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Normalized Solutions of Mass-Subcritical Klein-Gordon-Maxwell Systems

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Abstract

In this paper, we study the existence of normalized solutions to the Klein-Gordon-Maxwell systems. In the mass-subcritical case, we prove that the systems satisfying normalization conditions have a normalized ground state solution.

Subject Areas

Mathematics

Keywords

Normalized Solutions, Klein-Gordon-Maxwell Systems

1. Introduction

In this paper, we consider the following Klein-Gordon-Maxwell systems

$$\begin{cases} -\Delta u + V(x)u - (2\omega+\phi)\phi u + \lambda u = f(u) \ \text{ in } \mathbb{R}^N, \\ \Delta\phi = (\omega+\phi)u^2 \ \text{ in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where ω is a positive real constant, the parameter $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier, $\phi : \mathbb{R}^N \to \mathbb{R}$ and $N \geq 3$, $V \in C(\mathbb{R}^N, \mathbb{R})$. The field u and the electric potential ϕ are the unknowns of the systems. Moreover, the field u satisfies the normalization condition

$$\int_{\mathbb{R}^N} |u(x)|^2 \, \mathrm{d}x = a. \tag{1.2}$$

We shall dedicate to search for a solution $u \in \mathcal{H}$ of the problem

$$\begin{cases} -\Delta u + V(x)u - (2\omega+\phi)\phi u + \lambda u = f(u) \ \text{ in } \mathbb{R}^N, \\ \Delta\phi = (\omega+\phi)u^2 \ \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u(x)|^2 \, \mathrm{d}x = a,\ u \geq 0, \end{cases} \tag{1.3}$$

where we define the Hilbert space

$$\mathcal{H}=\left\{u\in H^1\left(\mathbb{R}^N\right):\int_{\mathbb{R}^N}|\nabla u|^2+V(x)u^2\mathrm{d}x<\infty\right\}.$$

Then, $\mathcal{H}$ is continuously embedded in $H^1\left(\mathbb{R}^N\right)$ and endowed with the norm

$$\|u\|_{\mathcal{H}}^2=\int_{\mathbb{R}^N}\left(|\nabla u|^2+V(x)u^2\right)\mathrm{d}x.$$

Let $D^{1,2}\equiv D^{1,2}\left(\mathbb{R}^N\right)=\left\{u\in L^{2^*}\left(\mathbb{R}^N\right):\nabla u\in L^2\left(\mathbb{R}^N\right)\right\}$ with the norm

$$\|u\|_{D^{1,2}}^2=\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x.$$

For any $2\le p<2^*$, $L^p\left(\mathbb{R}^N\right)$ is endowed with the norm

$$|u|_p^p=\int_{\mathbb{R}^N}|u|^p\mathrm{d}x.$$

Obviously, the embedding $\mathcal{H}\hookrightarrow L^p\left(\mathbb{R}^N\right)$ is continuous.

We define a functional $J:\mathcal{H}\times D^{1,2}\to\mathbb{R}$ by

$$J(u,\phi)=\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2-|\nabla\phi|^2+\left[V(x)-(2\omega+\phi)\phi\right]u^2\right)\mathrm{d}x-\int_{\mathbb{R}^N}F(u)\mathrm{d}x,\quad(1.4)$$

where $F(t)=\int_0^t f(s)\mathrm{d}s$ is a rather general nonlinearity. Then, we can know that the weak solutions of (1.1) $(u,\phi)\in\mathcal{H}\times D^{1,2}$ are critical points of the functional *J*. By standard arguments, the function *J* is C^1 on $\mathcal{H}\times D^{1,2}$.

We can search for a lot of known results about solutions of Klein-Gordon- Maxwell systems without the normalization condition. For instance, Paulo C. Carrião and Patrícia L. Cunha [1] combining the minimization of the corresponding Euler-Lagrange functional and Nirenberg technique proved the existence of positive ground state solutions for the following Klein-Gordon-Maxwell systems:

$$\begin{cases}-\Delta u+V(x)u-(2\omega+\phi)\phi u=\mu|u|^{p-2}u+|u|^{2^*-2}u & \text{in }\mathbb{R}^3,\\ \Delta\phi=(\omega+\phi)u^2 & \text{in }\mathbb{R}^3.\end{cases}$$

Xian Zhang and Chen Huang [2] combining a local linking argument and Morse theory investigated a nontrivial solution of the nonlinear Klein-Gordon-Maxwell systems. Daniele Cassani [3] overcame the lack of compactness using the Brezis-Nirenberg method to obtain the existence of solutions for the nonlinear Klein-Gordon-Maxwell systems. Viert Benct and Donato Fortunato [4] proved the existence of infinitely many pairs solutions for the nonlinear Klein-Gordon-Maxwell equation:

$$\begin{cases}-\Delta u+\left[m^2-(\omega+\phi)^2\right]u-|u|^{p-2}u=0 & \text{in }\mathbb{R}^3,\\ -\Delta\phi+e^2u^2\phi=-e\omega u^2 & \text{in }\mathbb{R}^3.\end{cases}$$

Similar to the above problems of Klein-Gordon-Maxwell systems without the normalization condition have been studied in many papers such as [5] [6] [7]. However, few works have been done on the problems satisfying the normalization condition. Thus, the main purpose of this paper is to study the existence of solutions satisfying the normalization condition for the Klein-Gordon-Maxwell systems.

In recent years, normalized solutions of Schrödinger equations have been widely studied. See, e.g. [8]-[13]. When searching for the existence of normalized solutions of Schrödinger equations in $\mathbb{R}^N$, appears a new mass-critical exponent

$$l = 2 + \frac{4}{N}.$$

Now, let us review the involved works. In the mass-subcritical case, Zuo Yang and Shijie Qi [14] using weak lower semmicontinuity of norm obtained the existence of a normalized ground state solution for the following Schrödinger equations with potentials and non-autonomous nonlinearities

$$\begin{cases} -\Delta u + V(x)u + \lambda u = f(x,u) \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u(x)|^2 \mathrm{d}x = a, u \in H^1(\mathbb{R}^N), \end{cases}$$

where $V(x) \to \infty$ as $|x| \to \infty$. Nicola Soave [15] in the mass-subcritical, mass-critical and mass-supercritical cases studied several existence/non-existence and stability/instability results of normalized ground state solutions of the nonlinear Schrödinger equation with combined power nonlinearities:

$$\begin{cases} -\Delta u = \lambda u + \mu |u|^{p-2} u + |u|^{2^*-2} u \text{ in } \mathbb{R}^N, N \geq 3, \\ \int_{\mathbb{R}^N} |u(x)|^2 \mathrm{d}x = a, u \in H^1(\mathbb{R}^N). \end{cases}$$

Masataka Shibata [16] studied the mass-subcritical case for the minimizing problem of nonlinear Schrödinger equations with a general nonlinear term:

$$E(a) = \inf \left\{ I(u) \,|\, u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} |u(x)|^2 \mathrm{d}x = a \right\},$$

where

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(|u|) \mathrm{d}x.$$

They showed that there exists $a_0 \in [0, \infty)$ such that $E(a)$ is attained for $a > a_0$. Moreover, Norihisa Ikoma and Yasuhito Miyamoto [17] also studied the existence of the minimizer of the L^2-constraint minimization problem $E(a)$, but $I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(|u|) \mathrm{d}x$, and $F(t) = \int_0^t f(s) \mathrm{d}s$ is a general nonlinear term, and $0 \not\equiv V(x) \leq 0$, $V(x) \to 0 (|x| \to \infty)$. For the existence of normalized solutions, they obtained the same conclusions to [16]. Zhen Chen and Wenming Zou [18] proved the existence of normalized solutions to the following system

$$\begin{cases} -\Delta u + (V_1(x) + \lambda_1) u = \mu_1 |u|^{p-2} u + \beta v \text{ in } \mathbb{R}^N, \\ -\Delta v + (V_2(x) + \lambda_2) v = \mu_2 |v|^{p-2} v + \beta u \text{ in } \mathbb{R}^N, \end{cases}$$

with the mass-subcritical condition $2 < p, q < 2 + \frac{4}{N}$. They studied the existence of a solution with prescribed L^2-*norm* under various conditions. The results were based on the refined energy estimates. In the mass-supercritical and Sobolev subcritical case, Thomas Bartsch and Riccardo Molle [19] by a new min-max argument studied the existence of normalized solutions of the nonlinear Schrödinger equation:

$$-\Delta u+V(x)u+\lambda u=|u|^{p-2}u \text{ in } \mathbb{R}^N.$$

Thomas Bartsch [20] showed the existence of infinitely many normalized solutions for the problem

$$-\Delta u-g(u)=\lambda u,\ u\in H^1(\mathbb{R}^N),$$

where $g(u)$ is a superlinear, subcritical, possibly nonhomogeneous, odd nonlinearity.

Inspired by the above works, we are dedicated to studying the solutions of Klein-Gordon-Maxwell systems satisfying normalization condition. In particular, the situation we consider will involve the presence of electric potential ϕ. In addition, $V(x)$ is a positive function satisfying the following appropriate assumptions, and the nonlinear term $f(u)$ is mass-subcritical. In this case, the functional *I* is bounded from below and coercive on $S(a)$, which will be proved in Lemma 2.3.

We assume the following conditions throughout the paper:

(f1) $f:\mathbb{R}^N\to\mathbb{R}$ is continuous.

(f2) $\lim\limits_{s\to 0}\dfrac{f(s)}{s}=0$ and $\lim\limits_{|s|\to+\infty}\dfrac{f(s)}{|s|^{l-1}}=0$ with $l=2+\dfrac{4}{N}$.

(V1) $V\in C(\mathbb{R}^N,\mathbb{R})$, and $\inf\limits_{x\in\mathbb{R}^N}V(x)\geq c>0$.

(V2) $\lim\limits_{|x|\to+\infty}V(x)=+\infty$.

Moreover, c and c_i are positive constants which may change from line to line.

Our main result is the following theorem:

Theorem 1.1. *Suppose* (*f*1) *and* (*f*2) *hold and* *V*(*x*) *satisfies* (V1) *and* (V2)*. Then, for any* $a>0$*, problem* (1.3) *has a normalized ground state solution.*

2. Proof of Main Results

Since the functional *J* exhibits a strong indefiniteness. To avoid the difficulty we use the reduction method. Thus, we introduce following the technical lemma that is described in [1] [3] [4] for details.

Lemma 2.1. *The Lax-Milgram theorem implies that for any* $u\in\mathcal{H}$*, there exists a unique solution* $\phi=\phi_u\in D^{1,2}$ *of* $\Delta\phi=(\omega+\phi)u^2$*. Moreover, in the set* $\{x:u(x)\neq 0\}$*, for* $\omega>0$*, we have*

$$-\omega\leq\phi_u\leq 0. \tag{2.1}$$

By Lemma 2.1, let any $v\in D^{1,2}$ be an admissible test function of the second equation of (1.1). Then, we have

$$\int_{\mathbb{R}^N}\nabla\phi_u\nabla v\mathrm{d}x=-\int_{\mathbb{R}^N}(\omega+\phi_u)u^2v\mathrm{d}x.$$

Using integration by parts, we have

$$\int_{\mathbb{R}^N} \nabla\phi_u \nabla v \mathrm{d}x = -\int_{\mathbb{R}^N} \Delta\phi_u v \mathrm{d}x.$$

Therefore,

$$\Delta\phi_u = (\omega + \phi_u) u^2. \tag{2.2}$$

According to Lemma 2.1, we define

$$\Phi : \mathcal{H} \to D^{1,2},$$

and we know that the map Φ is C^1.

Combining (1.4) and (2.2), we obtain

$$J'_\phi (u, \phi_u) = 0, \ \forall u \in \mathcal{H}. \tag{2.3}$$

Multiplying both members of (2.2) by ϕ_u and integrating by parts, we obtain

$$\int_{\mathbb{R}^N} |\nabla\phi_u|^2 \mathrm{d}x = -\int_{\mathbb{R}^N} \omega\phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^N} \phi_u^2 u^2 \mathrm{d}x. \tag{2.4}$$

Now let us consider the functional $I : \mathcal{H} \to \mathbb{R}^N$,

$$I(u) := J(u, \phi_u).$$

Then I is C^1, and by (2.3) we have $I'(u) = J'_u(u, \phi_u)$.

By the definition of J, we have

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 - |\nabla\phi_u|^2 + \left[V(x) - (2\omega + \phi_u)\phi_u \right] u^2 \right) \mathrm{d}x - \int_{\mathbb{R}^N} F(u) \mathrm{d}x.$$

By (2.4), the functional I can be written as

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x) u^2 - \omega\phi_u u^2 \right) \mathrm{d}x - \int_{\mathbb{R}^N} F(u) \mathrm{d}x. \tag{2.5}$$

Then u is a solution to (1.1) if and only if u is the critical point of the functional (2.5). The critical point will be obtained as the minimizer under the constrain of L^2-*sphere*

$$S(a) = \left\{ u \in \mathcal{H} : \int_{\mathbb{R}^n} u^2 \mathrm{d}x = a \right\}.$$

We are going to study the minimization problem with L^2-*constraint*

$$E(a) = \inf_{u \in S(a)} I(u). \tag{2.6}$$

The solution of (2.6) $u = \tilde{u}$ is called a normalized ground state solution satisfying problem (1.3) if it has minimal energy among all solutions:

$$dI|_{S(a)}(\tilde{u}) = 0 \text{ and } I(\tilde{u}) = \inf \left\{ I(u) : dI|_{S(a)}(\tilde{u}) = 0, \tilde{u} \in S(a) \right\}.$$

In this paper, we will be especially interested in the existence of normalized ground state solutions.

Lemma 2.2. *(Gagliardo-Nirenberg inequality). For all* $u \in H^1(\mathbb{R}^N)$*, we have*

$$\|u\|_p^p \le C(N) \|\nabla u\|_{p'}^2 \|u\|_2^{p-p'}, \ 2 \le p < 2^*,$$

where $C(N)$ is a positive constant depending on N and $p' = \dfrac{N(p-2)}{2p}$.

Lemma 2.3. *Suppose* (*f*1) *and* (*f*2) *hold. Then, for any* $a>0$ *, the functional* I *is bounded from below and coercive on S*(*a*).

Proof. Assumptions (f1) and (f2) imply that for any $\xi>0$, there exist $c(\xi)>0$ such that

$$F(s)\le c(\xi)|s|^2+\xi|s|^l,\ \forall s\in\mathbb{R}.$$

Hence, combining Lemma 2.2 and $p=l=2+\frac{4}{N}$ we obtain

$$\left|\int_{\mathbb{R}^N}F(u)\mathrm{d}x\right|\le c(\xi)|u|_2^2+\xi|u|_l^l$$
$$\le c(\xi)|u|_2^2+\xi C(N)|\nabla u|_2^2|u|_2^{\frac{4}{N}}.$$

Since we can choose ξ such that $\xi C(N)a^{\frac{2}{N}}=\frac{1}{4}$, and according to (2.1) we have

$$\begin{aligned}I(u)&=\frac{1}{2}\int_{\mathbb{R}^n}|\nabla u|^2+V(x)u^2-\omega\phi_u u^2\mathrm{d}x-\int_{\mathbb{R}^n}F(u)\mathrm{d}x\\&\ge\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2}V(x)\int_{\mathbb{R}^N}u^2\mathrm{d}x-\int_{\mathbb{R}^N}F(u)\mathrm{d}x\\&\ge\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2}\inf_{x\in\mathbb{R}^N}V(x)\int_{\mathbb{R}^N}u^2\mathrm{d}x-\int_{\mathbb{R}^N}F(u)\mathrm{d}x\\&\ge\frac{1}{4}\|\nabla u\|_2^2-ca>-\infty.\end{aligned}$$

Therefore, *I* is bounded from below and coercive on $S(a)$.□

The following lemma is Lemma 2.2 in [16].

Lemma 2.4. *Suppose* (f1) *and* (f2) *hold and* $\{u_n\}_{n\in N}$ *is a bounded sequence in* $\mathcal{H}$*. If* $\lim_{n\to\infty}|u_n|_2^2=0$ *holds, then it is true that*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}F(u_n)\mathrm{d}x=0.$$

Next, we collect a variant of Lemma 2.2 in [21]. The proof is similar, so we omit it.

Lemma 2.5. *Suppose* (f1) *and* (f2) *hold and* $\{u_n\}_{\in N}$ *is a bounded sequence in* $\mathcal{H}$*, then we have* $u_n\rightharpoonup u$ *in* $\mathcal{H}$*, thus*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left[F(u_n)-F(u)-F(u_n-u)\right]\mathrm{d}x=0.$$

When $V(x)$ satisfies (V1) and (V2), the functional *I* is weakly lower semi-continuous. Thus, the following compact embedding theorem holds, which follows from Lemma 3.4 in [22].

Lemma 2.6. *Suppose* (V1) *and* (V2) *hold. Then the embedding* $\mathcal{H}\hookrightarrow L^p(\mathbb{R}^N)$ *is compact for any* $p\in[2,2^*)$.

Proof of Theorem 1.1. Let $\{u_n\}\subset S(a)$ be a minimizing sequence of *I* with concerning $E(a)$. Then, by (2.5) we obtain

$$I(u_n)=\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u_n|^2+V(x)u_n^2-\omega\phi_{u_n}u_n^2\right)\mathrm{d}x-\int_{\mathbb{R}^N}F(u_n)\mathrm{d}x.$$

According to Lemma 2.3, we know that $\{u_n\}$ is bounded in $\mathcal{H}$. Moreover, letting u_0 be in $\mathcal{H}$, and by Lemma 2.6 we conclude

$$u_n \rightharpoonup u_0 \text{ in } \mathcal{H}, \tag{2.7}$$

$$u_n \to u_0 \text{ in } L^q(\mathbb{R}^N),\ 2 \le q < 2^*, \tag{2.8}$$

$$u_n \to u_0 \text{ a.e. in } \mathbb{R}^N.$$

We also have

$$I(u_0) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u_0|^2 + V(x)u_0^2 - \omega\phi_{u_0}u_0^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} F(u_0)\,\mathrm{d}x.$$

Since (2.8) holds, we have $\lim_{n\to\infty}|u_n - u_0|_2^2 = 0$. Then, by Lemma 2.4 we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F(u_n - u_0)\,\mathrm{d}x = 0.$$

Moreover, by Lemma 2.5 we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left[F(u_n) - F(u_0)\right]\mathrm{d}x = 0.$$

which implies

$$\int_{\mathbb{R}^N} F(u_n)\,\mathrm{d}x \to \int_{\mathbb{R}^N} F(u_0)\,\mathrm{d}x \text{ as } n\to\infty. \tag{2.9}$$

Hence, combining (2.1), (2.9) and weak lower semicontinuity of the norm $\|\cdot\|_{\mathcal{H}}$, we have

$$E(a) \le I(u_0) \le \liminf_{n\to\infty} I(u_n) = E(a),$$

which implies $I(u_0) = E(a)$. Then, u_0 satisfies

$$\begin{cases} -\Delta u_0 + V(x)u_0 - (2\omega + \phi)\phi u_0 = f(u_0) \text{ in } \mathbb{R}^N, \\ \Delta\phi = (\omega + \phi)u_0^2 \text{ in } \mathbb{R}^N, \end{cases}$$

and $\int_{\mathbb{R}^N}|u_0(x)|^2\,\mathrm{d}x = a$. Therefore, problem (1.3) has a normalized ground state solution.☐

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Carrião, P.C., Cunha, P.L. and Miyagaki, O.H. (2012) Positive Ground State Solutions for the Critical Klein-Gordon-Maxwell System with Potentials. *Nonlinear Analysis: Theory, Methods & Applications*, **75**, 4068-4078. https://doi.org/10.1016/j.na.2012.02.023

[2] Zhang, X. and Huang, C. (2022) Nontrivial Solutions for Klein-Gordon-Maxwell Systems with Sign-Changing Potentials. *Boundary Value Problems*, **10**, Article No. 83. https://doi.org/10.1186/s13661-022-01664-4

[3] Cassani, D. (2004) Existence and Non-Existence of Solitary Waves for the Critical Klein-Gordon Equation Coupled with Maxwell's Equations. *Nonlinear Analysis: Theory, Methods & Applications*, **58**, 733-747. https://doi.org/10.1016/j.na.2003.05.001

[4] Benci, V. and Fortunato, D. (2002) Solitary Waves of the Nonlinear Klein-Gordon

Equation Coupled with the Maxwell Equations. *Reviews in Mathematical Physics*, **14**, 409-420. https://doi.org/10.1142/S0129055X02001168

[5] He, C.-M., Li, L., Chen, S.-J. and O'Regan, D. (2022) Ground State Solution for the Nonlinear Klein-Gordon Equation Coupled with Born-Infeld Theory with Critical Exponents. *Analysis and Mathematical Physics*, **12**, Article No. 48. https://doi.org/10.1007/s13324-022-00661-1

[6] Wang, F. (2011) Ground-State Solutions for the Electrostatic Nonlinear Klein-Gordon-Maxwell System. *Nonlinear Analysis: Theory, Methods & Applications*, **74**, 4796-4803. https://doi.org/10.1016/j.na.2011.04.050

[7] Wang, F. (2011) Solitary Waves for the Klein-Gordon-Maxwell System with Critical Exponent. *Nonlinear Analysis: Theory, Methods & Applications*, **74**, 827-835. https://doi.org/10.1016/j.na.2010.09.033

[8] Bartsch, T. and Jeanjean, L. (2018) Normalized Solutions for Nonlinear Schrödinger Systems. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, **148**, 225-242. https://doi.org/10.1017/S0308210517000087

[9] Bartsch, T., Jeanjean, L. and Soave, N. (2016) Normalized Solutions for a System of Coupled Cubic Schrödinger Equations on $\mathbb{R}^3$. *Journal de Mathématiques Pures et Appliquées*, **106**, 583-614. https://doi.org/10.1016/j.matpur.2016.03.004

[10] Bartsch, T. and Soave, N. (2017) A Natural Constraint Approach to Normalized Solutions of Nonlinear Schrödinger Equations and Systems. *Journal of Functional Analysis*, **272**, 4998-5037. https://doi.org/10.1016/j.jfa.2017.01.025

[11] Bartsch, T. and Soave, N. (2019) Multiple Normalized Solutions for a Competing System of Schrödinger Equations. *Calculus of Variations and Partial Differential Equations*, **58**, Article No. 22. https://doi.org/10.1007/s00526-018-1476-x

[12] Gou, T. and Jeanjean, L. (2018) Multiple Positive Normalized Solutions for Nonlinear Schrödinger Systems. *Nonlinearity*, **31**, 2319-2345. https://doi.org/10.1088/1361-6544/aab0bf

[13] Ikoma, N. and Tanaka, K. (2019) A Note on Deformation Argument for L^2 Normalized Solutions of Nonlinear Schrödinger Equations and Systems. *Advances in Differential Equations*, **24**, 609-646. https://doi.org/10.57262/ade/1571731543

[14] Yang, Z., Qi, S. and Zou, W. (2022) Normalized Solutions of Nonlinear Schrödinger Equations with Potentials and Non-Autonomous Nonlinearities. *Journal of Geometric Analysis*, **32**, Article No. 159. https://doi.org/10.1007/s12220-022-00897-0

[15] Soave, N. (2020) Normalized Ground States for the NLS Equation with Combined Nonlinearities: The Sobolev Critical Case. *Journal of Functional Analysis*, **279**, Article ID: 108610. https://doi.org/10.1016/j.jfa.2020.108610

[16] Shibata, M. (2014) Stable Standing Waves of Nonlinear Schrödinger Equations with a General Nonlinear Term. *Manuscripta Mathematica*, **143**, 221-237. https://doi.org/10.1007/s00229-013-0627-9

[17] Ikoma, N. and Miyamoto, Y. (2020) Stable Standing Waves of Nonlinear Schrödinger Equations with Potentials and General Nonlinearities. *Calculus of Variations and Partial Differential Equations*, **59**, Article No. 48. https://doi.org/10.1007/s00526-020-1703-0

[18] Chen, Z. and Zou, W. (2021) Normalized Solutions for Nonlinear Schrödinger Systems with Linear Couples. *Journal of Mathematical Analysis and Applications*, **499**, Article ID: 125013. https://doi.org/10.1016/j.jmaa.2021.125013

[19] Bartsch, T., Molle, R., Rizzi, M. and Verzini, G. (2021) Normalized Solutions of Mass Supercritical Schrödinger Equations with Potential. *Communications in Partial Diffe-*

rential Equations, **46**, 1729-1756.
https://doi.org/10.1080/03605302.2021.1893747

[20] Bartsch, T. and de Valeriola, S. (2013) Normalized Solutions of Nonlinear Schrödinger Equations. *Archiv der Mathematik*, **100**, 75-83.
https://doi.org/10.1007/s00013-012-0468-x

[21] Zhu, X. and Cao, D. (1989) The Concentration-Compactness Principle in Nonlinear Elliptic Equations. *Acta Mathematica Scientia*, **9**, 307-328.
https://doi.org/10.1016/S0252-9602(18)30356-4

[22] Zou, W. and Schechter, M. (2006) Critical Point Theory and Its Applications. Springer, New York.