 192x_n^5\right)f_{n+\frac{1}{2}} \\ & + \left(53h^5 + 360x_nh^4 + 750x_n^2h^3 + 700x_n^3h^2 + 300x_n^4h + 48x_n^5\right)f_n \Big] \end{aligned}$$

$$\begin{aligned} a_2 = \frac{1}{18h^4}\Big[ & \left(3h^3x_n + 11x_n^2h^2 + 12x_n^3h + 4x_n^4\right)f_{n+2} \\ & - \left(16h^3x_n + 56x_n^2h^2 + 56x_nh^3 + 16x_n^4\right)f_{n+\frac{3}{2}} \end{aligned}$$

$$+\left(36h^3x_n+114x_n^2h^2+96x_n^3h+24x_n^4\right)f_{n+1}$$
$$-\left(48h^3x_n+104x_n^2h^2+72x_n^3h+16x_n^4\right)f_{n+\frac{1}{2}}$$
$$+\left(6h^4+25x_nh^3+35x_n^2h^2+20x_n^3h+4x_n^4\right)f_n\Big]$$

$$a_3=\frac{-1}{42h^4}\Big[\left(3h^3+22x_nh^2+36x_n^2h+16x_n^3\right)f_{n+2}$$
$$-\left(16h^3+112x_nh^2+168x_n^2h+64x_n^3\right)f_{n+\frac{3}{2}}$$
$$+(36h^3+228x_nh^2+288x_n^2h+96x_n^3)f_{n+1}$$
$$-\left(48h^3+208x_nh^2+216x_n^2h+64x_n^3\right)f_{n+\frac{1}{2}}$$
$$+\left(25h^3+70x_nh^2+60x_n^2h+16x_n^3\right)\Big]$$

$$a_4=\frac{1}{75h^4}\Big[\left(11h^2+36x_nh+24x_n^2\right)f_{n+2}-\left(56h^2+168x_nh+96x_n^2\right)f_{n+\frac{3}{2}}$$
$$+\left(114h^2+144x_nh+288x_n^2\right)f_{n+1}-\left(104h^2+216x_nh+96x_n^2\right)f_{n+\frac{1}{2}}$$
$$+\left(35h^2+60x_nh+24x_n^2\right)f_n\Big]$$

$$a_5=-\frac{4}{121h^4}\Big[\left(3h+4x_n\right)f_{n+2}-\left(14h+18x_n\right)f_{n+\frac{3}{2}}+\left(24x_n+24h\right)f_{n+1}$$
$$-\frac{h^4}{15}(18h+16x_n)f_{n+\frac{1}{2}}-\left(5h+4x_n\right)f_n\Big]$$

$$a_6=\frac{16}{721h^2}\left[f_{n+2}-4f_{n+\frac{3}{2}}+6f_{n+1}-4f_{n+\frac{1}{2}}+f_n\right] \tag{6}$$

Substituting the values of $a_j's,\ j=0(1)6$ into Equation (2) give the continuous hybrid method:

$$y_k(x)=\sum_{j=0}^{k-1}\alpha_j(x)y_{n+j}+h^2\sum_{j=0}^{k}\beta_j(x)f_{n+j}+\left\{\eta_1(x)y_{n+r}+\eta_2(x)y_{n+s}\right\} \tag{7}$$

Applying the transformation $t=\dfrac{x-x_{n+k-1}}{h}$ and $\dfrac{\mathrm{d}t}{\mathrm{d}x}=\dfrac{1}{h}$ in Obarhua and Kayode [12], the coefficients are given as follows

$$\alpha_0=-t$$
$$\alpha_1=(t+1)$$
$$\beta_0=h^2\left(\frac{1}{72}t+\frac{1}{36}t^3-\frac{1}{72}t^4-\frac{1}{30}t^5+\frac{1}{45}t^6\right)$$
$$\beta_{\frac{1}{2}}=h^2\left(\frac{13}{45}t-\frac{2}{9}t^3+\frac{2}{9}t^4+\frac{1}{15}t^5-\frac{4}{45}t^6\right)$$
$$\beta_1=h^2\left(\frac{13}{60}t+\frac{1}{2}t^2-\frac{5}{12}t^4+\frac{2}{15}t^6\right)$$
$$\beta_{\frac{3}{2}}=h^2\left(-\frac{1}{45}t+\frac{2}{9}t^3+\frac{2}{9}t^4-\frac{1}{15}t^5-\frac{4}{45}t^6\right)$$

$$\beta_2 = h^2\left(\frac{1}{360}t - \frac{1}{36}t^3 - \frac{1}{72}t^4 + \frac{1}{30}t^5 + \frac{1}{45}t^6\right) \quad (8)$$

The first derivative (8) gives

$$\alpha_0' = -\frac{1}{h}$$

$$\alpha_{\frac{3}{2}}' = \frac{1}{h}$$

$$\beta_0' = h\left(\frac{1}{72} + \frac{1}{12}t^2 - \frac{1}{18}t^3 - \frac{1}{6}t^4 + \frac{6}{45}t^5\right)$$

$$\beta_{\frac{1}{2}}' = h\left(\frac{13}{45} - \frac{2}{3}t^2 + \frac{8}{9}t^3 + \frac{1}{3}t^4 - \frac{24}{45}t^5\right)$$

$$\beta_1' = h\left(\frac{13}{60} + t - \frac{5}{3}t^3 + \frac{4}{5}t^5\right)$$

$$\beta_{\frac{3}{2}}' = h\left(-\frac{1}{45} + \frac{2}{3}t^2 + \frac{8}{9}t^3 - \frac{1}{3}t^4 - \frac{24}{45}t^5\right)$$

$$\beta_2' = h\left(\frac{1}{360} - \frac{1}{12}t^2 - \frac{1}{18}t^3 + \frac{1}{6}t^4 + \frac{6}{45}t^6\right) \quad (9)$$

Evaluating (8) and (9) at $t = 1$ yields the discrete scheme

$$y_{n+2} = 2y_{n+1} - y_n + \frac{h^2}{60}\left(f_{n+2} + 16f_{n+\frac{3}{2}} + 26f_{n+1} + 16f_{n+\frac{1}{2}} + f_n\right) \quad (10)$$

with its first derivative as

$$y_{n+2}' = \frac{1}{h}\left(-y_n + y_{n+\frac{3}{2}}\right) + \frac{h}{360}\left(59f_{n+2} + 240f_{n+\frac{3}{2}} - 450f_{n+1} + 112f_{n+\frac{1}{2}} + 3f_n\right) \quad (11)$$

The predictor-corrector method and its derivative in Equations (10) and (11) above are zero stable, consistent and of order six with error constant $C_8 = \frac{-1}{120960} = -8.267 \times 10^{-6}$ and $C_8 = \frac{-127}{8570880} = -1.984 \times 10^{-4}$ respectively.

## 3. Implementation and Analysis of the Method

### 3.1. Implementation of the Method

To overcome the intrinsic drawback of predictor-corrector mode with predictors of lower order of accuracy to implement, the same approach is used to construct a predictor and its derivative of the same order of accuracy.

$$y_{n+2} = -y_n - 16y_{n+\frac{1}{2}} + 34y_{n+1} - 16y_{n+\frac{3}{2}} + \frac{h^2}{3}\left(2f_{n+\frac{1}{2}} + 11f_{n+1} + 2f_{n+3}\right) \quad (12)$$

and

$$\begin{aligned} y_{n+2}' = &-\frac{149}{21h}y_n + \frac{2920}{21h}y_{n+\frac{1}{2}} + \frac{6245}{21h}y_{n+1} - \frac{3176}{21h}y_{n+\frac{3}{2}} \\ &-\frac{h}{630}\left(18f_n - 3394f_{n+\frac{1}{2}} - 19531f_{n+1} - 2818f_{n+\frac{3}{2}}\right) \end{aligned} \quad (13)$$

The main predictor and its derivative in Equations (12) and (13) above are of order six with error constant $C_8 = -2.3768\times10^{-5}$ and $C_8 = 2.2308\times10^{-4}$ respectively.

Other explicit systems were generated to evaluate the remaining values using Taylor series to evaluate the values for $y_{n+j}$, $j = 6$.

$$y_{n+j} = y_n + (jh)y'_n + \frac{(jh)^2}{2!}f_n + \frac{(jh)^3}{3!}\left\{\frac{\partial f_n}{\partial x_n} + y'_n\frac{\partial f_n}{\partial y_n} + f_n\frac{\partial f_n}{\partial y'_n}\right\} + 0\left(h^4\right) \tag{14}$$

and

$$y'_{n+j} = y'_n + (jh)f_n + \frac{(jh)^2}{2!}f_n\left\{\frac{\partial f_n}{\partial x_n} + y'_n\frac{\partial f_n}{\partial y_n} + f_n\frac{\partial f_n}{\partial y'_n}\right\} + 0\left(h^3\right) \tag{15}$$

## 3.2. Analysis of the Basic Properties of the Method

### 3.2.1. Order and Error Constant of the Methods

Let the linear difference operator $L$ associated with the continuous implicit hybrid method be defined as:

$$L\left[y(x);h\right] = \sum_{j=0}^{k}\left\{\alpha_j y(x_n + jh) - \alpha_{vi}y(x_n + vih) - h^2\beta_j y''(x_n + jh)\right.$$
$$\left. - h^2\beta_{v_i}y''\left(x^n + jh\right)\right\}; i = 1,2,\cdots,m$$

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a,b]$. Expanding $y(x_n + jh)$ and $y''(x_n + jh)$, $j = 0, v_i, 1$; $i = 1,2,\cdots,m$ in Taylor series about $x_n$ and collecting like terms in $h$ and $y$ gives;

$$L\left[y(x);h\right] = c_0 y(x) + c_1 h y^{(1)}(x) + c_2 h^2 y^{(2)}(x) + \cdots + c_p h^p y^{(p)}(x)$$

**Definition:** The difference operator $L$ and the associated continuous implicit hybrid one step method are said to be of order $p$ if in (3.1)

$c_0 = c_1 = c_2 = \cdots = c_p = c_{p+1} = 0$, $c_{p+2} \neq 0$.

Using the concept above, the method has order $p = 6$ and error constant $c_8 = \frac{-1}{120960} = -8.267\times10^{-6}$.

### 3.2.2. Region of Absolute Stability

Applying the boundary locus method,

$$\rho(r) = r^2 - 2r + 1 \quad \text{and} \quad \sigma(r) = \frac{1}{60}\left(r^2 - 16r^{\frac{3}{2}} + +26r^1 + 16r^{\frac{1}{2}} + 1\right)$$

$$h(r) = \frac{\rho(r)}{\sigma(r)} = \frac{\rho\left(e^{i\theta}\right)}{\sigma\left(e^{i\theta}\right)}$$

If $r = e^{i\theta} = \cos\theta + i\sin\theta$, substituting and simplifying yields

$$X(\theta) = \frac{60\cos 2\theta - 120\cos\theta + 60}{\cos 2\theta + \cos\frac{3}{2}\theta + 26\cos\theta + 16\cos\frac{1}{2}\theta + 1}$$

At $\theta = 0^{\circ}$ and $\theta = 180^{\circ}$ for $0^{\circ} \le \theta \le 180^{\circ}$ at an intervals $\left(0^{\circ}, 180^{\circ}\right)$ gives (−10.00, 0).

### 3.2.3. Consistency of the Methods

For our method to be consistent, the following conditions must be satisfied

1) order $p \ge 1$.

2) $\sum_{j=0}^{k} \alpha_j = 0$.

3) $\rho(1) = \rho'(1) = 0$.

4) $\rho''(1) = 2!\sigma(1)$.

$$y_{n+2} = 2y_{n+1} - y_n + \frac{h^2}{60}\left( f_{n+2} + 16 f_{n+\frac{3}{2}} + 26 f_{n+1} + 16 f_{n+\frac{1}{2}} - f_n \right)$$

Condition (1) is satisfied since the scheme is of order 6.

Condition (2) is satisfied since $\alpha_0 = 1, \alpha_1 = -2, \alpha_2 = 1$; $1 - 2 + 1 = 0$.

Condition (3) is satisfied when the first characteristic polynomial and its first derivative in the form, $\rho(r) = \rho'(r) = 0$, when $r = 1$. Therefore, $\rho(r) = r^2 - 2r + 1 = 0$ and $\rho'(r) = 2r - 2 = 0$ for $r = 1$.

Condition (4) is satisfied when $\rho''(r) = 2!\sigma(r)$.

Therefore, $\rho''(1) = 2$ and the second characteristic polynomial

$$\sigma(r) = \frac{1}{60}\left( r^2 + 16r^{\frac{3}{2}} + 26r + 16r^{\frac{1}{2}} + 1 \right)$$

when $r = 1$,

$$\sigma(1) = 2! \times \left( \frac{1}{60} + \frac{16}{60} + \frac{26}{60} + \frac{16}{60} + \frac{1}{60} \right) = 2 \times \frac{60}{60} = 2$$

Therefore $\rho''(r) = 2!\sigma(r) = 2$.

Hence the four conditions are satisfied, the method is consistent.

## 4. Numerical Examples

Using the proposed method to solve linear, nonlinear, and electric current circuit problems in the literature demonstrates the method's applicability and correctness.

**Problem 1**

$$y'' - x\left(y'\right)^2 = 0, \ y(0) = 1, \ y'(0) = \frac{1}{2}, x \in [0,1]$$

**Exact-solution:** $y(x) = 1 + \frac{1}{2}\ln\left[\frac{2+x}{2-x}\right], h = \frac{1}{100}$.

**Problem 2:**

$$y'' = \frac{\left(y'\right)^2}{2y} - 2y, \ y\left(\frac{\pi}{6}\right) = \frac{1}{4}, \ y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

**Exact-solution:** $y(x) = \sin^2 x, h = 0.003125$.

**Problem 3:**

Consider the direct circuit, DC of an electric current containing an inductance

$L$ (Henries), a resistance $R$ (Ohms), a condenser of capacitance $C$ (Faraday's) and an electromotive force $E(t)$ measured in volts:

$$Lq''(t)+Rq'+\frac{q(t)}{c}=E(t):\frac{1}{20}q''(t)+5q'+2500q(t)=110,$$
$$q(0)=q'(0)=0,\ h=0.01$$

**Exact solution:** $q(t)=-\frac{11}{250}\left(\cos 50\sqrt{19}t+\frac{\sin 50\sqrt{19}t}{\sqrt{19}}\right)e^{t}-50t+\frac{11}{250}$

where $E=110$ Volts, $R=5$, $L=\frac{1}{20}$, $C=4\times10^{-4}$ and $q$ is the charge in coulombs.

## 5. Discussion of Results

In Table 1, the results of the newly developed numerical method are presented and assessed with results in [13] [14]. The results revealed that our new method performed better than those authors in literature. Table 2 revealed the exact solution and the computed solution and the absolute errors showed the consistency of the results produced by the new method.

**Table 1.** Results of Problem 1, for $k = 2$, $p = 6$.

| $x$ | $y$-exact | $y$-computed | Error [13] | Error [14] | NMError |
|---|---|---|---|---|---|
| 0.1 | 1.050041729278491400 | 1.050041729278491400 | 6.661e−16 | 2.220e−16 | 0.00e+00 |
| 0.2 | 1.100335347731075800 | 1.100335347731075800 | 1.332e−15 | 2.220e−16 | 0.00e+00 |
| 0.3 | 1.151140435936466800 | 1.151140435936466800 | 4.441e−16 | 0.000e+00 | 0.00e+00 |
| 0.4 | 1.202732554054081900 | 1.202732554196769900 | 1.332e−15 | 2.220e−16 | 0.00e+00 |
| 0.5 | 1.255412811882994800 | 1.255412811882994842 | 3.775e−15 | 0.000e+00 | 4.20e−17 |
| 0.6 | 1.309519604203111000 | 1.309519604203111076 | 1.066e−14 | 2.220e−16 | 7.60e−17 |
| 0.7 | 1.365443754271395300 | 1.365443754271395300 | 2.642e−14 | 6.661e−16 | 0.00e+00 |
| 0.8 | 1.423648930193600600 | 1.423648930193600532 | 5.862e−14 | 1.332e−15 | 6.80e−16 |
| 0.9 | 1.484700278594050200 | 1.484700278594050200 | 1.266e−13 | 3.108e−15 | 0.00e+00 |
| 1.0 | 1.549306144334053000 | 1.549306144334051240 | 2.711e−13 | 6.217e−15 | 1.76e−15 |

Note: NME: New Method Error.

**Table 2.** Computed results and errors for Problem 2, $k = 2$, $p = 6$.

| $x$ | $y$-exact | $y$-computed | NMError |
|---|---|---|---|
| 1.0 | 0.711340357839 | 0.711340678290 | 7.528437e−08 |
| 1.1 | 0.797152508881 | 0.797152548656 | 5.648764e−07 |
| 1.2 | 0.871118127112 | 0.871118146890 | 5.596659e−07 |
| 1.3 | 0.930288436747 | 0.930288524186 | 4.568690e−07 |
| 1.4 | 0.972304504262 | 0.972304521571 | 3.793242e−06 |
| 1.5 | 0.995491281635 | 0.995496774918 | 3.369310e−06 |

**Table 3.** Numerical solution for Problem 3, $k = 2$, $p = 6$, $h = 0.01$.

| $x$ | $y$-exact | $y$-computed | NM Error |
|---|---|---|---|
| 0.10 | 0.0442774661415 | 0.0442795080355 | 6.041887e−06 |
| 0.20 | 0.0439983305605 | 0.0439981507157 | 7.985700e−08 |
| 0.30 | 0.0440000094666 | 0.0440000202637 | 7.811405e−10 |
| 0.40 | 0.0439999999506 | 0.0439999998435 | 6.698385e−12 |
| 0.50 | 0.0440000000002 | 0.0440000000410 | 5.308254e−14 |
| 0.60 | 0.0439999999999 | 0.0447899999999 | 3.747003e−16 |
| 0.70 | 0.0439999999999 | 0.0439980000000 | 2.081668e−17 |
| 0.80 | 0.0439999999999 | 0.0439999999999 | 0.0 |
| 0.90 | 0.0439999999999 | 0.0439999999999 | 0.0 |
| 1.00 | 0.0439999999999 | 0.0439999999999 | 0.0 |

The new method was applied on a real life problem in electronics to test and confirm its applicability and the results and the absolute errors produced → 0.

## 6. Conclusion

A linear multistep method implemented in predictor-corrector algorithms of order six is developed for direct integration of general second-order initial value problems of ordinary differential equations. The method is derived by interpolation and collocation using power series and an exponential basis function. The main predictor has the same order of accuracy with the method. The results of computed numerical examples with the method were compared with [13] and [14], and these were presented in **Tables 1-3**. The basis of comparison of results of this predictor-corrector with the two-step third-derivative block method [13] and the hybrid block method [14] is that they are all of order six. The absolute errors of the new method show that the new method outperformed the earlier ones.

## Conflicts of Interest

The authors declare no conflicts of interest.

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