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On Some Chaos Notions of Supra Topological Space

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Abstract

In this paper, we investigate the dynamics of supra topological space. We introduce some concepts of supra chaos notions, *i.e.*, supra transitive, supra minimal, supra totally transitive, supra mixing, supra locally everywhere onto (briefly, supra *l.e.o*), and supra weakly blending. First, we investigate some properties of supra transitive map, after that we figure out the relations of the supra chaos notions with the classical chaos notions and showed that supra *l.e.o*, supra mixing, supra totally transitive, and supra weakly blending implies *l.e.o*, topologically mixing, totally transitive, and weakly blending, respectively. Finally, we study the implication relations among the considered supra chaos notions and proved that supra *l.e.o* implies supra mixing, supra totally transitive, and supra weakly blending.

Subject Areas

Dynamical System

Keywords

Supra Transitive, Supra Mixing, Supra Locally Everywhere Onto

1. Introduction

Introducing new generalized topological spaces and exploring their topological properties in various approaches has become a phenomenon in the development of mathematical sciences. Levine (see [1]) in his paper "Semi-open Sets and Semi-continuity in Topological Spaces" generalized a topology by replacing open sets with semi-open sets and obtained some results. After that, different types of generalized open sets are introduced such as preopen, semi-preopen, etc. All these generalized sets have a common property which is closed under arbitrary union. In 1983, Mashhour *et al.* (see [2]) considered all of these sets and defined a generalized space called supra topological space. In other words, the family of open sets is replaced with a larger one. So, the supra open sets are defined where the supra topological spaces are presented.

In 2008, Jassim (see [3]) defined and studied compact, open cover, open sub cover concepts in supra topological spaces. In 2016 Al-Shami (see [4]) introduced and investigated some notions in supra topological spaces such as almost supra compact, supra Lindelof, supra regular and supra normal spaces.

In 2018, Jassim *et al.* (see [5]) introduced and defined a new class of topological transitive maps called topological semi-transitive, bi-supra transitive map by replacing open set in the definition of transitivity with semiopen and bisupra open sets (a subset A of a set X is called a bisupra-open set if $A = B \cap C$, where B is semiopen set and C is preopen set). As far as we know, the dynamics of supra topological space are not yet explored. So, motivated by this and the previously mentioned studies, we extend the study of supra topological space to a dynamical study. We define and introduce some supra chaos notions, *i.e.*, supra transitive, supra totally transitive, supra mixing, supra *l.e.o*, and supra weakly blending in analogue to chaos notions of topological spaces and investigate the relations among these chaos notions on supra topological space and proved supra *l.e.o* implies supra mixing, supra totally transitive, and supra weakly blending. Also, we figure out their relations with the classical chaos notions and we showed that supra *l.e.o*, supra mixing, supra totally transitive, and supra weakly blending imply *l.e.o*, topologically mixing, totally transitive, and weakly blending, respectively.

2. Preliminaries

Definition 2.1. (See [6]) A function $f: X \to X$ is said to be transitive if for any non-empty open subsets $U, V \subset X$, there exists $n > 0$ such that $f^n(U) \cap V \neq \phi$.

Definition 2.2. (See [7]) A function $f: X \to X$ is said to be totally transitive if f^n is transitive for all integers $n \geq 1$.

Definition 2.3. (See [8] A function) $f: X \to X$ is said to be mixing if for any non-empty open subsets $U, V \subset X$, there exists $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \phi$, for all $n > N$.

Definition 2.4. (See [9]) A function $f: X \to X$ is said to be locally everywhere onto or simply *l.e.o* if for every open subset $U \subseteq X$ there exists a positive integer n such that $f^n(U) = X$.

Definition 2.5. (See [10]) A function $f: X \to X$ is said to be weakly blending if for any pair of non-empty open sets U and V in X, there is some $n > 0$ such that $f^n(U) \cap f^n(V) \neq \varnothing$, and strongly blending if, for any pair of non-empty open sets U and V in X, there is some $n > 0$ such that $f^n(U) \cap f^n(V)$ contains a non-empty open subset.

Definition 2.6. (See [2]) Let X be a set and τ^* a family of subsets of X. τ^*

is said to be a supra topology on *X*, if the following axioms hold:

1. *X* and the empty set ϕ are in τ^*.

2. The union of an arbitrary family of members in τ^* is also in τ^*.

The members of τ^* are called supra open sets, and the complement of a supra open set is called a supra closed set.

Definition 2.7. (See [2]) Let (X,τ) be a topological space, and let τ^* be a supra topology on *X*. Then τ^* is said to be a supra topology associated with τ if $\tau \subset \tau^*$.

Definition 2.8. (See [11]) Let (X,τ^*) be a supra topological space and $A \subseteq X$. Then,

1. The supra closure of a set *A* is denoted by $Cl^s(A)$ and defined by $Cl^s(A) = \cap\{B : B \text{ is a supra closed and } A \subseteq B\}$.

2. The supra interior of a set *A* is denoted by $Int^s(A)$ and defined by $Int^s(A) = \cup\{G : G \text{ is a supra open and } A \supseteq G\}$.

Theorem 2.9. (See [2]) Let *X* be a set and τ^* be a supra topology defined on *X*. Then,

1. $Int^s(A \cap B) \subseteq Int^s(A) \cap Int^s(B)$.
2. $Cl^s(A) \cup Cl^s(B) \subseteq Cl^s(A \cup B)$.
3. $D^s(A) \cup D^s(B) \subseteq D^s(A \cup B)$.

Proposition 2.10. (See [4]) Let (X,τ^*) be a supra topological space. Then for any subset *A* of *X*, the following holds;

1. $Cl^s(\varnothing) = \varnothing$.
2. $A \subseteq Cl^s(A)$.
3. $Cl^s(Cl^s(A)) = Cl^s(A)$.

Definition 2.11. (See [2]) Let (X,τ_1), (Y,τ_2) be topological spaces and τ_1^* be a supra topology associated with τ_1. A function $f : X \to Y$ is said to be *S*-continuous if for each open set *U* in *Y*, $f^{-1}(U)$ is τ_1^*-supra open set in *X*.

Definition 2.12. (See [3]) For a supra topology (X,τ^*), a supra open cover of a subset *A* of *X* is a collection B_α of supra open sets such that $A \subseteq \bigcup_\alpha B_\alpha$.

Definition 2.13. (See [3]) A supra topology (X,τ^*) is said to be supra compact if every supra open cover of *X* has a finite subcover.

Theorem 2.14. (See [3])

1. Every supra closed subspace of a supra compact space is supra compact.

2. If *X* is a finite supra topological space. Then *X* is supra compact.

3. Dynamics of Supra Topological Space

Throughout this paper, a pair (X, f) of a supra compact space *X* and *S'*-continuous $f : X \to X$ is said to be a supra dynamical system. A subset $A \subseteq X$ is invariant if $f(A) \subseteq A$. It is supra dense if for every supra open set *U*, $A \cap U \neq \varnothing$ and nowhere supra dense if $Int^s(Cl^s(A)) = \varnothing$. It is of supra second category, if *A* cannot be written as the countable union of subsets which are nowhere supra dense in *X*, *i.e.*, if writing *A* as a union $A = \bigcup_{n\in\mathbb{N}} A_n$ implies that at least one subset $A_n \subset X$ fails to be nowhere supra dense in *X*.

Definition 3.1. A supra topological space (X,τ^*) is said to be supra separable if it has a supra dense subset which is countable.

Definition 3.2. A point $x\in X$ is called supra non-wandering point if for any supra neighbourhood *U* of *x*, there exists $n\ge 1$ such that $f^n(U)\cap U\ne\phi$. The set of supra non-wandering points is denoted by $\Omega^s(f)$.

Proposition 3.3. For the set $\Omega^s(f)$, we have,

1. $\Omega^s(f)$ is supra closed.
2. $\Omega^s(f)$ is *f*-invariant.
3. If *f* is invertible, then $\Omega^s(f)=\Omega^s(f^{-1})$ and $f(\Omega^s(f))=\Omega^s(f)$.

Proof.

(1) To see that $\Omega^s(f)$ is supra closed, we show its complement is supra open. If $x\notin\Omega^s(f)$, then there exists a supra neighbourhood *U* of *x* such that $f^n(U)\cap U\ne\phi$, for all $n\ge 1$, and hence if $V\subset U$ is any smaller supra neighbourhood of *x*, then all points $y\in V$ also do not belong to $\Omega^s(f)$. So, for every $x\notin\Omega^s(f)$, there exists $V_x\in\tau^*$ such that $x\in V_x$ and $V_x\subset X-\Omega^s(f)$. Thus $X-\Omega^s(f)$ is a union of supra open sets in *X*. Therefore, $\Omega^s(f)$ is supra closed.

(2) To show $\Omega^s(f)$ is invariant, let $x\in\Omega^s(f)$. Let *V* be a supra neighbourhood of $f(x)$. Then $U=f^{-1}(V)$ is a supra neighbourhood of *x*, and hence there exists some $n\ge 1$ such that $f^n(U)\cap U\ne\phi$. The image of this intersection under *f* is contained in $f^n(V)\cap V$, and hence $f^n(V)\cap V\ne\phi$. Thus $f(x)\in\Omega^s(f)$.

(3) Let *f* be invertible and let $x\in\Omega^s(f)$, then for every supra neighbourhood *U* of *x* there exists $n\ge 1$ such that $f^n(U)\cap U\ne\phi$. The f^{-n} image of this intersection is contained in $U\cap f^{-n}(U)$, which is nonempty, and hence
$x\in\Omega^s(f^{-1})$. Thus $\Omega^s(f)\subset\Omega^s(f^{-1})$. By the same argument, we can show that $\Omega^s(f^{-1})\subset\Omega^s(f)$, and then $\Omega^s(f)=\Omega^s(f^{-1})$. Thus
$f^{-1}(\Omega^s(f))=f^{-1}(\Omega^s(f^{-1}))\subset\Omega^s(f^{-1})=\Omega^s(f)$. Hence
$f(\Omega^s(f))=\Omega^s(f)$. □

Definition 3.4. A supra dynamical system (X,f) is called supra minimal if the orbit of each point of *X* is supra dense in *X*.

Theorem 3.5 A dynamical system (X,f) is called supra minimal system if one of the three equivalent conditions hold:

1. The orbit of each point of *X* is supra dense in *X*,
2. $Cl^s(Orb_f(x))=X$ for each $x\in X$,
3. For $x\in X$ and a nonempty supra open *U* in *X*, there exists $n\in\mathbb{N}$ such that $f^n(x)\in U$.

Proof. If (1) holds, then by Definition 3.4 *f* is supra minimal. If (2) holds, then $Orb_f(x)$, the orbit of each point $x\in X$ is supra dense in *X*. Therefore *f* is supra minimal. If (3) holds, then $Orb_f(x)$, the orbit of each point $x\in X$ is supra dense in *X*. Therefore *f* is supra minimal. □

Theorem 3.6. *For a supra dynamical system* (X,f), *the following are equivalent*:

1. (X, f) is supra minimal,
2. If F is a supra closed subset of X with $f(F) \subseteq F$, then $F = X$ or $F = \phi$,
3. For a nonempty supra open set U of X, $X = \bigcup_{n=0}^{\infty} f^{-n}(U)$.

Proof.

(i) (1) $\Rightarrow$ (2): Let f be a supra minimal map, and let F be a supra closed subset of X with $f(F) \subseteq F$. If $F \neq \phi$, let $x \in F$, since F is a supra closed subset of X, then $Cl^s(F) = F$, hence $Cl^s(Orb_f(x)) \subset F$. But by (2) of Theorem 3.5 we have $Cl^s(Orb_f(x)) = X$. Therefore $F = X$.

(ii) (2) $\Rightarrow$ (3): Let U be a nonempty supra open set of X. Let $B = X - \bigcup_{n=0}^{\infty} f^{-n}(U)$. Since U is a nonempty set then $B \neq X$. Since f is S'-continuous and U is a nonempty supra open set, then B is a supra closed set and $f(B) \subset B$, so B must be empty. Therefore $X = \bigcup_{n=0}^{\infty} f^{-n}(U)$.

(iii) (3) $\Rightarrow$ (1): Let U be any nonempty supra open set of X, and let $x \in X$. Since $x \in X = \bigcup_{n=0}^{\infty} f^{-n}(U)$, then $x \in \bigcup_{n=0}^{\infty} f^{-n}(U)$. Hence $f^n(x) \in U$ for some $n > 0$, *i.e.*, the orbit of every point x in X is supra dense in X. Therefore, (X, f) is supra minimal. □

Definition 3.7. A supra dynamical system (X, f) is said to be supra transitive if for any non-empty supra open subsets $U, V \subset X$, there exists $n > 0$ such that $f^n(U) \cap V \neq \phi$.

Proposition 3.8. A function $f : X \to X$ is supra transitive if and only if for every nonempty supra open set U in X, $\bigcup_{n=0}^{\infty} f^n(U)$ is supra dense in X.

Proof. Let f be a supra transitive function, and let U be a nonempty supra open set in X. Suppose that $\bigcup_{n=0}^{\infty} f^n(U)$ is not supra dense in X, then there exists a supra open set V such that $\bigcup_{n=0}^{\infty} f^n(U) \cap V = \phi$. This implies that $f^n(U) \cap V = \phi$ for all $n \in \mathbb{N}$, which is a contradiction since f is a supra transitive. Conversely, Let U and V be two nonempty supra open sets in X. Since $\bigcup_{n=0}^{\infty} f^n(U)$ is supra dense in X, then for every supra open set V we have $\bigcup_{n=0}^{\infty} f^n(U) \cap V \neq \phi$. Hence there exist an integer $k > 0$ such that $f^k(U) \cap V \neq \phi$. Therefore f is supra transitive. □

Proposition 3.9. Let (X, f) be a supra dynamical space. If there exists a supra dense orbit, then f is supra transitive.

Proof. Let U and V be two nonempty supra open sets in X, and let $x \in X$ such that the set $Orb_f(x)$ is supra dense. Since $Orb_f(x)$ is supra dense, there exists $n > 0$ such that $f^n(x) \in U$. Since $Orb_f(x)$ is supra dense, then $Orb_f(f^n(x))$ is also supra dense. Hence there exists m such that $f^m(f^n(x)) \in V$. Therefore $f^{m+n}(x) \in f^m(U) \cap V$, and then $f^m(U) \cap V \neq \phi$. So f is topological supra transitive. □

Proposition 3.10. Let (X, f) be a supra dynamical space. If X is supra separable and of supra second category, then supra transitivity of f implies that f has supra dense orbit.

Proof. Let X be a supra separable and of supra second category. Let $\{U_i\}_{i=1}^{\infty}$ be a countable supra base for X, and suppose that f has no supra dense orbit, then for each $x \in X$ there exists $U_{i(x)}$ such that $f^n(x) \notin U_{i(x)}$ for every

$n \in \mathbb{N}$. Let $U = \bigcup_{n=0}^{\infty} f^{-n}\left(U_{i(x)}\right)$, then U is a supra open and meet every supra open set since f is supra transitive. Let

$$C_{i(x)} = X - \bigcup_{n=0}^{\infty} f^{-n}\left(U_{i(x)}\right),$$

then $C_{i(x)}$ is a supra closed set since it is a complement of supra open set and nowhere supra dense. However, $X = \bigcup_{x \in X} C_{i(x)}$ is a countable union of nowhere supra dense sets which contradicts the fact that X is of supra second category. Therefore f has supra dense orbit. □

Proposition 3.11. Let (X, f) be a supra dynamical system. If f is supra transitive then X does not contain two disjoint supra open invariant subsets of X.

Proof. Let f be a supra transitive map, and let U and V be two disjoint, supra open, invariant subsets of X. Since U is invariant then $f^n(U) \subseteq U$, and hence $f^n(U) \cap V = \varnothing$, which is a contradiction since f is supra transitive. Therefore X does not contain two disjoint supra open invariant subsets of X. □

Proposition 3.12. Let (X, f) be a supra dynamical system. If f is supra transitive then X is not a union of two proper supra closed invariant subsets of X.

Proof. Let f be a supra transitive map, then by Proposition 3.11, X does not contain two disjoint, supra open, invariant subsets. Suppose that B_1 and B_2 are two proper, supra closed, invariant subsets such that $X = B_1 \cup B_2$. This means that we can write $X = B_1 \cup B_2$ where B_1 and B_2 are nonempty proper supra closed, invariant subsets if and only if we can find nonempty proper supra closed, invariant subsets B_1 and B_2 such that $(X - B_1) \cap (X - B_2) = \varnothing$. Since B_1 and B_2 are supra closed subsets, then $U = X - B_1$ and $V = X - B_2$ are two disjoint, supra open, invariant subsets of X, which is a contradiction since X does not contain two disjoint, supra open, invariant subsets. Therefore, X is not a union of two proper supra closed invariant subsets of X. □

Definition 3.13. A supra dynamical system (X, f) is said to be supra totally transitive if f^n is supra transitive for all integers $n \geq 1$.

Definition 3.14. A supra dynamical system (X, f) is said to be supra mixing if whenever U and V are nonempty supra open subsets of X, there exists an $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \phi$, for all $n > N$.

Definition 3.15. A supra dynamical system (X, f) is said to be supra locally everywhere onto or simply supra *l.e.o* if for every supra open subset $U \subseteq X$ there exists a positive integer n such that $f^n(U) = X$.

Definition 3.16. A supra dynamical system (X, f) is said to be supra weakly blending if for any pair of non-empty supra open sets U and V in X, there is some $n > 0$ such that $f^n(U) \cap f^n(V) \neq \varnothing$.

Next, we show the relation between each supra chaos notion and its analogue in topological space.

Proposition 3.17 Let (X, f) be a supra dynamical system. If f is supra *l.e.o*, then it is *l.e.o*

Proof. Let U be any nonempty open subset of X. Since every open set is supra open set, and since f is supra *l.e.o*, then there exists an integer $n > 0$ such that $f^n(U) = X$. Therefore f is *l.e.o* □

Proposition 3.18. Let (X, f) be a supra dynamical system. Then

1. If *f* is supra totally transitive, then it is totally transitive.
2. If *f* is supra mixing, then it is mixing.
3. If *f* is supra weakly blending, then it is weakly blending.

Proof. By the same argument in the proof of Proposition 3.17. □

After showing the relation between each supra chaos notion and its analogue in topological space, we will show the relation between the supra chaos notions, *i.e.*, supra *l.e.o*, supra topologically mixing, supra totally transitive, and supra weakly blending.

Proposition 3.19. Let (X, f) be a supra dynamical system. If $f : X \to X$ is a supra *l.e.o*, then it is supra transitive.

Proof. Let *U* and *V* be any nonempty supra open subsets of *X*. Since *f* is supra *l.e.o*, then for any supra open *U*, there exits integer $n > 0$ such that $f^n(U) = X$. So for any supra open *V* of *X*, we have $f^n(U) \cap V \neq \phi$, for some $n > 0$. Therefore *f* is supra transitive. □

Proposition 3.20. Let (X, f) be a supra dynamical system. If $f : X \to X$ is a supra *l.e.o*, then it is supra totally transitive.

Proof. Let U, V be any two nonempty supra open sets in *X*. Since *f* is supra *l.e.o*, then for every supra open set *U* there exists a positive integer *n* such that $f^n(U) = X$. Let $r > 0$ be any integer, then we have

$$\left(f^r\right)^n(U) = f^r f^n(U) = f^r(X) = X,$$

and so

$$\left(f^r\right)^n(U) \cap V = V \neq \phi.$$

Hence *f* is supra totally transitive. □

Proposition 3.21. Let (X, f) be a supra dynamical system. If $f : X \to X$ is a supra *l.e.o*, then it is supra mixing.

Proof. Let U, V be any two nonempty supra open sets in *X*. Since *f* is supra *l.e.o* then for every supra open set *U* there exists a positive integer *n* such that $f^n(U) = X$, and then $f^n(U) \cap V \neq \phi$. So we can choose $N > n$ such that $f^k(U) \cap V \neq \phi$ for every $k \geq N$. Hence *f* is supra mixing. □

Proposition 3.22. Let (X, f) be a supra dynamical system. If $f : X \to X$ is a supra *l.e.o*, then it is supra weakly blending.

Proof. Let U, V be any two nonempty supra open sets in *X*. Since *f* is supra *l.e.o*, then there exists $n_1, n_2 > 0$ such that

$$f^{n_1}(U) = f^{n_2}(V) = X.$$

Without lose of generality, let $n_1 > n_2$. Then

$$f^{n_1}(U) = f^{n_1}(V) = X.$$

So

$$f^{n_1}(U) \cap f^{n_1}(V) \neq \phi.$$

Hence *f* is supra weakly blending. □

4. Conclusion

In this paper, we introduced some concepts of supra chaos notions, *i.e.*, supra transitive, supra totally transitive, supra mixing, supra *l.e.o*, and supra weakly blending. Firstly, we studied the properties of supra transitive map and after that we figured out the relation between the classical chaos notions and supra chaos notions, and proved that supra *l.e.o*, supra mixing, supra totally transitive, and supra weakly blending imply *l.e.o*, topologically mixing, totally transitive, and weakly blending, respectively. Secondly, we showed that supra *l.e.o* implies supra mixing, supra totally transitive, and supra weakly blending.

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Conflicts of Interest

The authors declare no conflicts of interest.

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