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Broadly Derivation on Fuzzy Banach Algebra Involving Functional Equations and General Cauchy-Jensen Functional Inequalities with 3*k*-Variables

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Abstract

In this paper, I use the research fixed point method to establish derivatives on fuzzy Banach algebras based on functional equations and Cauchy-Jensen functional inequalities with 3*k*-variables. These are the main results of this paper.

Subject Areas

Mathematics

Keywords

General Cauhy-Jensen-Type Additive Function Equation, Cauchy-Jensen Functional Inequalities, Fuzzy Banach Algebras, Fixed Point Method, Fuzzy Derivatives

1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ are two fuzzy normed vector spaces on the same field $\mathbb{K}$, and mapping $f:\mathbf{X}\to\mathbf{Y}$ be continuously on $\mathbf{X}$. I use the notation N', N for corresponding the norms on $\mathbf{X}$ and $\mathbf{Y}$. In this paper, I study the setting of derivatives on fuzzy algebras involving functional equations and Cauchy-Jensen additive functional inequalities with 3k-variables when $\mathbf{X}$ is a fuzzy Banach algebra with the norm N or (X,N). Indeed, when $\mathbf{X}$ is a fuzzy normal Banach algebra with N norm, I construct the derivative on a Banach fuzzy algebra that involves functional equations and Cauchy-Jensen additive functional inequalities with the following 3k-variables:

$$2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)=\sum_{j=1}^{k}f\left(x_j\right)+\sum_{j=1}^{k}f\left(y_j\right)+2k\sum_{j=1}^{k}f\left(z_j\right) \tag{1}$$

and

$$\left\|\sum_{j=1}^{k}f\left(x_j\right)+\sum_{j=1}^{k}f\left(y_j\right)+f\left(2k\sum_{j=1}^{k}z_j\right)\right\|\le\left\|2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)\right\| \tag{2}$$

The study construct the derivative on a Banach fuzzy algebra that involves functional equations and general Cauchy-Jensen additive functional inequalities originated from a question of S. M. Ulam [1], concerning the stability of group homomorphisms.

Let $(\mathbf{G},*)$ be a group and let $(\mathbf{G}',\circ,d)$ be a metric group with metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that if $f:\mathbf{G}\to\mathbf{G}'$ satisfies

$$d\left(f(x*y),f(x)\circ f(y)\right)<\delta,\forall x\in\mathbf{G}$$

then there is a homomorphism $h:\mathbf{G}\to\mathbf{G}'$ with

$$d\left(f(x),h(x)\right)<\varepsilon,\forall x\in\mathbf{G}$$

Since Hyers' answer to Ulam's question [2], many ideas have arisen from mathematicians who have built theories about space such as the Theory of fuzzy space has much progressed as developing the theory of randomness. Some mathematicians have defined fuzzy norms on a vector space from various points of view. Following Bag and Samanta [3] and Cheng and Mordeson [4] gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [5] and investigated some properties of fuzzy normed spaces. I use the definition of fuzzy normed spaces given in [3] [6] [7] [8] to investigate a fuzzy version of the Hyers-Ulam stability for the Jensen functional equation in the fuzzy normed algebra setting.

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called a quadratic functional equation. The Hyers-Ulam stability of the quadratic functional equation was proved by Skof [9] for mappings $f:\mathbf{X}\to\mathbf{Y}$, where $\mathbf{X}$ is a normed space and $\mathbf{Y}$ is a Banach space. Cholewa [10] noticed that the theorem of Skof is still true if the relevant domain $\mathbf{X}$ is replaced by an Abelian group. Czerwik [11] proved the Hyers-Ulam stability of the quadratic functional equation.

The stability problems for several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. Such as in 2008 Choonkil Park [12] have established the and investigated the *Hyers-Ulam-Rassias* stability of homomorphisms in quasi-Banach algebras the following Jensen functional equation

$$2f\left(\frac{x+y}{2}\right)=f(x)+f(y) \tag{3}$$

and next in 2009, M. Éhaghi Gordji and M. Bavand Savadkouhi [13] have established the and investigated the approximation of generalized stability of homomorphisms in quasi-Banach algebras the following Jensen functional equation

$$rf\left(\frac{x+y}{r}\right)=f(x)+f(y). \tag{4}$$

Next in 2022, Ly Van An [14] have established the and investigated the approximation of generalized stability of homomorphisms in quasi-Banach algebras the following Jensen type functional equation

$$mf\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}x_{k+j}}{m}\right)=\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(x_{k+j}) \tag{5}$$

Next in 2023, the author [15] have established the and investigated the Extension of Homomorphisms-Isomorphisms and Derivatives on Quasi-Banach Algebra Based on the General Additive Cauchy-Jensen Equation

$$2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)=\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(y_j)+2k\sum_{j=1}^{k}f(z_j) \tag{6}$$

Next in 2023, Ly Van An [16] have established the and investigated the approximation of generalized stability of homomorphisms in on fuzzy Banach algebras the following Jensen type functional equation

$$mf\left(\frac{\alpha\sum_{j=1}^{k}x_j+\alpha\sum_{j=1}^{k}y_j}{m}\right)=\sum_{j=1}^{k}\alpha f(x_j)+\sum_{j=1}^{k}\alpha f(y_j) \tag{7}$$

Recently, the author continues to conduct extensive research on the derivative for (1) and (2) on the fuzzy Banach algebra for the following functional equation and inequalities.

$$2kf\left(\sum_{j=1}^{k}\frac{qx_j+y_j}{2k}+\sum_{j=1}^{k}qz_j\right)=\sum_{j=1}^{k}f(qx_j)+\sum_{j=1}^{k}f(qy_j)+2k\sum_{j=1}^{k}f(qz_j) \tag{8}$$

and

$$\left\|\sum_{j=1}^{k}f(qx_j)+\sum_{j=1}^{k}f(qy_j)+f\left(2k\sum_{j=1}^{k}qz_j\right)\right\|\le\left\|2kf\left(\sum_{j=1}^{k}\frac{qx_j+qy_j}{2k}+\sum_{j=1}^{k}qz_j\right)\right\| \tag{9}$$

i.e., the functional equation and inequalities with $3k$-variables. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, I will prove that the mappings satisfying the functional equation and equation inequalities (8) and (9). Thus, the results in this paper are generalization of those in [12] [13] [14] [15] [16] [29] for functional equation with $2k$-variables.

In this paper, I build a general homomorphism based on Jensen equation with $2k$-variables on fuzzy Banach algebra. This is an expansion bracket for the research field of exploiting unlimited Math problems on variables to build this problem based on the ideas of mathematicians around the world. See [1]-[32]. Allow me to express my deep thanks to the mathematicians.

The paper is organized as follows:

In section preliminaries, I remind some basic notations in [3] [6] [7] [8] [18] [27] [32] such as Fuzzy normed spaces, Extended metric space theorem and solutions of the Jensen function equation.

Section 3: Using the fixed point method, extend the derivative for the functional Equation (1) on the fuzzy Banach algebra.

Section 4: Using the fixed point method, extend the derivative for the functional inequality (2) on the fuzzy Banach algebra.

2. Preliminaries

2.1. Fuzzy Normed Spaces

Let X be a real vector space. A function $N : X \times R \to [0,1]$ is called a fuzzy norm on X if for all $x, y \in X$ and $s, t \in \mathbb{R}$,

1) (N1) $N(x,t) = 0$ for $t \le 0$;

2) (N2) $x = 0$ if and only if $N(x,t) = 1$ for $t > 0$;

3) (N3) $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ if $c \ne 0$;

4) (N4) $N(x+y, s+t) \ge \min\{N(x,s), N(y,t)\}$;

5) (N5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;

6) (N6) for $x \ne 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed vector space

1) Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ with $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and I denote it by $N - \lim_{n\to\infty} x_n = x$.

2) Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in N$ such that with $n = n_0$ and all $p > 0$, I have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space. I say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X.

Let X be algebra and (X,N) a fuzzy normed space.

1) The fuzzy normed space (X,N) is called a fuzzy normed algebra if

$$N(xy, st) \ge N(x,s) \cdot N(y,t),$$

for all $x, y \in X$ and with all real s, t positive.

2) A complete fuzzy normed algebra is called a fuzzy Banach algebra.

Let (X, N_X) and (Y,N) be fuzzy normed algebras. Then a multiplicative $\mathbb{R}$-linear mapping $H : (X, N_X) \to (Y,N)$ is called a fuzzy algebra homomorphism.

Let (X,N) be a fuzzy normed Algebra. Then an $\mathbb{R}$-linear mapping $H : (X,N) \to (X,N)$ is call derivation if

$$H(xy) := H(x)y + xH(y)$$

with all $x, y \in X$.

EXAMPLE

Let $(X, \|\cdot\|)$ be a normed algebra. Let

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|} & t > 0 \\ 0 & t \le 0 \end{cases}, \quad x \in X$$

Then $N(x,t)$ is a fuzzy norm on X and $(X, N(x,t))$ is a fuzzy normed algebra.

2.2. Extended Metric Space Theorem

Theorem 1. Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n, J^{n+1}) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

1) $d(J^n, J^{n+1}) < \infty, \quad \forall n \ge n_0$;
2) The sequence $\{J^n x\}$ converges to a fixed point y^* of J;
3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^n, J^{n+1}) < \infty\}$;
4) $d(y, y^*) \le \frac{1}{1-l} d(y, Jy) \quad \forall y \in Y$.

2.3. Solutions of the Equation

The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be a *Cauchy-additive mapping*.

The functional equation

$$2f\left(\frac{x+y}{2} + z\right) = f(x) + f(y) + 2f(z)$$

is called the Cauchy-Jensen equation. In particular, every solution of the Cauchy equation is said to be a Cauchy-Jensen additive mapping.

The functional inequality

$$\|f(x) + f(y) + f(2z)\| \le \left\|2f\left(\frac{x+y}{2} + z\right)\right\|$$

is called the functional inequality Jensen-Cauchy. In particular, every solution of the functional inequality Jensen-Cauchy is said to be a Cauchy-Jensen additive mapping.

3. Using the Fixed Point Method, Extend the Derivative for the Functional Equation (1) on the Fuzzy Banach Algebra

Now I study extended derivation by fixed point method when $\mathbf{X}$ is a fuzzy Banach algebra with norm *N*. Under this setting, I need to show that the mapping must satisfy (1). These results are given in the following.

Theorem 2. Let $\psi:\mathbf{X}^{3k}\to[0,\infty)$ be a function such that there exists an $L<\frac{1}{2k}$

$$\begin{aligned}&\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\\&\le\frac{L}{2k}\psi(2kx_1,\cdots,2kx_k,2ky_1,\cdots,2ky_k,2kz_1,\cdots,2kz_k)\end{aligned}\tag{10}$$

with all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$.

Let $f:\mathbf{X}\to\mathbf{X}$ be a mapping satisfying

$$\begin{aligned}&N\left(2kf\left(\sum_{j=1}^{k}\frac{qx_j+qy_j}{2k}+\sum_{j=1}^{k}qz_j\right)-\sum_{j=1}^{k}qf(x_j)-\sum_{j=1}^{k}qf(y_j)-2k\sum_{j=1}^{k}qf(z_j),t\right)\\&\ge\frac{t}{t+\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)}\end{aligned}\tag{11}$$

$$\begin{aligned}&N\left(f\left(\prod_{j=1}^{k}x_j\cdot y_j\right)-\prod_{j=1}^{k}f(x_j)\cdot\prod_{j=1}^{k}y_j-\prod_{j=1}^{k}x_j\cdot\prod_{j=1}^{k}f(y_j),t\right)\\&\ge\frac{t}{t+\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,0,\cdots,0)}\end{aligned}\tag{12}$$

with all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$, all $t>0$. And all $q\in\mathbb{R}$.

Then

$$H(x)=N-\lim_{n\to\infty}(2k)^n f\left(\frac{x}{(2k)^n}\right)\tag{13}$$

exists each $x\in\mathbf{X}$ and defines a fuzzy derivation $H:\mathbf{X}\to\mathbf{X}$.

Such that

$$N(f(x)-H(x),t)\ge\frac{(1-L)t}{(1-L)+L\psi(x_1,\cdots,x_k,0,\cdots,0)}\tag{14}$$

for all $x\in\mathbf{X}$ and all $t>0$.

Proof. Letting $q=1$ and I replace $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (11), I get

$$N\left(2kf\left(\frac{x}{2k}\right)-f(x),t\right)\ge\frac{t}{1+\varphi(x,\cdots,0,0,\cdots,0,0,\cdots,0)}\tag{15}$$

with all $x\in\mathbf{X}$. Now I consider the set

$$\mathbb{M}:=\{h:\mathbf{X}\to\mathbf{X}\}$$

and introduce the generalized metric on $\mathbb{M}$ as follows:

$$\begin{aligned}d(g,h)&:=\inf\left\{\beta\in\mathbb{R}_+:N(g(x)-h(x),\beta t)\right.\\&\left.\ge\frac{t}{t+\varphi(x,0,\cdots,0,0,\cdots,0,0,\cdots,0)},\forall x\in\mathbf{X},\forall t>0\right\},\end{aligned}\tag{16}$$

where, as usual, $\inf \phi = +\infty$. That has been proven by mathematicians $(\mathbb{M}, d)$ is complete (see [20]).

Now I consider the linear mapping $T : \mathbb{M} \to \mathbb{M}$ such that

$$Tg(x) := 2kg\left(\frac{x}{2k}\right)$$

with all $x \in \mathbf{X}$. Let $g, h \in \mathbb{M}$ be given such that $d(g,h) = \varepsilon$ then

$$N\left(g(x) - h(x), \varepsilon t\right) \geq \frac{t}{t + \varphi(x, 0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0)}, \forall x \in \mathbf{X}, \forall t > 0.$$

Hence

$$\begin{aligned}
N\left(g(x) - h(x), \varepsilon t\right) &= N\left(2kg\left(\frac{x}{2k}\right) - 2kh\left(\frac{x}{2k}\right), L\varepsilon t\right) \\
&= N\left(g\left(\frac{x}{2k}x\right) - h\left(\frac{x}{2k}x\right), \frac{L}{2k}\varepsilon t\right) \\
&\geq \frac{\frac{Lt}{2k}}{\frac{Lt}{2k} + \varphi\left(\frac{x}{2k}, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0\right)} \\
&\geq \frac{\frac{Lt}{2k}}{\frac{Lt}{2k} + \frac{L}{2k}\varphi(x, 0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0)} \\
&= \frac{t}{t + \varphi(x, x, \cdots, x, x, \cdots, x)}, \forall x \in \mathbf{X}, \forall t > 0.
\end{aligned} \tag{17}$$

Therefore $d(g,h) = \varepsilon$ implies that $d(Tg, Th) \leq L \cdot \varepsilon$. This means that

$$d(Tg, Th) \leq Ld(g,h)$$

for all $g, h \in \mathbb{M}$. It follows from (15) that with all $x \in \mathbb{X}$. So $d(f, Tf) \leq 1$. By Theorem 1, there exists a mapping $H : \mathbf{X} \to \mathbf{Y}$ satisfying the following:

1) H is a fixed point of T, *i.e.*,

$$H\left(\frac{x}{2k}\right) = \frac{1}{2k}H(x) \tag{18}$$

With all $x \in \mathbf{X}$. The mapping H is a unique fixed point T in the set

$$\mathbb{Q} = \{g \in \mathbb{M} : d(f, g) < \infty\}$$

This implies that H is a unique mapping satisfying (18) such that there exists a $\beta \in (0, \infty)$ satisfying

$$N\left(f(x) - H(x), \beta t\right) \geq \frac{t}{t + \varphi(x, 0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0)}, \forall x \in \mathbf{X}.$$

2) $d(T^l f, H) \to 0$ as $l \to \infty$. This implies equality

$$N - \lim_{l \to \infty} (2k)^l f\left(\frac{x}{(2k)^l}\right) = H(x)$$

with everyone $x \in \mathbb{X}$.

3) $d(f,H) \le \frac{1}{1-L} d(f,Tf)$, which implies the inequality.

4)

$$d(f,H) \le \frac{1}{1-L}.$$

This implies that the inequality (15) holds

By (12)

$$N\left((2k)^{p+1} f\left(\sum_{j=1}^{k} \frac{qx_j + qy_j}{(2k)^{p+1}} + \sum_{j=1}^{k} \frac{qz_j}{(2k)^p}\right) - (2k)^p \sum_{j=1}^{k} qf\left(\frac{x_j}{(2k)^p}\right)\right.$$
$$\left. -(2k)^p \sum_{j=1}^{k} qf\left(\frac{y_j}{(2k)^p}\right) - (2k)^p 2k \sum_{j=1}^{k} qf\left(\frac{z_j}{(2k)^p}\right), t\right) \tag{19}$$
$$\ge \frac{t}{t + \psi\left(\frac{x_1}{(2k)^p}, \cdots, \frac{x_k}{(2k)^p}, \frac{y_1}{(2k)^p}, \cdots, \frac{y_k}{(2k)^p}, \frac{z_1}{(2k)^p}, \cdots, \frac{z_k}{(2k)^p}\right)}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathbf{X}$, all $t > 0$ and all $q \in \mathbb{R}$. Then

$$N\left((2k)^{p+1} f\left(\sum_{j=1}^{k} \frac{qx_j + qy_j}{(2k)^{p+1}} + \sum_{j=1}^{k} \frac{qz_j}{(2k)^p}\right) - (2k)^p \sum_{j=1}^{k} qf\left(\frac{x_j}{(2k)^p}\right)\right.$$
$$\left. -(2k)^p \sum_{j=1}^{k} qf\left(\frac{y_j}{(2k)^p}\right) - (2k)^p 2k \sum_{j=1}^{k} qf\left(\frac{z_j}{(2k)^p}\right), t\right) \tag{20}$$
$$\ge \frac{\frac{t}{(2k)^p}}{\frac{t}{(2k)^p} + \frac{L^p}{(2k)^p} \psi(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathbf{X}$, all $t > 0$ and all $q \in \mathbb{R}$.

Since

$$\lim_{n \to \infty} \frac{\frac{t}{(2k)^p}}{\frac{t}{(2k)^p} + \frac{L^p}{(2k)^p} \psi(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)} = 1$$

For all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathbf{X}$, all $t > 0$ and $q \in \mathbb{R}$. Thus

$$N\left(2kH\left(\sum_{j=1}^{k} \frac{qx_j + qy_j}{2k} + \sum_{j=1}^{k} qz_j\right) - \sum_{j=1}^{k} qH(x_j) - \sum_{j=1}^{k} qH(y_j) - 2k \sum_{j=1}^{k} qH(z_j), t\right) = 1 \tag{21}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathbf{X}$, all $t > 0$ and $q \in \mathbb{R}$. Thus

$$2kH\left(\sum_{j=1}^{k} \frac{qx_j + qy_j}{2k} + \sum_{j=1}^{k} qz_j\right) - \sum_{j=1}^{k} qH(x_j) - \sum_{j=1}^{k} qH(y_j) - 2k \sum_{j=1}^{k} qH(z_j) = 0 \tag{22}$$

Thus the mapping

$$H:\mathbf{X}\to\mathbf{X}$$

is additive and $\mathbf{R}$-linear by (12), I have

$$N\left((2k)^{2p} f\left(\prod_{j=1}^{k}\frac{x_j\cdot y_j}{(2k)^{2p}}\right)-(2k)^{p}\prod_{j=1}^{k} f\left(\frac{x_j}{(2k)^{p}}\right)\cdot\prod_{j=1}^{k} y_j\right.$$
$$\left.-\prod_{j=1}^{k} x_j\cdot(2k)^{p}\prod_{j=1}^{k} f\left(\frac{y_j}{(2k)^{p}}\right),t\right) \quad (23)$$
$$\geq\frac{t}{t+\psi\left(\frac{x_1}{(2k)^{p}},\cdots,\frac{x_k}{(2k)^{p}},\frac{y_1}{(2k)^{p}},\cdots,\frac{y_k}{(2k)^{p}},0,\cdots,0\right)}$$

with all $x_1,\cdots,x_k,y_1,\cdots,y_k\in\mathbf{X}$, all $t>0$. Then

$$N\left((2k)^{2p} f\left(\prod_{j=1}^{k}\frac{x_j\cdot y_j}{(2k)^{2p}}\right)-(2k)^{p}\prod_{j=1}^{k} f\left(\frac{x_j}{(2k)^{p}}\right)\cdot\prod_{j=1}^{k} y_j\right.$$
$$\left.-\prod_{j=1}^{k} x_j\cdot(2k)^{p}\prod_{j=1}^{k} f\left(\frac{y_j}{(2k)^{p}}\right),t\right) \quad (24)$$
$$\geq\frac{\frac{t}{(2k)^{2p}}}{\frac{t}{(2k)^{2p}}+\frac{L^p}{(2k)^{p}}\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,0,\cdots,0\right)}$$

with all $x_1,\cdots,x_k,y_1,\cdots,y_k\in\mathbf{X}$, for all $t>0$. Since

$$\lim_{p\to\infty}\frac{\frac{t}{(2k)^{2p}}}{\frac{t}{(2k)^{2p}}+\frac{L^p}{(2k)^{p}}\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,0,\cdots,0\right)}=1 \quad (25)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k\in\mathbf{X}$, all $t>0$.

Thus

$$N\left(f\left(\prod_{j=1}^{k} x_j\cdot y_j\right)-\prod_{j=1}^{k} f\left(x_j\right)\cdot\prod_{j=1}^{k} y_j-\prod_{j=1}^{k} x_j\cdot\prod_{j=1}^{k} f\left(y_j\right),t\right)=1 \quad (26)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k\in\mathbf{X}$, all $t>0$. Thus

$$f\left(\prod_{j=1}^{k} x_j\cdot y_j\right)-\prod_{j=1}^{k} f\left(x_j\right)\cdot\prod_{j=1}^{k} y_j-\prod_{j=1}^{k} x_j\cdot\prod_{j=1}^{k} f\left(y_j\right)=0 \quad (27)$$

So the mapping $H:\mathbf{X}\to\mathbf{X}$ is a fuzzy derivation, as desired.

□

Theorem 3. Let $\psi:\mathbf{X}^{3k}\to[0,\infty)$ be a function such that there exists an $L<1$

$$\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)\leq 2k\psi\left(\frac{x_1}{2k},\cdots,\frac{x_k}{2k},\frac{y_1}{2k},\cdots,\frac{y_k}{2k},\frac{z_1}{2k},\cdots,\frac{z_k}{2k}\right) \quad (28)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$.

Let $f:\mathbf{X}\to\mathbf{X}$ be a mapping satisfying

$$N\left(2kf\left(\sum_{j=1}^{k}\frac{qx_j+qy_j}{2k}+\sum_{j=1}^{k}qz_j\right)-\sum_{j=1}^{k}qf\left(x_j\right)-\sum_{j=1}^{k}qf\left(y_j\right)-2k\sum_{j=1}^{k}qf\left(z_j\right),t\right)\geq\frac{t}{t+\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)}\tag{29}$$

$$N\left(f\left(\prod_{j=1}^{k}x_j\cdot y_j\right)-\prod_{j=1}^{k}f\left(x_j\right)\cdot\prod_{j=1}^{k}y_j-\prod_{j=1}^{k}x_j\cdot\prod_{j=1}^{k}f\left(y_j\right),t\right)\geq\frac{t}{t+\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,0,\cdots,0\right)}\tag{30}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k\in\mathbf{X}$, all $t>0$. Then

$$\beta(x)=N-\lim_{n\to\infty}\frac{1}{(2k)^n}f\left((2k)^n x\right)\tag{31}$$

exists each $x\in\mathbf{X}$ and defines a fuzzy derivation $H:\mathbf{X}\to\mathbf{X}$. Such that

$$N\left(f(x)-H(x),t\right)\geq\frac{(1-L)t}{(1-L)+L\psi\left(x_1,\cdots,x_k,0,\cdots,0\right)}\tag{32}$$

for all $x\in\mathbf{X}$, all $t>0$.

Proof. Let $(\mathbb{M},d)$ be the generalized metric space defined in the proof of Theorem 2. Consider the linear mapping $T:\mathbb{M}\to\mathbb{M}$ such that

$$Tg(x):=\frac{x}{2k}g(2kx)$$

for all $x\in\mathbf{X}$. I have

$$N\left(f(x)-\frac{1}{2k}f(2kx),\frac{1}{2k}t\right)\geq\frac{t}{t+\varphi(2kx,0,\cdots,0,0,\cdots,0,0,\cdots,0)}\geq\frac{t}{t+2k\varphi(x,0,\cdots,0,0,\cdots,0,0,\cdots,0)}\tag{33}$$

with everyone $x\in\mathbf{X}$. And all $t>0$. So

$$d(f,Tf)\leq L$$

The rest of the proof is similar to the proof of Theorem 2.

□

Theorem 4. Let $\psi:\mathbf{X}^{3k}\to[0,\infty)$ be a function such that there exists an $L<2k$

$$\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)\leq 2k\psi\left(\frac{x_1}{2k},\cdots,\frac{x_k}{2k},\frac{y_1}{2k},\cdots,\frac{y_k}{2k},\frac{z_1}{2k},\cdots,\frac{z_k}{2k}\right)\tag{34}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$, all $t>0$. Let $f:\mathbf{X}\to\mathbf{X}$ be a mapping satisfying

$$N\left(2kf\left(\sum_{j=1}^{k}\frac{qx_j+qy_j}{2k}+\sum_{j=1}^{k}qz_j\right)-\sum_{j=1}^{k}qf\left(x_j\right)-\sum_{j=1}^{k}qf\left(y_j\right)-2k\sum_{j=1}^{k}qf\left(z_j\right),t\right)\geq\frac{t}{t+\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)}\tag{35}$$

$$N\left(f\left(\prod_{j=1}^{k} x_j \cdot y_j\right) - \prod_{j=1}^{k} f(x_j) \cdot \prod_{j=1}^{k} y_j - \prod_{j=1}^{k} x_j \cdot \prod_{j=1}^{k} f(y_j), t\right) \geq \frac{t}{t + \psi(x_1, \cdots, x_k, y_1, \cdots, y_k, 0, \cdots, 0)} \tag{36}$$

with all $x_1, \cdots, x_k, y_1, \cdots, y_k \in \mathbf{X}$, all $t > 0$ and all $q \in \mathbb{R}$. Then

$$\beta(x) = N - \lim_{n\to\infty} \frac{1}{(2k)^n} f\left((2k)^n x\right) \tag{37}$$

exists each $x \in \mathbf{X}$ and defines a fuzzy derivation $H : \mathbf{X} \to \mathbf{X}$.

So that

$$N\left(f(x) - H(x), t\right) \geq \frac{(2k - 2kL)t}{(2k - 2kL)t + \psi(x_1, \cdots, x_k, 0, \cdots, 0)} \tag{38}$$

for all $x \in \mathbf{X}$, all $t > 0$.

Proof. Letting $q = 1$ and I replace $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $(2kx, 0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0)$ in (35), I get

$$N\left(2kf(x) - f(2kx), t\right) \geq \frac{t}{t + \varphi(2kx, 0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0)} \tag{39}$$

for all $x \in \mathbf{X}$, all $t > 0$.

Now I consider the set

$$\mathbb{M} := \{h : \mathbf{X} \to \mathbf{Y}\}$$

so introduce the generalized metric on $\mathbb{M}$ as follows:

$$d(g,h) := \inf\left\{\beta \in \mathbb{R}_+ : N\left(g(x) - h(x), \beta t\right) \geq \frac{t}{t + \varphi(2kx, 0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0)}, \forall x \in \mathbf{X}, \forall t > 0\right\}, \tag{40}$$

where, as usual, $\inf \phi = +\infty$. That has been proven by mathematicians $(\mathbb{M}, d)$ is complete (see [20]).

Now I consider the linear mapping $T : \mathbb{M} \to \mathbb{M}$ such that

$$Tg(x) := \frac{1}{2k} g(2kx)$$

with everyone $x \in \mathbf{X}$.

It follows from (41) that

$$N\left(f(x) - \frac{1}{2k} f(2kx), \frac{t}{2k}\right) \geq \frac{t}{t + \varphi(2kx, 0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0)} \tag{41}$$

The rest of the proof is similar to the proof of Theorem 2.

□

Theorem 5. Let $\psi : \mathbf{X}^{3k} \to [0, \infty)$ be a function such that there exists an $L < \frac{1}{2k}$

$$\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k) \le \frac{L}{2k}\psi(2kx_1,\cdots,2kx_k,2ky_1,\cdots,2ky_k,2kz_1,\cdots,2kz_k) \tag{42}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{X}$.

Let $f:\mathbf{X}\to\mathbf{X}$ be a mapping satisfying

$$N\left(2kf\left(\sum_{j=1}^{k}\frac{qx_j+qy_j}{2k}+\sum_{j=1}^{k}qz_j\right)-\sum_{j=1}^{k}f(x_j)-\sum_{j=1}^{k}f(y_j)-2kq\sum_{j=1}^{k}f(z_j),t\right) \ge \frac{t}{t+\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)} \tag{43}$$

$$N\left(f\left(\prod_{j=1}^{k}x_j\cdot y_j\right)-\prod_{j=1}^{k}f(x_j)\cdot\prod_{j=1}^{k}y_j-\prod_{j=1}^{k}x_j\cdot\prod_{j=1}^{k}f(y_j),t\right) \ge \frac{t}{t+\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,0,\cdots,0)} \tag{44}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k \in \mathbf{X}$, all $t>0$ and all $q\in\mathbb{R}$. Then

$$H(x) = N-\lim_{n\to\infty}(2k)^n f\left(\frac{x}{(2k)^n}\right) \tag{45}$$

exists each $x\in\mathbf{X}$ and defines a fuzzy derivation $H:\mathbf{X}\to\mathbf{X}$.

Such that

$$N(f(x)-H(x),t) \ge \frac{(2-2)t}{(2-2L)+L\psi(2kx,0,\cdots,0,0,\cdots,0,0,\cdots,0)} \tag{46}$$

for all $x\in\mathbf{X}$ and all $t>0$.

Proof. Let $(\mathbb{M},d)$ be the generalized metric space defined in the proof of Theorem 2. Consider the linear mapping $T:\mathbb{M}\to\mathbb{M}$ such that

$$Tg(x) := 2kg\left(\frac{x}{2k}\right)$$

with everyone $x\in\mathbf{X}$. I have

$$N(f(2kx)-2kf(x),t) \ge \frac{t}{t+\varphi(2kx,0,\cdots,0,0,\cdots,0,0,\cdots,0)} \tag{47}$$

with everyone $x\in\mathbf{X}$, and all $t>0$. So

$$d(f,Tf) \le \frac{L}{2k}$$

the rest of the proof is similar to the proof of Theorem 2.

□

4. Using the Fixed Point Method, Extend the Derivative for the Functional Inequalities (2) on the Fuzzy Banach Algebra

Now I study extended homomorphism by fixed point method.

With $\mathbf{X}$ is a fuzzy Banach algebras with quasi-norm N and that $(\mathbf{Y},N)$ be a fuzzy normed vector space. Under this setting, I need to show that the mapping must satisfy (2). These results are given in the following.

Lemma 1. Let $(\mathbf{X},N')$ and $(\mathbf{Y},N)$ be a fuzzy normed vector space and $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that

$$N\left(\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(y_j)+f\left(2k\sum_{j=1}^{k}z_j\right),t\right)\ge N\left(2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right),t\right) \quad (48)$$

for all $x_1,\ldots,x_k,y_1,\ldots,y_k\in\mathbf{X}$, all $t>0$, then *f* is Cauchy additive.

Then *f* is Cauchy additive.

Proof. I replace $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (48), I have

$$N\left((2k+1)f(0),t\right)=N\left(f(0),\frac{t}{2k+1}\right)\ge N\left(2kf(0),t\right)=N\left(f(0),\frac{t}{2k}\right)=1 \quad (49)$$

with everyone $t>0$. By N_5 and N_6, $N\left(f(0),\frac{t}{2k}\right)=1$. It follows N_2 that $f(0)=0$.

Next I replace $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(-y,\cdots,-y,y,\cdots,y,0,\cdots,0)$ in (48). I have

$$N\left(kf(-y)+kf(y)+f(0),t\right)=N\left(f(-y)+f(y),\frac{t}{k}\right)\ge N\left(f(0),\frac{t}{2k}\right) \quad (50)$$

It follows N_2 that $f(-y)+f(y)=0$.

So

$$f(-y)=-f(y)$$

Next I replace $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $\left(x,0,\cdots,0,y,0,\cdots,0,\frac{-x-y}{2k},0,\cdots,0\right)$ in (48), I have

$$N\left(f(x)+f(y)+f(-x-y),t\right)\ge N\left(f(0),\frac{t}{2k}\right)=1 \quad (51)$$

for all $x,y\in\mathbf{X}$, all $t>0$. It follows N_2 that

$$f(x)+f(y)+f(-x-y)=0$$

with everyone $x,y\in\mathbf{X}$.

Thus

$$f(x)+f(y)=f(x+y)$$

with everyone $x,y\in\mathbf{X}$, as desired.

□

Theorem 6. Let $\psi:\mathbf{X}^{3k}\to[0,\infty)$ be a function such that there exists an $L<\frac{1}{2k}$

$$\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\le\frac{L}{2k}\psi(2kx_1,\cdots,2kx_k,2ky_1,\cdots,2ky_k,2kz_1,\cdots,2kz_k) \quad (52)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k \in \mathbf{X}$.

Let $f:\mathbf{X}\to\mathbf{X}$ be an odd mapping satisfying

$$N\left(\sum_{j=1}^{k} qf\left(x_j\right)+\sum_{j=1}^{k} qf\left(y_j\right)+f\left(2kq\sum_{j=1}^{k} z_j\right),t\right)$$
$$\geq \min\left\{N\left(2kf\left(\sum_{j=1}^{k}\frac{qx_j+qy_j}{2k}+\sum_{j=1}^{k} qz_j\right),\frac{2kt}{3}\right),\frac{t}{t+\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)}\right\} \tag{53}$$

$$N\left(f\left(\prod_{j=1}^{k} x_j\cdot y_j\right)-\prod_{j=1}^{k} f\left(x_j\right)\cdot\prod_{j=1}^{k} y_j-\prod_{j=1}^{k} x_j\cdot\prod_{j=1}^{k} f\left(y_j\right),t\right)$$
$$\geq\frac{t}{t+\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,0,\cdots,0\right)} \tag{54}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k \in \mathbf{X}$, all $t>0$ and all $q\in\mathbb{R}$. Then

$$H\left(x\right)=N-\lim_{n\to\infty}\left(2k\right)^n f\left(\frac{x}{\left(2k\right)^n}\right) \tag{55}$$

exists each $x\in\mathbf{X}$ and defines a fuzzy derivation $H:\mathbf{X}\to\mathbf{X}$.

So that

$$N\left(f\left(x\right)-H\left(x\right),t\right)\geq\frac{\left(2k-2kL\right)t}{\left(2k-2kL\right)+L\psi\left(x,\cdots,x,x,\cdots,x,-x,0,\cdots,0\right)} \tag{56}$$

for all $x\in\mathbf{X}$, and all $t>0$.

Proof. Letting $q=1$ and I replace $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(x,\cdots,x,x,\cdots,x,-x,0,\cdots,0\right)$ in (53), I get

$$N\left(2kf\left(x\right)-f\left(2kx\right),t\right)\geq\frac{t}{1+\varphi\left(x,\cdots,x,x,\cdots,x,-x,0,\cdots,0\right)} \tag{57}$$

$$x\in\mathbf{X}.$$

Now I consider the set

$$\mathbb{M}:=\left\{h:\mathbf{X}\to\mathbf{X}\right\}$$

so introduce the generalized metric on $\mathbb{M}$ as follows:

$$d\left(g,h\right):=\inf\left\{\beta\in\mathbb{R}_+:N\left(g\left(x\right)-h\left(x\right),\beta t\right)\right.$$
$$\left.\geq\frac{t}{t+\varphi\left(x,\cdots,x,x,\cdots,x,-x,0,\cdots,0\right)},\forall x\in\mathbf{X},\forall t>0\right\}, \tag{58}$$

where, as usual, $\inf\phi=+\infty$. That has been proven by mathematicians $\left(\mathbb{M},d\right)$ is complete (see [16]).

Now I consider the linear mapping $T:\mathbb{M}\to\mathbb{M}$ such that

$$Tg\left(x\right):=2kg\left(\frac{x}{2k}\right)$$

with everyone $x\in\mathbf{X}$.

It follows from (59) that

$$N\left(f(x)-2kf\left(\frac{x}{2k}\right),t\right)\ge\frac{t}{1+\varphi\left(\frac{x}{2k},\cdots,\frac{x}{2k},\frac{x}{2k},\cdots,\frac{x}{2k},-\frac{x}{2k},0,\cdots,0\right)}$$
$$\ge\frac{t}{t+\frac{L}{2k}\varphi(x,\cdots,x,x,\cdots,x,-x,0,\cdots,0)} \tag{59}$$

for all $x\in\mathbb{X}$ and all $t>0$. So $d(f,Tf)\le\frac{L}{2k}$. By Theorem 1, there exists a mapping $H:\mathbf{X}\to\mathbf{Y}$ satisfying the following:

1) H is a fixed point of T, *i.e.*,

$$H\left(\frac{x}{2k}\right)=\frac{1}{2k}H(x) \tag{60}$$

with everyone $x\in\mathbf{X}$. The mapping H is a unique fixed point T in the set

$$\mathbb{Q}=\{g\in\mathbb{M}:d(f,g)<\infty\}$$

This implies that H is a unique mapping satisfying (60) such that there exists a $\beta\in(0,\infty)$ satisfying

$$N(f(x)-H(x),\beta t)\ge\frac{t}{t+\varphi(x,\cdots,x,x,\cdots,x,-x,0,\cdots,0)},\forall x\in\mathbf{X}.$$

2) $d(T^l f,H)\to 0$ as $l\to\infty$. This implies equality

$$N-\lim_{l\to\infty}(2k)^l f\left(\frac{x}{(2k)^l}\right)=H(x)$$

with everyone $x\in\mathbb{X}$.

3) $d(f,H)\le\frac{1}{1-L}d(f,Tf)$, which implies the inequality

4)

$$d(f,H)\le\frac{L}{2k-2kL}$$

this implies that the inequality (59) holds

By (54)

$$N\left((2k)^p\sum_{j=1}^k qf\left(\frac{x_j}{(2k)^p}\right)-(2k)^p\sum_{j=1}^k qf\left(\frac{y_j}{(2k)^p}\right)-f\left(q\sum_{j=1}^k\frac{z_j}{(2k)^{p-1}}\right),(2k)^p t\right)$$
$$\ge\min\left\{N\left((2k)^{p+1}f\left(\sum_{j=1}^k\frac{qx_j+qy_j}{(2k)^{p+1}}+\sum_{j=1}^k\frac{qz_j}{(2k)^p}\right),\frac{(2k)^{p+1}}{(2k)^p}\right),\right. \tag{61}$$
$$\left.\frac{t}{t+\psi\left(\frac{x_1}{(2k)^p},\cdots,\frac{x_k}{(2k)^p},\frac{y_1}{(2k)^p},\cdots,\frac{y_k}{(2k)^p},\frac{z_1}{(2k)^p},\cdots,\frac{z_k}{(2k)^p}\right)}\right\}$$

with all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$, all $t>0$, all $q\in\mathbb{R}$ and $p\in\mathbb{N}$. So

$$N\left((2k)^p \sum_{j=1}^{k} qf\left(\frac{x_j}{(2k)^p}\right) - (2k)^p \sum_{j=1}^{k} qf\left(\frac{y_j}{(2k)^p}\right) - f\left(q\sum_{j=1}^{k}\frac{z_j}{(2k)^{p-1}}\right), t\right)$$

$$\geq \min\left\{ N\left((2k)^{p+1} f\left(\sum_{j=1}^{k}\frac{qx_j+qy_j}{(2k)^{p+1}} + \sum_{j=1}^{k}\frac{qz_j}{(2k)^p}\right), \frac{2t}{3}k\right), \right. \tag{62}$$

$$\left. \frac{\frac{t}{(2k)^p}}{\frac{t}{(2k)^p} + \frac{L^p}{(2k)^p}\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)} \right\}$$

with all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{X}$, all $t>0$, all $q\in\mathbb{R}$ and $p\in\mathbb{N}$.

Since

$$\lim_{n\to\infty}\frac{\frac{t}{(2k)^p}}{\frac{t}{(2k)^p} + \frac{L^p}{(2k)^p}\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)} = 1$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{X}$, all $t>0$ and $p\in\mathbb{N}$. So

$$N\left(\sum_{j=1}^{k} qH(x_j) + \sum_{j=1}^{k} qH(y_j) + H\left(2kq\sum_{j=1}^{k} z_j\right)\right)$$
$$\geq N\left(2kH\left(\sum_{j=1}^{k}\frac{qx_j+qy_j}{2k} + \sum_{j=1}^{k} qz_j\right), \frac{2t}{3}k\right) \tag{63}$$

with all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{X}$, all $t>0$ and all $q\in\mathbb{R}$.

Let $q=1$ in (63). By Lemma 1, the mapping $H:\mathbf{X}\to\mathbf{X}$ is Cauchy additive. Next I replace $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,-x,0,\cdots,0)$ in (63), I get

$$2kqH(x) - H(2kqx) = 0$$

with all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{X}$, all $t>$ and all $q\in\mathbb{R}$.

So the mapping $H:\mathbf{X}\to\mathbf{X}$ is $\mathbb{R}$-linear.

The rest of the proof is similar to the proof of Theorem 2.

□

Theorem 7. Let $\psi:\mathbf{X}^{3k}\to[0,\infty)$ be a function such that there exists an $L<\frac{1}{2k}$

$$\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$$
$$\leq 2kL\psi\left(\frac{x_1}{2k},\cdots,\frac{x_k}{2k}k,\frac{y_1}{2k},\cdots,\frac{y_k}{2k},\frac{z_1}{2k},\cdots,\frac{z_k}{2k}\right) \tag{64}$$

with all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{X}$.

Let $f:\mathbf{X}\to\mathbf{X}$ be an odd mapping satisfying

$$N\left(\sum_{j=1}^{k} qf\left(x_j\right)+\sum_{j=1}^{k} qf\left(y_j\right)+f\left(2kq\sum_{j=1}^{k} z_j\right),t\right)$$
$$\geq \min\left\{N\left(2kf\left(\sum_{j=1}^{k}\frac{qx_j+qy_j}{2k}+\sum_{j=1}^{k} qz_j\right),\frac{2kt}{3}\right),\frac{t}{t+\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)}\right\} \tag{65}$$

$$N\left(f\left(\prod_{j=1}^{k} x_j\cdot y_j\right)-\prod_{j=1}^{k} f\left(x_j\right)\cdot\prod_{j=1}^{k} y_j-\prod_{j=1}^{k} x_j\cdot\prod_{j=1}^{k} f\left(y_j\right),t\right)$$
$$\geq\frac{t}{t+\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,0,\cdots,0\right)} \tag{66}$$

with all $x_1,\cdots,x_k,y_1,\cdots,y_k\in\mathbf{X}$, all $t>0$ and all $q\in\mathbb{R}$. Then

$$H\left(x\right)=N-\lim_{n\to\infty}\frac{1}{\left(2k\right)^n} f\left(\left(2k\right)^n x\right) \tag{67}$$

exists each $x\in\mathbf{X}$ and defines a fuzzy derivation $H:\mathbf{X}\to\mathbf{X}$.

So that

$$N\left(f\left(x\right)-H\left(x\right),t\right)\geq\frac{\left(2k-2kL\right)t}{\left(2k-2kL\right)+\psi\left(x,\cdots,x,x,\cdots,x,-x,0,\cdots,0\right)} \tag{68}$$

for all $x\in X$, all $t>0$.

The rest of the proof is similar to the proof of Theorem 2 and Theorem 6.

5. Conclusion

In this paper, I build the existence of the extended derivative on fuzzy Banach algebra for the Cauchy-Jensen equation with $3k$-variables above by applying the fixed point method to check that existence.

Conflicts of Interest

The author declares no conflicts of interest.

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