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Exploring Cauchy-Jensen μ_j-Function Inequality with $3k$-Variables on Complex Banach Spaces and Application to Establish Isomorphism between Unital Banach Algebras

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Abstract

In this paper, I study to establish general Cauchy-Jensen μ_j-function inequalities by relying on general Cauchy-Jensen equations with $3k$-variables on complex Banach spaces. First, I investigated the Cauchy-Jensen μ_j-function inequalities in complex Banach spaces and then I establish Isomorphisms between Unital Banach Algebras. These are the main results of this paper.

Subject Areas

Mathematics

Keywords

Cauchy-Jensen Equation with Variables, Cauchy-Jensen μ_j-Function Inequalities, Complex Banach Space, Isomorphisms between Unital Banach Algebras

1. Introduction

Let $\mathbb{A}$ and $\mathbb{B}$ be vector spaces on the same field $\mathbb{K}$, and $\phi : \mathbb{A} \to \mathbb{B}$. I use the notation $\|\cdot\|$ for all the norms on both $\mathbb{A}$ and $\mathbb{B}$. In this paper, I investigate additive functional inequalities when $\mathbb{A}$ is a normed vector space and $\mathbb{B}$ is a Banach space.

In fact, when $\mathbb{A}$ is a complex normed space and $\mathbb{B}$ is a complex Banach space, I solve and prove the general Cauchy-Jensen stability for the following additive functional inequalities.

$$\left\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\| \leq \left\| g(\mu_1)\left(k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(y_i)\right)\right\|_{\mathbb{B}} \tag{1}$$

and

$$\left\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(y_i)\right\| \leq \left\| g(\mu_2)\left(2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right)\right\|_{\mathbb{B}} \tag{2}$$

Final

$$\left\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\| \leq \left\| g(\mu_3)\left(2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right)\right\|_{\mathbb{B}} \tag{3}$$

Based on the general Cauchy-Jensen equations with the following 3*k*-variables.

$$k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}\phi(x_i)+2k\sum_{i=1}^{k}\phi(z_i) \tag{4}$$

$$k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}\phi(y_i) \tag{5}$$

$$2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}\phi(x_i)+\sum_{i=1}^{k}\phi(y_i)+2k\sum_{i=1}^{k}\phi(z_i) \tag{6}$$

Note: The $g(\mu_i)$-functional inequality.

The study of the functional equation stability is originated from a question of S. M. Ulam [1], concerning the stability of group homomorphisms. Let $(\mathbf{G},*)$ be a group and let $(\mathbf{G}',\circ,d)$ be a metric group with metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, there exists a $\delta>0$ such that if $f:\mathbf{G}\to\mathbf{G}'$ satisfies

$$d\left(f(x*y),f(x)\circ f(y)\right)<\delta$$

for all $x,y\in\mathbf{G}$, then there is a homomorphism $h:\mathbf{G}\to\mathbf{G}'$ with

$$d\left(f(x),h(x)\right)<\varepsilon$$

for all $x\in\mathbf{G}$, if the answer is affirmative, I would say that equation of homomophism $h(x*y)=h(y)\circ h(y)$ is stable. The concept of stability for a functional equation arises when I replace a functional equation with an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is how do the solutions of the inequality differ from those of the given function equation? Hyers gave a first affirmative answer to the question of Ulam as follows:

In 1941, D. H. Hyers [2], let $\varepsilon\geq 0$ and let $f:\mathbf{E}_1\to\mathbf{E}_2$ be a mapping between Banach space such that

$$\left\| f\left(x+y\right)-f\left(x\right)-f\left(y\right)\right\| \le \varepsilon,$$

for all $x, y \in \mathbf{E}_1$ and some $\varepsilon \ge 0$. It was shown that the limit

$$T\left(x\right)=\lim_{n\to\infty}\frac{f\left(2^n x\right)}{2^n}$$

exists for all $x \in \mathbf{E}_1$ and that $T:\mathbf{E}_1 \to \mathbf{E}_2$ is that unique additive mapping satisfying

$$\left\| f\left(x\right)-T\left(x\right)\right\| \le \varepsilon, \forall x \in \mathbf{E}_1.$$

Next in 1978, Th. M. Rassias [3] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded:

Consider $\mathbf{E}, \mathbf{E}'$ to be two Banach spaces, and let $f:\mathbf{E}\to\mathbf{E}'$ be a mapping such that $f\left(tx\right)$ is continuous in *t* for each fixed *x*. Assume that there exist $\theta \ge 0$ and $p \in \left[0,1\right)$ such that

$$\left\| f\left(x+y\right)-f\left(x\right)-f\left(y\right)\right\| \le \varepsilon\left(\left\|x\right\|^p+\left\|y\right\|^p\right), \forall x, y \in \mathbb{E}.$$

then there exists a unique linear $L:\mathbf{E}\to\mathbf{E}'$ satisfies

$$\left\| f\left(x\right)-L\left(x\right)\right\| \le \frac{2\theta}{2-2^p}\left\|x\right\|^p, x \in \mathbf{E}.$$

Next J. M. Rassias [4] followed the spirit of the innovative approach of Th. M. Rassias for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\left\|x\right\|^p+\left\|y\right\|^p$ by $\left\|x\right\|^p\left\|y\right\|^p$ for $p, q \in \mathbb{R}$ with $p+q \ne 1$.

Next in 1992, a generalized of Rassias' Theorem was obtained by Găvruta [5] Gilányi [6] and Fechner [7], proving the Hyers-Ulam stability of the functional inequality.

Next is about the development of γ-function inequalities of mathematicians in the world.

In 2020, Ly Van An studied the inequalities of the function on the group and the ring see [8]

$$\left\| f\left(\sum_{j=1}^{n} x_j+\frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right)-\sum_{j=1}^{n} f\left(x_j\right)-\sum_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}} \le \varepsilon, \forall \varepsilon \ge 0 \tag{7}$$

and

$$\left\| f\left(\prod_{j=1}^{n} x_j+\frac{1}{n}\prod_{j=1}^{n} x_{n+j}\right)-\prod_{j=1}^{n} f\left(x_j\right)-\prod_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}} \le \delta, \forall \delta \ge 0 \tag{8}$$

Next, in 2020, Ly Van An continued to study additive β-functional inequality in complex Banach spaces see [9]

$$\begin{aligned}&\left\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{k}+\sum_{j=1}^{k} z_j\right)-\sum_{j=1}^{k} f\left(\frac{x_j+y_j}{k}\right)-\sum_{j=1}^{k} f\left(z_j\right)\right\| \\ &\le \left\| \beta\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{k^2}+\frac{1}{k}\sum_{j=1}^{k} z_j\right)-\frac{1}{k}\sum_{j=1}^{k} f\left(\frac{x_j+y_j}{k}\right)-\frac{1}{k}\sum_{j=1}^{k} f\left(z_j\right)\right)\right\|\end{aligned} \tag{9}$$

and

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{k^2}+\frac{1}{k}\sum_{j=1}^{k}z_j\right)-\frac{1}{k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{k}\right)-\frac{1}{k}\sum_{j=1}^{k}f\left(z_j\right)\right\|\\&\leq\left\|\beta\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{k}+\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{k}\right)-\sum_{j=1}^{k}f\left(z_j\right)\right)\right\|\end{aligned}\tag{10}$$

Next, in 2021, Ly Van An continued to study additive functional inequality investigated in non-Archimedean Banach spaces see [10]

$$\begin{aligned}&\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}F\left(x_j\right)\right\|_{\mathbf{X}_2}\\&\leq\left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{1}{k}\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\frac{1}{k}\sum_{j=1}^{k}F\left(x_j\right)\right\|_{\mathbf{X}_2}\end{aligned}\tag{11}$$

and

$$\begin{aligned}&\left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{1}{k}\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\frac{1}{k}\sum_{j=1}^{k}F\left(x_j\right)\right\|_{\mathbf{X}_2}\\&\leq\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}F\left(\frac{x_{k+1}}{k}\right)-\sum_{j=1}^{k}F\left(x_j\right)\right\|_{\mathbf{X}_2}\end{aligned}\tag{12}$$

Recently, Ly Van An continues to give the general Cauchy-Jensen see [11] functional equations after I study the μ_j-function inequalities (1), (2) and (3) based on the functional Equations (4)-(6) on a complex Banach space. In this paper, Isolve and proved the μ_j-function inequalities (1), (2) and (3) based on the functional Equations (4)-(6) on a complex Banach space, *i.e.* the μ_j-functional inequalities with $3k$ variables. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, I will prove that the mappings satisfy the (1), (2) and (3). Thus, the results in this paper are generalization of those in [8] [9] [10] [11] [12].

To overcome the limitation on the number of variables in the classical Cauchy-Jensen p-function inequalities I introduce three general Cauchy-Jensen μ_j-function inequalities with $3k$-variables on complex Banach spaces to help math researchers in the space they navigate. To get the above idea, I rely on the thinking of world mathematicians, see [1]-[23]. First, I build the general Cauchy-Jensen equations, and then build the functional inequalities.

The paper is organized as follows: In the section preliminaries, I remind some basic notations such as: Cauchy equation, Cauchy-Jensen equation, Classical Cauchy-Jensen *βj*-functional equation and Classical Cauchy-Jensen *βj*-functional inequalities.

Section 3: The basis for building a solution for the Cauchy-Jensen μ_j-function inequality.

Section 4: Establishing Solutions for general Cauchy-Jensen μ_j-function inequalities.

Section 5: Establish Isomorphisms between Unital Banach Algebras.

2. Preliminaries

2.1. Solutions of the Equation

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive *mapping*.

The functional equations

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)=f(x)+2f(z) \tag{13}$$

$$f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x-y}{2}+z\right)=f(y) \tag{14}$$

$$2f\left(\frac{x+y}{2}+z\right)=f(x)+f(y)+2f(z) \tag{15}$$

is called the Cauchy-Jensen equation. In particular, every solution of the equation is said to be Cauchy-Jensen additive mapping and the functional equations

$$\begin{aligned}&f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)-f(x)-2f(z)\\&=\beta_1\left(f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x-y}{2}+z\right)-f(y)\right)\end{aligned} \tag{16}$$

$$\begin{aligned}&f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x-y}{2}+z\right)-f(y)\\&=\beta_2\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right)\end{aligned} \tag{17}$$

$$\begin{aligned}&f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)-f(x)-2f(z)\\&=\beta_3\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right)\end{aligned} \tag{18}$$

is called the Classical Cauchy-Jensen βj-functional equation. In particular, every solution of the βj-functional equation is said to be an additive mapping.

2.2. Solutions of the Functional Inequalities

The functional inequalities

$$\begin{aligned}&\left\|f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)-f(x)-2f(z)\right\|\\&\le\left\|\beta_1\left(f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x-y}{2}+z\right)-f(y)\right)\right\|\end{aligned} \tag{19}$$

$$\begin{aligned}&\left\|f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x-y}{2}+z\right)-f(y)\right\|\\&\le\left\|\beta_2\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right)\right\|\end{aligned} \tag{20}$$

$$\begin{aligned}&\left\|f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)-f(x)-2f(z)\right\|\\&\le\left\|\beta_3\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right)\right\|\end{aligned}\tag{21}$$

is called the Classical Cauchy-Jensen βj-functional inequalities. In particular, every solution of the βj-functional inequalities is said to be an additive mapping.

$$\mathbf{D}:=\left\{h:\mathbb{C}\to\mathbb{C}:h(\eta_i)=\eta_i,\left|h(\eta_i)\right|<1\text{ as }i=1\text{ and }\left|h(\eta_i)\right|<\frac{1}{2}\text{ as }i>1,i\in\mathbb{N}^*\right\}$$

3. Basis for Building Solutions for Cauchy-Jensen. P-Function Inequalities

Note Here I assume that $\mathbb{A}$, $\mathbb{B}$ be real or complex vector spaces and $g\in\mathbf{D}$.

Lemma 1. *Suppose that* $\mathbb{A}$, $\mathbb{B}$ *be real or complex vector space. If the mapping* $\phi_1,\phi_2,\phi_3:\mathbb{A}\to\mathbb{B}$ *satisfies of the following functional inequalities*

$$\begin{aligned}&\left\|k\phi_1\left(\sum_{i=1}^k\frac{x_i+y_i}{2k}+\sum_{i=1}^k z_i\right)+k\phi_1\left(\sum_{i=1}^k\frac{x_i-y_i}{2k}+\sum_{i=1}^k z_i\right)-\sum_{i=1}^k\phi_1(x_i)-2k\sum_{i=1}^k\phi_1(z_i)\right\|\\&\le\left\|g(\mu_1)\left(k\phi_1\left(\sum_{i=1}^k\frac{x_i+y_i}{2k}+\sum_{i=1}^k z_i\right)-k\phi_1\left(\sum_{i=1}^k\frac{x_i-y_i}{2k}+\sum_{i=1}^k z_i\right)-\sum_{i=1}^k\phi_1(y_i)\right)\right\|_{\mathbb{B}}\end{aligned}\tag{22}$$

$$\begin{aligned}&\left\|k\phi_2\left(\sum_{i=1}^k\frac{x_i+y_i}{2k}+\sum_{i=1}^k z_i\right)-k\phi_2\left(\sum_{i=1}^k\frac{x_i-y_i}{2k}+\sum_{i=1}^k z_i\right)-\sum_{i=1}^k\phi_2(y_i)\right\|\\&\le\left\|g(\mu_2)\left(2k\phi_2\left(\sum_{i=1}^k\frac{x_i+y_i}{2k}+\sum_{i=1}^k z_i\right)-\sum_{i=1}^k\phi_2(x_i)-\sum_{i=1}^k\phi_2(y_i)-2k\sum_{i=1}^k\phi_2(z_i)\right)\right\|_{\mathbb{B}}\end{aligned}\tag{23}$$

$$\begin{aligned}&\left\|k\phi_3\left(\sum_{i=1}^k\frac{x_i+y_i}{2k}+\sum_{i=1}^k z_i\right)+k\phi_3\left(\sum_{i=1}^k\frac{x_i-y_i}{2k}+\sum_{i=1}^k z_i\right)-\sum_{i=1}^k\phi_3(x_i)-2k\sum_{i=1}^k\phi_3(z_i)\right\|\\&\le\left\|g(\mu_3)\left(2k\phi_3\left(\sum_{i=1}^k\frac{x_i+y_i}{2k}+\sum_{i=1}^k z_i\right)-\sum_{i=1}^k\phi_3(x_i)-\sum_{i=1}^k\phi_3(y_i)-2k\sum_{i=1}^k\phi_3(z_i)\right)\right\|_{\mathbb{B}}\end{aligned}\tag{24}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$, if and only the mappings $\phi_1,\phi_2,\phi_3:\mathbb{A}\to\mathbb{B}$ is additive.

Note: Here I prove (22) while (23) and (24) are completely similar proofs.

Proof. Assume that $f:\mathbb{A}\to\mathbb{B}$ satisfies (22).

I replace $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,z,\cdots,0)$ in (22), I have

$$\left\|k\phi(x+z)-k\phi(x)-k\phi(z)\right\|\le\left\|g(\mu_1)\left(k\phi(x+z)-k\phi(x)-k\phi(z)\right)\right\|_{\mathbb{B}}\tag{25}$$

for all $x,z\in\mathbf{X}$. So

$$\phi(x+z)=\phi(x)+\phi(z)$$

Hence $\phi:\mathbb{A}\to\mathbb{B}$ is Cauchy additive.

The remaining (23) and (24) are completely similar proofs. □

From the proof of the lemma, I have the following corollary:

Corollary 1. *Suppose that* $\mathbb{X},\mathbb{Y}$ *be real or complex vector space. If the map-*

ping $\phi_1,\phi_2,\phi_3:\mathbb{A}\to\mathbb{B}$ *satisfies the following functional equations*

$$\begin{aligned}&k\phi_1\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi_1\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi_1(x_i)-2k\sum_{i=1}^{k}\phi_1(z_i)\\&=g(\mu_1)\left(k\phi_1\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi_1\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi_1(y_i)\right)\end{aligned}\tag{26}$$

$$\begin{aligned}&k\phi_2\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi_2\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi_2(y_i)\\&=g(\mu_2)\left(2k\phi_2\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi_2(x_i)-\sum_{i=1}^{k}\phi_2(y_i)-2k\sum_{i=1}^{k}\phi_2(z_i)\right)\end{aligned}\tag{27}$$

$$\begin{aligned}&k\phi_3\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi_3\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi_3(x_i)-2k\sum_{i=1}^{k}\phi_3(z_i)\\&=g(\mu_3)\left(2k\phi_3\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi_3(x_i)-\sum_{i=1}^{k}\phi_3(y_i)-2k\sum_{i=1}^{k}\phi_3(z_i)\right)\end{aligned}\tag{28}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$, if and only the mappings $\phi_1,\phi_2,\phi_3:\mathbb{A}\to\mathbb{B}$ is additive.

4. Establishing Solutions for General Cauchy-Jensen μ_j-Function Inequalities

Now, I first study the solutions of (1), (2) and (3). Note that for this μ_j-function inequalities, $\mathbb{A}$ be real or complex vector space with norm $\|\cdot\|_{\mathbb{A}}$ and that $\mathbb{B}$ is a Banach space with norm $\|\cdot\|_{\mathbb{B}}$. Under this setting, I can show that the mappings satisfying (1), (2) and (3) is additive.

Theorem 1. *Suppose that* $\phi:\mathbb{A}\to\mathbb{B}$ *be a mapping. If there is a function* $\varphi:\mathbb{A}^{3k}\to[0,\infty)$ *such that satisfying*

$$\begin{aligned}&\left\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right.\\&\left.-g(\mu_1)\left(k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(y_i)\right)\right\|_{\mathbb{B}}\\&\le\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\end{aligned}\tag{29}$$

and

$$\tilde{\varphi}(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)=\sum_{j=1}^{\infty}2^j\varphi\left(\frac{x_1}{2^j},\cdots,\frac{x_k}{2^j},\frac{y_1}{2^j},\cdots,\frac{y_k}{2^j},\frac{z_1}{2^j},\cdots,\frac{z_k}{2^j}\right)<\infty\tag{30}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. Then there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\le\frac{1}{k}\left(\frac{1}{2|1-g(\mu_1)|}\right)\tilde{\varphi}(x,\cdots,x,x,\cdots,x,x,\cdots,0)\tag{31}$$

for all $x\in\mathbf{X}$.

Proof. I replace $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,x,\cdots,0)$ in (29), I have

$$\left\|k\phi(2x)-2k\phi(x)\right\|_{\mathbb{B}}\le\left(\frac{1}{\left|1-g(\mu_1)\right|}\right)\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,0) \tag{32}$$

for all $x\in\mathbb{A}$. So

$$\left\|\phi(x)-2\phi\left(\frac{x}{2}\right)\right\|_{\mathbb{B}}\le\frac{1}{k}\left(\frac{1}{\left|1-g(\mu_1)\right|}\right)\varphi\left(\frac{x}{2},\cdots,\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2},\frac{x}{2},\cdots,0\right)$$

for all $x\in\mathbb{A}$. Hence

$$\begin{aligned}&\left\|2^l\phi\left(\frac{x}{2^l}\right)-2^m\phi\left(\frac{x}{2^m}\right)\right\|_{\mathbb{B}}\\&\le\frac{1}{k}\sum_{j=m}^{l-1}\left\|2^l\phi\left(\frac{x}{2^j}\right)-2^{j+1}\phi\left(\frac{x}{2^{j+1}}\right)\right\|_{\mathbb{B}}\\&\le\frac{1}{k}\sum_{j=m}^{l-1}\left(\frac{2^j}{\left|1-g(\mu_1)\right|}\right)\varphi\left(\frac{x}{2^{j+1}},\cdots,\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\cdots,\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\cdots,0\right)\\&=\frac{1}{k}\left(\frac{1}{2\left|1-g(\mu_1)\right|}\right)\sum_{j=m}^{l-1}2^{j+1}\varphi\left(\frac{x}{2^{j+1}},\cdots,\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\cdots,\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\cdots,0\right)\\&=S_{l-1}-S_{m-1}\end{aligned} \tag{33}$$

At here

$$S_p=\frac{1}{k}\left(\frac{1}{2\left|1-g(\mu_1)\right|}\right)\sum_{j=1}^{p}2^{j+1}\varphi\left(\frac{x}{2^{j+1}},\cdots,\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\cdots,\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\cdots,0\right)<\infty \tag{34}$$

and so there exists $q\ge 0$ such that $S_p\to r$ as $m\to\infty$. Therefore so when I give $\lim_{l,m\to\infty}$ in (33), I have

$$\left\|2^l\phi\left(\frac{x}{2^l}\right)-2^m\phi\left(\frac{x}{2^m}\right)\right\|_{\mathbb{B}}\to 0,\ \text{as } l,m\to\infty. \tag{35}$$

for all nonnegative integers m and l with $l>m$ and for all $x\in\mathbb{A}$. It follows (30) and (33) that the sequence $\left\{2^n\phi\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all $x\in\mathbb{A}$. Since $\mathbb{B}$ is complete, the sequence $\left\{2^n\phi\left(\frac{x}{2^n}\right)\right\}$ converges, one can define the mapping $\psi:\mathbb{A}\to\mathbb{B}$ by

$$\psi(x)=\lim_{n\to\infty}2^n\phi\left(\frac{x}{2^n}\right)$$

for all $x\in\mathbb{A}$. By (30) and (29),

$$\begin{aligned}&\Bigg\|k\psi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\psi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\psi(x_i)-2k\sum_{i=1}^{k}\psi(x_i)\\&-g(\mu_1)\left(k\psi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\psi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\psi(x_i)\right)\Bigg\|_{\mathbb{B}}\\&=\lim_{n\to\infty}2^n\Bigg\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2^n\cdot 2k}+\sum_{i=1}^{k}\frac{z_i}{2^n}\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2^n\cdot 2k}+\sum_{i=1}^{k}\frac{z_i}{2^n}\right)-\sum_{i=1}^{k}\phi\left(\frac{x_i}{2^n}\right)-2k\sum_{i=1}^{k}\phi\left(\frac{z_i}{2^n}\right)\end{aligned}$$

$$-g(\mu_1)\left(k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2^n\cdot 2k}+\sum_{i=1}^{k}\frac{z_i}{2^n}\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2^n\cdot 2k}+\sum_{i=1}^{k}\frac{z_i}{2^n}\right)-\sum_{i=1}^{k}\phi\left(\frac{y_i}{2^n}\right)\right)\Bigg\|_{\mathbb{B}}$$

$$\le \lim_{n\to\infty}2^n\varphi\left(\frac{x_1}{2^n},\cdots,\frac{x_k}{2^n},\frac{y_1}{2^n},\cdots,\frac{y_k}{2^n},\frac{z_1}{2^n},\cdots,\frac{z_k}{2^n}\right)=0$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. So

$$k\psi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\psi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)\sum_{i=1}^{k}\psi(x_i)+2k\sum_{i=1}^{k}\psi(z_i)$$

$$=g(\mu_1)\left(k\psi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\psi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)\sum_{i=1}^{k}\psi(x_i)\right)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$.

By Corollary 1, the mapping $\psi:\mathbb{A}\to\mathbb{B}$ is additive mapping.

Now, let $\psi:\mathbb{A}\to\mathbb{B}$ be another generalized Cauchy-Jensen additive mapping satisfying (31). Then I have

$$\begin{aligned}\|\psi(x)-\psi'(x)\|_{\mathbb{B}}&=2^n\left\|\psi\left(\frac{x}{2^n}\right)-\psi'\left(\frac{x}{2^n}\right)\right\|_{\mathbb{B}}\\&\le 2^n\left(\left\|\psi\left(\frac{x}{2^n}\right)-\phi\left(\frac{x}{2^n}\right)\right\|_{\mathbb{B}}+\left\|\psi'\left(\frac{x}{2^n}\right)-\phi\left(\frac{x}{2^n}\right)\right\|_{\mathbb{B}}\right)\\&\le 2\frac{2^n}{k}\left(\frac{1}{2|1-g(\mu_1)|}\right)\tilde{\varphi}\left(\frac{x}{2^n},\cdots,\frac{x}{2^n},\frac{x}{2^n},\cdots,\frac{x}{2^n},\frac{x}{2^n},\cdots,0\right)\\&=\frac{2^n}{k}\left(\frac{1}{|1-g(\mu_1)|}\right)\tilde{\varphi}\left(\frac{x}{2^n},\cdots,\frac{x}{2^n},\frac{x}{2^n},\cdots,\frac{x}{2^n},\frac{x}{2^n},\cdots,0\right)\end{aligned}\tag{36}$$

which tends to zero as $n\to\infty$ for all $x\in\mathbb{A}$. So I can conclude that $\psi(x)=\psi'(x)$ for all $x\in\mathbb{A}$. This proves the uniqueness of ψ'. □

Corollary 2. Suppose p and θ be positive real numbers with $p>1$, and let $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping such that

$$\left\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right.$$

$$\left.-g(\mu_1)\left(k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(y_i)\right)\right\|_{\mathbb{B}}$$

$$\le\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|y_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|z_i\|_{\mathbb{A}}^p\right)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. The there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\le\left(2+\frac{1}{k}\right)\frac{\theta}{(2^p-2)|1-g(\mu_1)|}\|x\|_{\mathbb{A}}^p$$

for all $x\in\mathbb{A}$.

Corollary 3. Suppose $p_1,p_2,\cdots,p_k$ and θ be positive real numbers with $3p_1+2p_2+\cdots+2p_k>1$, and let $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping such that

$$\begin{aligned}&\left\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right.\\&\left.-g(\mu_1)\left(k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(y_i)\right)\right\|_{\mathbb{B}}\\&\le\theta\prod_{i=1}^{k}\|x_i\|_{\mathbb{A}}^{p_i}\cdot\prod_{i=1}^{k}\|y_i\|_{\mathbb{A}}^{p_i}\cdot\|z_1\|_{\mathbb{A}}^{p_1}\cdot\left(1+\prod_{i=2}^{k}\|z_i\|_{\mathbb{A}}^{p_i}\right)\end{aligned}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. The there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\le\frac{\theta}{k\left(2^{3p_1+2p_2+\cdots+2p_k}-2\right)|1-g(\mu_1)|}\|x\|_{\mathbb{A}}^{3p_1+2p_2+\cdots+2p_k}$$

for all $x\in\mathbb{A}$.

Theorem 2. *Suppose* $\phi:\mathbb{A}\to\mathbb{B}$ *be a mapping. If there is a function* $\varphi:\mathbb{A}^{3k}\to[0,\infty)$ *such that satisfying*

$$\begin{aligned}&\left\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right.\\&\left.-g(\mu_1)\left(k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(y_i)\right)\right\|_{\mathbb{B}}\\&\le\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\end{aligned}\tag{37}$$

and

$$\begin{aligned}&\tilde{\varphi}(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\\&=\sum_{j=1}^{\infty}\frac{1}{2^j}\varphi\left(2^jx_1,\cdots,2^jx_k,2^jy_1,\cdots,2^jy_k,2^jz_1,\cdots,2^jz_k\right)<\infty\end{aligned}\tag{38}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$.

Then there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\le\frac{1}{k}\left(\frac{1}{2|1-g(\mu_1)|}\right)\tilde{\varphi}(x,\cdots,x,x,\cdots,x,x,\cdots,0)\tag{39}$$

for all $x\in\mathbf{X}$.

Proof. I replace $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,x,\cdots,0)$ in (37), I have

$$\|k\phi(2x)-2k\phi(x)\|_{\mathbb{B}}\le\frac{1}{|1-g(\mu_1)|}\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,0)\tag{40}$$

for all $x\in\mathbb{A}$. So

$$\left\|\phi(x)-\frac{1}{2}\phi(2x)\right\|_{\mathbb{B}}\le\frac{1}{2k}\frac{1}{|1-g(\mu_1)|}\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,0)\tag{41}$$

for all $x\in\mathbb{A}$. Hence

$$\begin{aligned}&\left\|\frac{1}{2^l}\phi(2^lx)-\frac{1}{2^m}\phi(2^mx)\right\|_{\mathbb{B}}\\&\le\frac{1}{2k}\frac{1}{|1-g(\mu_1)|}\sum_{j=m}^{l-1}\frac{1}{2^j}\varphi\left(2^jx,\cdots,2^jx,2^jx,\cdots,2^jx,2^jx,\cdots,0\right)\end{aligned}\tag{42}$$

for all nonnegative integers m and l with $l > m$ and for all $x \in \mathbb{A}$. It follows (38) and (42) that the sequence $\left\{\frac{1}{2^n}\phi\left(2^n x\right)\right\}$ is a Cauchy sequence for all $x \in \mathbb{A}$. Since $\mathbb{B}$ is complete, the sequence $\left\{\frac{1}{2^n}\phi\left(2^n x\right)\right\}$ converges. So one can define the mapping $\psi : \mathbb{A} \to \mathbb{B}$ by

$$\psi(x) = \lim_{n\to\infty}\frac{1}{2^n} f\left(2^n x\right)$$

for all $x \in \mathbb{A}$. The proof is similar to the proof of Theorem 1. □

Corollary 4. Suppose p and θ be positive real numbers with $p < 1$, and let $\phi : \mathbb{A} \to \mathbb{B}$ be a mapping such that

$$\begin{aligned}&\Bigg\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\\&-g(\mu_1)\left(k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(y_i)\right)\Bigg\|_{\mathbb{B}}\\&\le\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|y_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|z_i\|_{\mathbb{A}}^p\right)\end{aligned}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbb{A}$. The there exists a unique additive mapping $\psi : \mathbb{A} \to \mathbb{B}$ such that

$$\begin{aligned}&\|\phi(x)-\psi(x)\|_{\mathbb{B}}\\&\le\left(2+\frac{1}{2k}\right)\frac{\theta}{\left(2-2^p\right)\left|1-g(\mu_1)\right|}\|x\|_{\mathbb{A}}^p\end{aligned}$$

for all $x \in \mathbb{A}$.

Corollary 5. *Let* $p_1, p_2, \cdots, p_k$ *and* θ *be positive real numbers with* $3p_1 + 2p_2 + \cdots + 2p_k < 1$, *and let* $\phi : \mathbb{A} \to \mathbb{B}$ *be a mapping such that*

$$\begin{aligned}&\Bigg\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\\&-g(\mu_1)\left(k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(y_i)\right)\Bigg\|_{\mathbb{B}}\\&\le\theta\prod_{i=1}^{k}\|x_i\|_{\mathbb{A}}^{p_i}\cdot\prod_{i=1}^{k}\|y_i\|_{\mathbb{A}}^{p_i}\cdot\|z_1\|_{\mathbb{A}}^{p_1}\cdot\left(1+\prod_{i=2}^{k}\|z_i\|_{\mathbb{A}}^{p_i}\right)\end{aligned}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbb{A}$. The there exists a unique additive mapping $\psi : \mathbb{A} \to \mathbb{B}$ such that

$$\begin{aligned}&\|\phi(x)-\psi(x)\|_{\mathbb{B}}\\&\le\frac{\theta}{2k\left(2-2^{3p_1+2p_2+\cdots+2p_k}\right)\left|1-g(\mu_1)\right|}\|x\|_{\mathbb{A}}^{3p_1+2p_2+\cdots+2p_k}\end{aligned}$$

for all $x \in \mathbb{A}$.

Theorem 3. *Let* $\phi : \mathbb{A} \to \mathbb{B}$ *be a mapping. If there is a function* $\varphi : \mathbb{A}^{3k} \to [0,\infty)$ *such that satisfying*

$$\begin{aligned}&\Bigg\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(y_i)\\&-g(\mu_2)\left(2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right)\Bigg\|_{\mathbb{B}}\\&\le\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\end{aligned} \tag{43}$$

and

$$\tilde{\varphi}(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)=\sum_{j=1}^{\infty}2^j\varphi\left(\frac{x_1}{2^j},\cdots,\frac{x_k}{2^j},\frac{y_1}{2^j},\cdots,\frac{y_k}{2^j},\frac{z_1}{2^j},\cdots,\frac{z_k}{2^j}\right)<\infty \tag{44}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$.

Then there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|\le\frac{1}{k}\left(\frac{1}{2|1-2g(\mu_2)|}\right)\tilde{\varphi}(x,\cdots,x,x,\cdots,x,x,\cdots,0) \tag{45}$$

for all $x\in\mathbb{A}$.

Proof. I replace $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,x,\cdots,0)$ in (43), I have

$$\|k\phi(2x)-2k\phi(x)\|_{\mathbb{B}}\le\frac{1}{|1-2g(\mu_2)|}\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,0) \tag{46}$$

for all $x\in\mathbb{A}$. So

$$\left\|\phi(x)-2\phi\left(\frac{x}{2}\right)\right\|_{\mathbb{B}}\le\frac{1}{k}\left(\frac{1}{|1-2g(\mu_2)|}\right)\varphi\left(\frac{x}{2},\cdots,\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2},\frac{x}{2},\cdots,0\right) \tag{47}$$

for all $x\in\mathbb{A}$. □

The rest of the proof is similar to the proof of Theorem 1.

Corollary 6. Suppose p and θ be positive real numbers with $p>1$, and let $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping such that

$$\begin{aligned}&\Bigg\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(y_i)\\&-g(\mu_2)\left(2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right)\Bigg\|_{\mathbb{B}}\\&\le\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|y_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|z_i\|_{\mathbb{A}}^p\right)\end{aligned}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. The there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\le\left(2+\frac{1}{k}\right)\frac{\theta}{(2^p-2)|1-2g(\mu_2)|}\|x\|_{\mathbb{A}}^p$$

for all $x\in\mathbb{A}$.

Corollary 7. Suppose $p_1,p_2,\cdots,p_k$ and θ be positive real numbers with $3p_1+2p_2+\cdots+2p_k>1$, and let $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping such that

$$\Bigg\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(y_i)$$
$$-g(\mu_2)\left(2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right)\Bigg\|_{\mathbb{B}}$$
$$\le\theta\prod_{i=1}^{k}\|x_i\|_{\mathbb{A}}^{p_i}\cdot\prod_{i=1}^{k}\|y_i\|_{\mathbb{A}}^{p_i}\cdot\|z_1\|_{\mathbb{A}}^{p_1}\cdot\left(1+\prod_{i=2}^{k}\|z_i\|_{\mathbb{A}}^{p_i}\right)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. The there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\le\frac{\theta}{k\left(2^{3p_1+2p_2+\cdots+2p_k}-2\right)|1-2\mu_2|}\|x\|_{\mathbb{A}}^{3p_1+2p_2+\cdots+2p_k}$$

for all $x\in\mathbb{A}$.

Theorem 4. Suppose $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping. If there is a function $\varphi:\mathbb{A}^{3k}\to[0,\infty)$ such that satisfying

$$\Bigg\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(y_i)$$
$$-g(\mu_2)\left(2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right)\Bigg\|_{\mathbb{B}} \quad (48)$$
$$\le\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$$

and

$$\tilde{\varphi}(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$$
$$=\sum_{j=1}^{\infty}\frac{1}{2^j}\varphi\left(2^jx_1,\cdots,2^jx_k,2^jy_1,\cdots,2^jy_k,2^jz_1,\cdots,2^jz_k\right)<\infty \quad (49)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$.

Then there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|\le\frac{1}{2k}\left(\frac{1}{2|1-2g(\mu_2)|}\right)\tilde{\varphi}(x,\cdots,x,x,\cdots,x,x,\cdots,0) \quad (50)$$

for all $x\in\mathbb{A}$.

Proof. I replace $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,x,\cdots,0)$ in (29), I have

$$\|k\phi(2x)-2k\phi(x)\|_{\mathbb{B}}\le\left(\frac{1}{|1-2g(\mu_2)|}\right)\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,0) \quad (51)$$

for all $x\in\mathbb{A}$. So

$$\left\|\phi(x)-2\phi\left(\frac{x}{2}\right)\right\|_{\mathbb{B}}\le\frac{1}{2k}\left(\frac{1}{|1-2g(\mu_2)|}\right)\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,0)$$

for all $x\in\mathbb{A}$. The rest of the proof is similar to the proof of Theorem 1, Theorem 3. □

Corollary 8. Suppose p and θ be positive real numbers with $p<1$, and let $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping such that

$$\left\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(y_i)\right.$$
$$\left.-g(\mu_2)\left(2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right)\right\|_{\mathbb{B}}$$
$$\le\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbb{A}}^{p}+\sum_{i=1}^{k}\|y_i\|_{\mathbb{A}}^{p}+\sum_{i=1}^{k}\|z_i\|_{\mathbb{A}}^{p}\right)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. The there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\le\left(2+\frac{1}{k}\right)\frac{\theta}{(2-2^p)|1-2g(\mu_2)|}\|x\|_{\mathbb{A}}^{p}$$

for all $x\in\mathbb{A}$.

Corollary 9. Suppose $p_1,p_2,\cdots,p_k$ and θ be positive real numbers with $3p_1+2p_2+\cdots+2p_k<1$, and let $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping such that

$$\left\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(y_i)\right.$$
$$\left.-g(\mu_2)\left(2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right)\right\|_{\mathbb{B}}$$
$$\le\theta\prod_{i=1}^{k}\|x_i\|_{\mathbb{A}}^{p_i}\cdot\prod_{i=1}^{k}\|y_i\|_{\mathbb{A}}^{p_i}\cdot\|z_1\|_{\mathbb{A}}^{p_1}\cdot\left(1+\prod_{i=2}^{k}\|z_i\|_{\mathbb{A}}^{p_i}\right)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. The there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\le\frac{\theta}{k\left(2-2^{3p_1+2p_2+\cdots+2p_k}\right)|1-2g(\mu_2)|}\|x\|_{\mathbb{A}}^{3p_1+2p_2+\cdots+2p_k}$$

for all $x\in\mathbb{A}$.

Theorem 5. *Suppose* $\phi:\mathbb{A}\to\mathbb{B}$ *be a mapping. If there is a function* $\varphi:\mathbb{A}^{3k}\to[0,\infty)$ *such that satisfying*

$$\left\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right.$$
$$\left.-g(\mu_3)\left(2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right)\right\|_{\mathbb{B}} \tag{52}$$
$$\le\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$$

and

$$\tilde{\varphi}(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)=\sum_{j=1}^{\infty}2^j\varphi\left(\frac{x_1}{2^j},\cdots,\frac{x_k}{2^j},\frac{y_1}{2^j},\cdots,\frac{y_k}{2^j},\frac{z_1}{2^j},\cdots,\frac{z_k}{2^j}\right)<\infty \tag{53}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$.

Then there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\le\frac{1}{k}\left(\frac{1}{2|1-2g(\mu_2)|}\right)\tilde{\varphi}(x,\cdots,x,x,\cdots,x,x,\cdots,0) \tag{54}$$

for all $x \in \mathbb{A}$.

The rest of the proof is the same as in the proof of Theorem 4.

Corollary 10. Suppose p and θ be positive real numbers with $p > 1$, and let $\phi : \mathbb{A} \to \mathbb{B}$ be a mapping such that

$$\begin{aligned}&\left\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right.\\&\left.-g(\mu_3)\left(2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right)\right\|_{\mathbb{B}}\\&\le\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|y_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|z_i\|_{\mathbb{A}}^p\right)\end{aligned}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbb{A}$. The there exists a unique additive mapping $\psi : \mathbb{A} \to \mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}} \le \left(2+\frac{1}{k}\right)\frac{\theta}{(2^p-2)|1-2\mu_3|}\|x\|_{\mathbb{A}}^p$$

for all $x \in \mathbb{A}$.

Suppose $p_1, p_2, \cdots, p_k$ and θ be positive real numbers with $3p_1+2p_2+\cdots+2p_k > 1$, and let $\phi : \mathbb{A} \to \mathbb{B}$ be a mapping such that

$$\begin{aligned}&\left\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right.\\&\left.-g(\mu_3)\left(2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right)\right\|_{\mathbb{B}}\\&\le\theta\prod_{i=1}^{k}\|x_i\|^{p_i}\cdot\prod_{i=1}^{k}\|y_i\|^{p_i}\cdot\|z_1\|^{p_1}\cdot\left(1+\prod_{i=1}^{k}\|z_k\|^{p_i}\right)\end{aligned}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbb{A}$. The there exists a unique additive mapping $\psi : \mathbf{X} \to \mathbf{Y}$ such that

$$\|\phi(x)-\psi(x)\| \le \frac{1}{k}\cdot\frac{\theta}{\left(2^{3p_1+2p_2+\cdots+2p_k+1}-2\right)|1-2g(\mu_3)|}\|x\|^{3p_1+2p_2+\cdots+2p_k}$$

for all $x \in \mathbf{X}$.

Theorem 6. Let $\phi : \mathbb{A} \to \mathbb{B}$ be a mapping. If there is a function $\varphi : \mathbb{A}^{3k} \to [0,\infty)$ such that satisfying

$$\begin{aligned}&\left\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right.\\&\left.-g(\mu_3)\left(2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right)\right\|_{\mathbb{B}}\\&\le\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\end{aligned}\tag{55}$$

and

$$\begin{aligned}&\tilde{\varphi}(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\\&=\sum_{j=1}^{\infty}\frac{1}{2^j}\varphi\left(2^jx_1,\cdots,2^jx_k,2^jy_1,\cdots,2^jy_k,2^jz_1,\cdots,2^jz_k\right)<\infty\end{aligned}\tag{56}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$.

Then there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|\le\frac{1}{k}\left(\frac{1}{2|1-2g(\mu_3)|}\right)\tilde{\varphi}(x,\cdots,x,x,\cdots,x,x,\cdots,0) \tag{57}$$

for all $x\in\mathbb{A}$.

Proof. The rest of the proof is the same as in the proof of Theorems 1 and 4. □

Corollary 11. Suppose p and θ be positive real numbers with $p<1$, and let $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping such that

$$\begin{aligned}&\Bigg\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\\&-g(\mu_3)\left(2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right)\Bigg\|_{\mathbb{B}}\\&\le\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|y_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|z_i\|_{\mathbb{A}}^p\right)\end{aligned}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. The there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\le\left(2+\frac{1}{k}\right)\frac{\theta}{(2-2^p)|1-2g(\mu_3)|}\|x\|_{\mathbb{A}}^p$$

for all $x\in\mathbb{A}$.

Corollary 12. *Suppose* $p_1,p_2,\cdots,p_k$ *and* θ *be positive real numbers with* $3p_1+2p_2+\cdots+2p_k<1$, *and let* $\phi:\mathbb{A}\to\mathbb{B}$ *be a mapping such that*

$$\begin{aligned}&\Bigg\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\\&-g(\mu_3)\left(2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right)\Bigg\|_{\mathbb{B}}\\&\le\theta\prod_{i=1}^{k}\|x_i\|^{p_i}\cdot\prod_{i=1}^{k}\|y_i\|^{p_i}\cdot\|z_1\|^{p_1}\cdot\left(1+\prod_{i=1}^{k}\|z_k\|^{p_i}\right)\end{aligned}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. The there exists a unique additive mapping $\psi:\mathbf{X}\to\mathbf{Y}$ such that

$$\|\phi(x)-\psi(x)\|\le\frac{1}{k}\cdot\frac{\theta}{\left(2-2^{3p_1+2p_2+\cdots+2p_k+1}\right)|1-2\mu_2|}\|x\|^{3p_1+2p_2+\cdots+2p_k}$$

for all $x\in\mathbf{X}$.

5. Isomorphisms between Unital Banach Algebras

Now, I first study the Isomorphisms between Unital Banach Algebras. Note that for this μ_j-function inequalities, $\mathbb{M}$ be Unital Banach Algebras over a Field $\mathbb{K}=(\mathbb{R},\mathbb{C})$ with unit e and norm $\|\cdot\|$ and that $\mathbb{W}$ be Unital Banach Algebras over a Field $\mathbb{K}=(\mathbb{R},\mathbb{C})$ with unit e' over a Field $\mathbb{K}=(\mathbb{R},\mathbb{C})$.

Note: here I construct the isomorphism for the μ_j-function inequality (3), the rest of the μ_j-function inequalities (1) and (2) I prove exactly the same.

Theorem 7. *Let* $\phi:\mathbb{M}\to\mathbb{W}$ *be a mapping if there is a function* $\varphi:\mathbb{M}^{3k}\to[0,\infty)$ *such that satisfying*

$$\begin{aligned}&\left\|k\phi\left(\beta\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\beta\sum_{i=1}^{k}z_i\right)+\beta k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\beta\sum_{i=1}^{k}\phi(x_i)-2\beta\sum_{i=1}^{k}\phi(z_i)\right.\\&\left.-g(\mu_4)\left(2k\phi\left(\beta\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\beta\sum_{i=1}^{k}z_i\right)-\beta\sum_{i=1}^{k}\phi(x_i)-\beta\sum_{i=1}^{k}\phi(y_i)-2k\beta\sum_{i=1}^{k}\phi(z_i)\right)\right\|_{\mathbb{W}}\\&\le\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\end{aligned}\tag{58}$$

$$\tilde{\varphi}(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)=\sum_{j=1}^{\infty}2^j\varphi\left(\frac{x_1}{2^j},\cdots,\frac{x_k}{2^j},\frac{y_1}{2^j},\cdots,\frac{y_k}{2^j},\frac{z_1}{2^j},\cdots,\frac{z_k}{2^j}\right)<\infty\tag{59}$$

and

$$\lim_{n\to\infty}2^n\phi\left(\frac{e}{2^n}\right)=e'\tag{60}$$

for all $\beta\in\mathbb{K}$, $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{M}$. Then the mapping $\phi:\mathbb{M}\to\mathbb{W}$ is an isomorphism.

Proof. Let $\beta=1$ in (58). By Theorem 6, there is a unique additive mapping $\psi:\mathbb{M}\to\mathbb{W}$ satisfying the additive mapping $\psi:\mathbb{M}\to\mathbb{W}$ is given by

$$\psi(x)=\lim_{n\to\infty}2^n\phi\left(\frac{x}{2^n}\right)\tag{61}$$

for all $x\in\mathbb{M}$ and satisfying

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\le\frac{1}{k}\left(\frac{1}{2|1-2\mu_2|}\right)\tilde{\varphi}(x,\cdots,x,x,\cdots,x,x,\cdots,0)\tag{62}$$

for all $x\in\mathbb{M}$.

By (58) and (60) I have

$$\begin{aligned}&|1-2g(\mu_4)|\|\psi(2\beta x)-2\beta\psi(x)\|_{\mathbb{W}}\\&=\lim_{n\to\infty}2^n\left\|\phi\left(\frac{2\beta x}{2^n}\right)-2\beta\phi\left(\frac{x}{2^n}\right)-\mu_4\left(\phi\left(\frac{2\beta x}{2^n}\right)-2\beta\phi\left(\frac{x}{2^n}\right)\right)\right\|_{\mathbb{W}}\\&\le\lim_{n\to\infty}2^n\varphi\left(\frac{x}{2^n},\cdots,\frac{x}{2^n},\frac{x}{2^n},\cdots,\frac{x}{2^n},\frac{x}{2^n},\cdots,0\right)=0\end{aligned}$$

for all $|g(\mu_4)|<\frac{1}{2},\beta\in\mathbb{K}$ and $x\in\mathbb{M}$.

So

$$k\psi(2\beta kx)-2k\beta\psi(kx)=0.$$

So

$$\psi(2\beta kx)=2\beta\psi(kx).$$

for all $\beta\in\mathbb{K}$ and $x\in\mathbb{M}$. Since ψ is additive,

$$\psi(2\beta kx)=2\beta\psi(kx).$$

$$\psi(\beta kx)=\beta\psi(kx),$$

for all $\beta \in \mathbb{K}$ and for all $x \in \mathbb{M}$. Hence the additive mapping $\psi : \mathbb{M} \to \mathbb{W}$ is an $\mathbb{K}$-*linear* mapping.

Since ϕ is multiplicative,

$$\begin{aligned}\psi\left(\prod_{i=1}^{k} x_i y_i\right) &= \lim_{n\to\infty} 2^{2nk}\phi\left(\prod_{i=1}^{k}\frac{x_i y_i}{n}\right)\\ &= \lim_{n\to\infty} 2^{2nk}\phi\left(\prod_{i=1}^{k}\frac{x_i}{2^n}\right)\phi\left(\prod_{i=1}^{k}\frac{y_i}{2^n}\right)\\ &= \prod_{i=1}^{k}\psi(x_i)\cdot\prod_{i=1}^{k}\psi(y_i)\end{aligned} \tag{63}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k \in \mathbb{M}$. By (60)

$$\psi(e) = \lim_{n\to\infty} 2^n\phi\left(\frac{e}{2^n}\right) = e', \tag{64}$$

so by (63) and (64) I have

$$\psi\left(\prod_{i=1}^{k} x_i\right) = \psi\left(e\prod_{i=1}^{k} x_i\right) = \psi(e)\cdot\phi\left(\prod_{i=1}^{k} x_i\right) = e'\cdot\phi\left(\prod_{i=1}^{k} x_i\right) = \phi\left(\prod_{i=1}^{k} x_i\right), \tag{65}$$

for all $x \in \mathbb{M}$. Therefore, the mapping $\phi : \mathbb{M} \to \mathbb{W}$ is an isomorphism, as desired. □

Corollary 13. Let p and θ be positive real numbers with $p > 1$, and let $\phi : \mathbb{M} \to \mathbb{W}$ be a mapping such that

$$\Bigg\| k\phi\left(\beta\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\beta\sum_{i=1}^{k} z_i\right)+\beta k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k} z_i\right)-\beta\sum_{i=1}^{k}\phi(x_i)-2\beta\sum_{i=1}^{k}\phi(z_i)$$
$$-g(\mu_4)\left(2k\phi\left(\beta\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\beta\sum_{i=1}^{k} z_i\right)-\beta\sum_{i=1}^{k}\phi(x_i)-\beta\sum_{i=1}^{k}\phi(y_i)-2k\beta\sum_{i=1}^{k}\phi(z_i)\right)\Bigg\|_{\mathbb{W}}$$
$$\le \theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbb{M}}^{p}+\sum_{i=1}^{k}\|y_i\|_{\mathbb{M}}^{p}+\sum_{i=1}^{k}\|z_i\|_{\mathbb{M}}^{p}\right)$$

$$\lim_{n\to\infty} 2^n\phi\left(\frac{e}{2^n}\right) = e' \tag{66}$$

for all $\beta \in \mathbb{K}$, $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbb{M}$.

Then the mapping $\phi : \mathbb{M} \to \mathbb{W}$ is an isomorphism.

Corollary 14. *Let* $p_1, p_2, \cdots, p_{3k}$ *and* θ *be positive real numbers with* $3p_1 + 3p_2 + \cdots + 2p_{3k} > 1$, *and let* $\phi : \mathbb{M} \to \mathbb{W}$ *be a mapping such that*

$$\Bigg\| k\phi\left(\beta\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\beta\sum_{i=1}^{k} z_i\right)+\beta k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k} z_i\right)-\beta\sum_{i=1}^{k}\phi(x_i)-2\beta\sum_{i=1}^{k}\phi(z_i)$$
$$-g(\mu_4)\left(2k\phi\left(\beta\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\beta\sum_{i=1}^{k} z_i\right)-\beta\sum_{i=1}^{k}\phi(x_i)-\beta\sum_{i=1}^{k}\phi(y_i)-2k\beta\sum_{i=1}^{k}\phi(z_i)\right)\Bigg\|_{\mathbb{W}}$$
$$\le \theta\prod_{i=1}^{k}\|x_i\|^{p_i}\cdot\prod_{i=1}^{k}\|y_i\|^{p_i}\cdot\|z_1\|^{p_1}\cdot\left(1+\prod_{i=1}^{k}\|z_k\|^{p_i}\right)$$

$$\lim_{n\to\infty} 2^n\phi\left(\frac{e}{2^n}\right) = e' \tag{67}$$

for all $\beta \in \mathbb{K}$, $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbb{M}$.

Then the mapping $\phi : \mathbb{M} \to \mathbb{W}$ is an isomorphism.

6. Conclusion

In this paper, I construct general Cauchy-Jensen μ_j-function inequalities and give the conditions for the existence of solutions and from there, I construct them on complex Banach spaces. The aim is to improve the classical Cauchy-Jensen inequalities on the unlimited space of the number of variables. It is convenient for researchers in the field of Mathematics.

Conflicts of Interest

The author declares no conflicts of interest.

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