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Spectral Analysis of the Derivation of Green's Function of Helmholtz Integral Equation via Dirac-Delta Function and Cauchy Residual Approach

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Abstract

In this study, we have determined Green's functions for Helmholtz integral equations in a spherical polar coordinate system in the whole plane domain with the aid of spectral Fourier transform technique. Our intended Green's function solution has a dominant role to represent wave propagation with a high quantum wave number. The Dirac-delta function also plays an important role here to represent the scattering region for wave propagation. The evaluation of the improper double integrals in the complex plane furnishes our desired Green's functions. The applied technique allows us to obtain all of the possible Green's functions by using Somerfield radiation condition. With the help of computational software package MATLAB, we have drawn the solution plot that can express the analogue of wave propagation features. By using the MATLAB software package, we have drawn the solution plot that can express the analogue of wave propagation features.

Subject Areas

Mathematical Analysis, Mathematics, Ordinary Differential Equation, Partial Differential Equation

Keywords

Helmholtz Integral Equation, Summerfield Radiation Condition, Cauchy Residual Approach, Classical Schrödinger Wave Equation, Green's Function

1. Introduction

In the present world, an integral equation is observed to be a well-built branch of science and technology providing outcomes which advantageously employed by experts working in various fields such as wave acoustics, electromagnetics, plasma physics, and fluid dynamics and so on. One of the most important integral equations of special status is Helmholtz integral equation which displays such kind of wave phenomena of the fundamental particle with Somerfield radiation condition. Besides, scattering of buildings always governs contrary intention in the domain of noise fences.

The Green's function solution is one of the most significant analytical tools for solving partial differential equations in the field of electromagnetism, wave mechanics, optics, quantum mechanics, fluid dynamics, relativistic particle dynamics, general relativity and so on [1] [2]. In order to exclude the integration over an infinite plane, one can use the formulations [3] [4] [5] utilizing the half-space Green's function. In physical or mathematical, acoustics can be utilized for the study of solutions of the wave equation [6]. The wave phenomenon has been protracted to include even more particles and quasi-particles such as photons denoted by optical waves [7] [8]. On the other hand, phonons have represented by acoustic or elastic waves [9]. The acoustic or elastic wave phenomena of wave equation are the science of sound that deals with the generation of sound, propagation, and collaboration with the matter. Sources of quantum particles are known to be an indispensable element in all scattering experiments.

In previous, researchers have utilized different schemes to clarify the wave propagation of an acoustic wave. A phase-shift method is an accurate scheme for laterally invariant velocity structure but breaks down for arbitrary velocity variation, due to approximations made to the solutions of the wave equation [10]. In underwater acoustics, propagation, radiation, and scattering of sound are usually modeled by obtaining analytical and/or numerical solutions of a wave equation for the acoustic pressure, complemented by appropriate initial and boundary conditions [11].

When the source moves even closer to the observer, then analysis of a wave field produce a non-asymptotic form of Green's function that help us to elucidate near field analysis. It is vital to attain a correct causal form for the 3D transform, as otherwise the outline of forward and backward propagating waves does not follow logically [12] [13] [14]. This ability to treat modeling of Green's function solution is achieved by making use of some early ideas on the non-iterative solution of the Lippmann-Schrödinger equation in quantum scattering [15] [16] as well as some very simple standard ideas from differential equations [17]. However, many studies have been already carried out on wave propagation throughout the world. In this work, we have considered the classical time-independent Schrödinger equation of finding the Green's function solution for Helmholtz integral equation subject to Sommerfeld radiation conditions.

2. Green's Function for Helmholtz Integral Wave Equation

Helmholtz boundary value problems can be put into one of three types: direct, eigenvalue or inverse. Among these methods, inverse is appropriate for analytical series methods that are established for a direct problem, with Dirichlet boundary conditions [18]. Helmholtz equation can be explained in many methods, but in our study, we introduce complex valued Green's Function to solve non-homogeneous acoustic Helmholtz integral equation. Moreover, the general way to solve non-homogeneous acoustic Helmholtz integral equation for wave propagation is to form a Green's function [19]. We used the one-dimensional (1D) Schrödinger classical wave equation as the following form

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{1}$$

Using separation of variables $u(x,t) = \psi(x) f(t)$ (where $\psi(x)$ is the Schrödinger wave function), the suitable wave equation solutions for $f(t)$ such as $e^{i\omega t}$, take the following form

$$\frac{d^2\psi(x)}{dx^2} = -\frac{\omega^2}{v^2}\psi(x) \tag{2}$$

The above equation signifies an ordinary differential equation describing the spatial amplitude of the matter wave as a function of position. The entire energy of a particle is the sum of kinetic and potential energy and mathematically

$$E = \frac{p^2}{2m} + V(x) \tag{3}$$

But we know that $\omega = 2\pi\upsilon$ and $v = \upsilon\lambda$. Using these values in Equation (2), we get $\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}\left[E - V(x)\right]\psi(x) = 0$

$$\text{or,}\ \left(\nabla^2 + k^2\right)\psi(\boldsymbol{r}) = \lambda U(\boldsymbol{r})\psi(\boldsymbol{r}) \tag{4}$$

where, $U(\boldsymbol{r}) = \frac{2m}{\hbar^2}V(\boldsymbol{r})$

With Green's function $G(\boldsymbol{r},\boldsymbol{r}')$, we can write

$$\left(\nabla^2 + k^2\right)G(\boldsymbol{r},\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}')$$

$$\text{or,}\ \left(\nabla^2 + k^2\right)G(\boldsymbol{R}) = -\delta^3(\boldsymbol{R}) \tag{5}$$

This is Helmholtz integral equation. Where, δ means Dirac delta function and $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$.

The Fourier integral transform is defined as

$$\hat{G}(\xi_1,\xi_2,\xi_3) = \iiint_{-\infty} G(x_1,x_2,x_3)\,e^{-i(\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3)}\,dx_1 dx_2 dx_3$$

$$\Rightarrow \hat{G}(\xi) = \int_{R^3} G(\boldsymbol{x})\exp(-i\boldsymbol{\xi}\cdot\boldsymbol{x})\,d\boldsymbol{x} \tag{6}$$

By utilizing Fourier integral formula (6), the solution of the above Equation (5) is

$$G(\boldsymbol{x})=\frac{1}{8\pi^3}\int_{R^3}\hat{G}(\xi)\exp(i\boldsymbol{\xi}\cdot\boldsymbol{x})\mathrm{d}\boldsymbol{\xi} \tag{7}$$

Assume that

$$\lim_{x_1\to\pm\infty}\frac{\partial^n}{\partial x_1^n}G(\boldsymbol{x})=0,(n=0,1,2,\cdots)\ \text{Sommerfeld radiation condition} \tag{8}$$

The Fourier Transform of $\frac{\partial}{\partial x_1}G(\boldsymbol{x})$ is defined by

$$=\int_{R^3}\frac{\partial}{\partial x_1}G(\boldsymbol{x})\mathrm{e}^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{x}$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-i(\xi_2x_2+\xi_3x_3)}\mathrm{d}x_2\mathrm{d}x_3\int_{-\infty}^{\infty}\frac{\partial}{\partial x_1}G(\boldsymbol{x})\mathrm{e}^{-i\xi_1x_1}\mathrm{d}x_1$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-i(\xi_2x_2+\xi_3x_3)}\mathrm{d}x_2\mathrm{d}x_3\times\left[\lim_{M\to\infty}\left[G(\boldsymbol{x})\mathrm{e}^{-i\xi_1x_1}\right]_{x_1=-M}^{x_1=M}-\int_{-\infty}^{\infty}G(\boldsymbol{x})(-i\xi_1)\mathrm{e}^{-i\xi_1x_1}\mathrm{d}x_1\right]$$

(Integral by Parts)

$$=(i\xi_1)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-i(\xi_2x_2+\xi_3x_3)}\mathrm{d}x_2\mathrm{d}x_3\int_{-\infty}^{\infty}G(\boldsymbol{x})\mathrm{e}^{-i\xi_1x_1}\mathrm{d}x_1$$
$$=(i\xi_1)\int_{R^3}G(\boldsymbol{x})\mathrm{e}^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{x}$$
$$=(i\xi_1)\hat{G}(\boldsymbol{\xi})$$

Now the Fourier Transform of $\frac{\partial^2}{\partial x_1^2}G(\boldsymbol{x})$ is

$$=\int_{R^3}\frac{\partial^2}{\partial x_1^2}G(\boldsymbol{x})\exp(-i\boldsymbol{\xi}\cdot\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$
$$=i\xi_1\frac{\partial}{\partial x_1}\int_{R^3}\frac{\partial}{\partial x_1}G(\boldsymbol{x})\exp(-i\boldsymbol{\xi}\cdot\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$
$$=i\xi_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-i(\xi_2x_2+\xi_3x_3)}\mathrm{d}x_2\mathrm{d}x_3\int_{-\infty}^{\infty}\frac{\partial}{\partial x_1}G(\boldsymbol{x})\mathrm{e}^{-i\xi_1x_1}\mathrm{d}x_1 \tag{9a}$$
$$=(i\xi_1)\cdot(i\xi_1)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-(\xi_2x_2+\xi_3x_3)}\mathrm{d}x_2\mathrm{d}x_3\int_{-\infty}^{\infty}G(\boldsymbol{x})\mathrm{e}^{-i\xi_1x_1}\mathrm{d}x_1$$
$$=-\xi_1^2\int_{R^3}G(\boldsymbol{x})\exp(-i\boldsymbol{\xi}\cdot\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$
$$=-\xi_1^2\hat{G}(\boldsymbol{\xi})$$

In this similar way we can write

$$\frac{\partial^2}{\partial x_2^2}G(\boldsymbol{x})=-\xi_2^2\hat{G}(\boldsymbol{\xi}) \tag{9b}$$

$$\frac{\partial^2}{\partial x_3^2}G(\boldsymbol{x})=-\xi_3^2\hat{G}(\boldsymbol{\xi}) \tag{9c}$$

Plugging Fourier transform on both sides of equation (5), we can write

$$F\left\{\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}+k^2\right)G(x_1,x_2,x_3)\right\}=-F\left\{\delta(x_1)\delta(x_2)\delta(x_3)\right\}$$

$$\Rightarrow\int_{R^3}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)G(\boldsymbol{x})\exp(-i\boldsymbol{\xi}\cdot\boldsymbol{x})\mathrm{d}\boldsymbol{x}+k^2\int_{R^3}G(\boldsymbol{x})\exp(-i\boldsymbol{\xi}\cdot\boldsymbol{x})\mathrm{d}\boldsymbol{x}=-1$$

$$\Rightarrow-\left(\xi_1^2+\xi_2^2+\xi_3^2\right)\hat{G}(\boldsymbol{\xi})+k^2\hat{G}(\boldsymbol{\xi})=-1$$

$$\Rightarrow -\xi^2\hat{G}(\boldsymbol{\xi}) + k^2\hat{G}(\boldsymbol{\xi}) = -1$$

$$\Rightarrow \hat{G}(\boldsymbol{\xi})\left\{k^2 - \xi^2\right\} = -1 \quad \left[i^2 = -1\right]$$

Therefore,

$$\hat{G}(\boldsymbol{\xi}) = \frac{1}{\xi^2 - k^2} \tag{10}$$

The above equation satisfies the spherical symmetric potential and even function criteria respectively (Details in appendix A and B).

Now, the Equation (7) can be written as

$$G(\boldsymbol{x}) = \frac{1}{8\pi^3}\int_{R^3}\frac{1}{\xi^2 - k^2}\exp(i\boldsymbol{\xi}\cdot\boldsymbol{x})\mathrm{d}\boldsymbol{\xi}$$

$$\Rightarrow G(\boldsymbol{x}) = \frac{1}{4\pi^2}\int_0^\infty \frac{\xi^2}{\xi^2 - k^2}\mathrm{d}\xi\int_0^\pi \mathrm{e}^{i\xi r\cos\phi}\sin\phi\mathrm{d}\phi$$

$$\Rightarrow G(\boldsymbol{x}) = \frac{1}{4\pi^2}\int_0^\infty \frac{\xi^2}{\xi^2 - k^2}G(\xi)\mathrm{d}\xi\,;$$

where, $G(\xi) = \frac{1}{i\xi r}\left[\mathrm{e}^{i\xi r} - \mathrm{e}^{-i\xi r}\right]$

$$\Rightarrow G(\boldsymbol{x}) = \frac{1}{4\pi^2}\int_0^\infty \frac{\xi^2}{\xi^2 - k^2}\frac{1}{i\xi r}\left[\mathrm{e}^{i\xi r} - \mathrm{e}^{-i\xi r}\right]\mathrm{d}\xi$$

$$\Rightarrow G(\boldsymbol{x}) = \frac{1}{4\pi^2}\frac{1}{2}\int_{-\infty}^\infty \frac{\xi^2}{\xi^2 - k^2}\frac{1}{i\xi r}\left[\mathrm{e}^{i\xi r} - \mathrm{e}^{-i\xi r}\right]\mathrm{d}\xi$$

Therefore

$$G(\boldsymbol{x}) = \begin{cases} \frac{1}{4\pi}\left(\frac{\cos kr + i\sin kr}{r}\right); & I_m(k) > 0 \\ \frac{1}{4\pi}\left(\frac{\cos kr - i\sin kr}{r}\right); & I_m(k) < 0 \end{cases} \tag{11}$$

(Details in appendix C) Equation (11) is the desired Green's function solution for Helmholtz integral equation.

3. Results & Discussion

Our derived Green's function solution for acoustic Helmholtz integral wave equation that yield good outcomes at adequately high wave numbers. This wave number k is often taking some real or complex constant, and Green's function gets more accuracy for increasing the wave number, as depicted in **Figures 1(a)-(d)**. The wave number can be measured complex if the medium of propagation is energy absorbing or a function of space. We have seen that, at very low wave numbers, it behaves very much like the Laplace equation and solutions at large wave number are highly oscillatory. This causes a great increase in complexity of analytical and numerical method. We also imposed Sommerfeld radiation condition or scattered condition for the solution of Helmholtz equation where confirms that the scattered wave is outgoing *i.e.* propagates away from the obstacle, as shown in the **Figure 2(a)** & **Figure 2(b)**. **Figure 3(a)** & **Figure 3(b)**

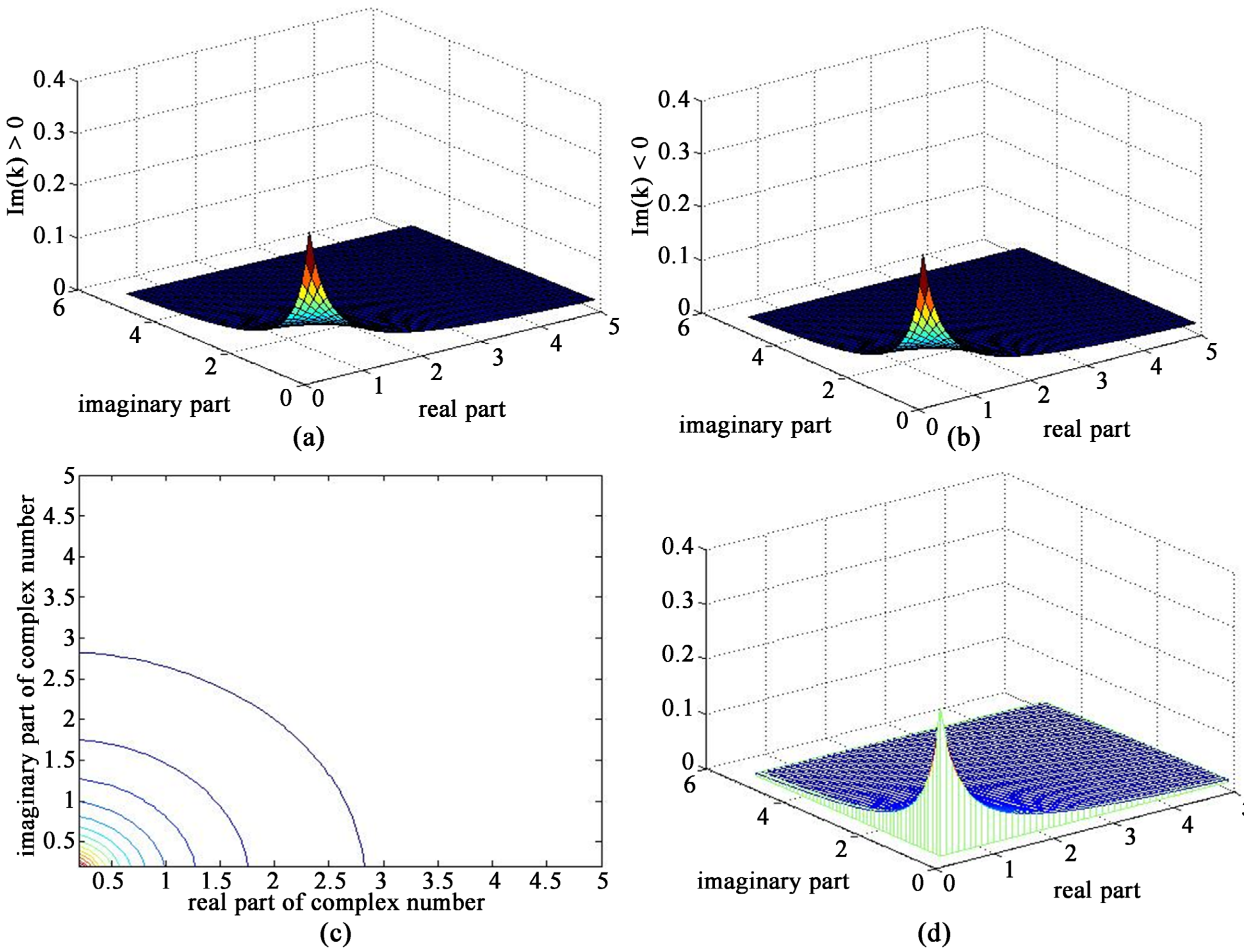

Figure 1. Graphical Green's function (a) 3D Plot of Green's function for $k=100$, $\operatorname{Im}(k)>0$; (b) 3D Plot of Green's function for $k=100$, $\operatorname{Im}(k)<0$; (c) Contour two dimensional; (d) curtain mesh plot of Green's function when wave number $k=10000$.

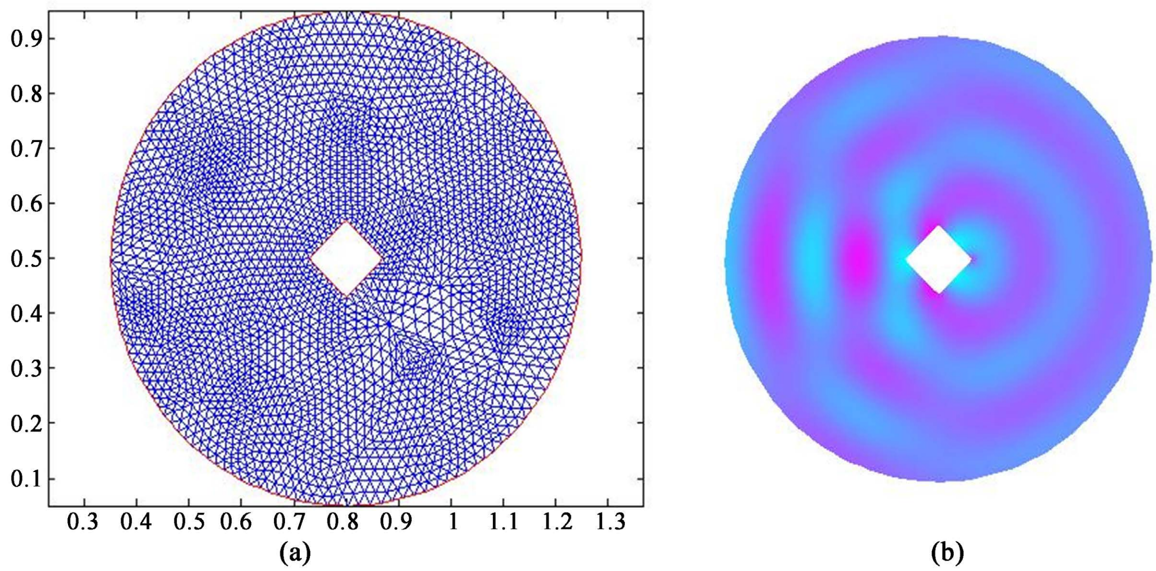

Figure 2. Scattering region of Helmholtz integral wave equation for wave number $k = 50$ and $k = 500$ respectively.

shows, the Helmholtz integral wave equation is an equation of the elliptic type, for which it is usual to consider boundary value problems. The wave front emitted from the source can no longer be regarded as a sphere. If the light emitted from a source exhibits sharp directivity then the wave front near the source point will act like a superimposed form of a plane wave and a spherical wave, but, it is clear that we need more and more time period at very high wave number to get accurate result for the solution of Helmholtz equation, as shown in Figure 4. Overall, Green's function of Helmholtz integral equation for wave propagation are the solution of monochromatic mechanical and longitudinal plane waves (sink) when combined with e^{-ikr} and diverging wave (source) when combined with e^{ikr} respectively.

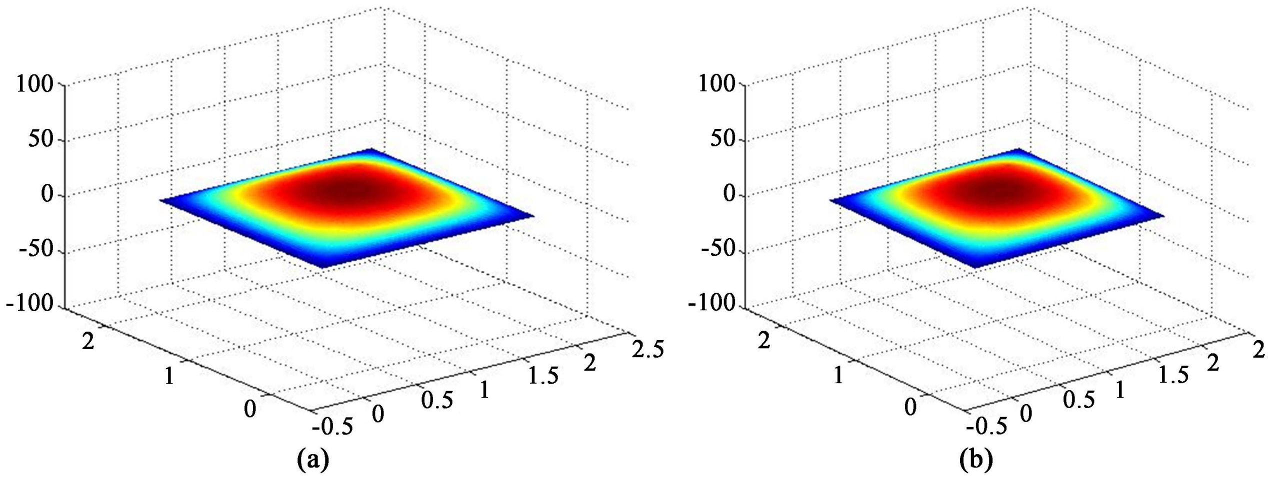

Figure 3. (a) Green's function solution for Helmholtz integral equation when k = 10 and number of iteration n = 1000 and (b) when k = 25 and number of iteration n = 1000.

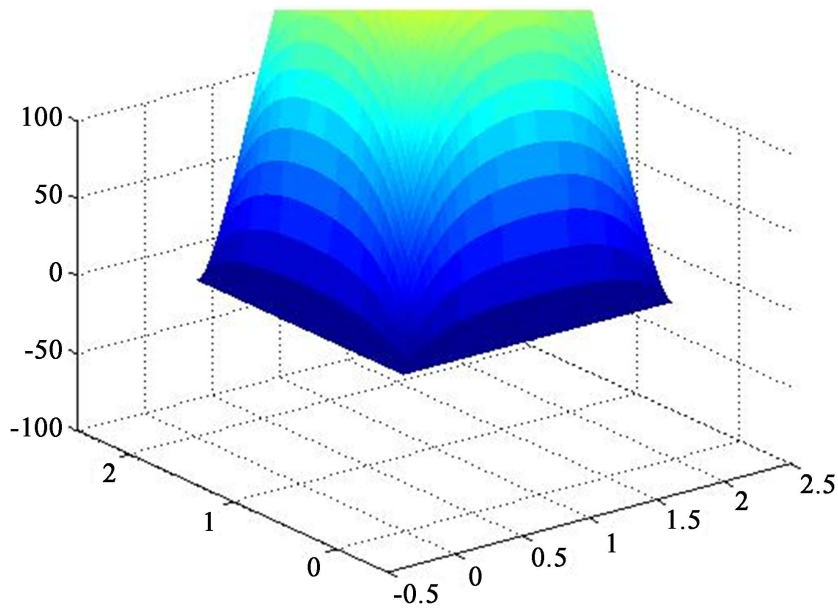

Figure 4. Green's function solution for Helmholtz integral equation when k =200 and number of iteration n = 1000.

4. Conclusion

We have applied here an arguably more direct and general scheme to determine the generalized form of Green's functions. A potential advantage of this approach is the fact that it may allow one to find a general form for Green's functions of a given partial differential equation with a high quantum wave number k. The imposition of suitable Sommerfeld radiation condition allows us to find Green's function solution that helps to represent wave propagation of Helmholtz integral equation. Our observation of this study allows us that monochromatic longitudinal plane wave is an idealization and represents a complex valued physical situation. The skills for proving the required facts about the involved generalized functions are likely to be beneficial or solution of many similar problems of acoustic, electrodynamics and mathematical physics.

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Conflicts of Interest

The authors declare no conflicts of interest.

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Appendix

A. Spherically Symmetric Potential Test

The Fourier transform of a function is spherically symmetric if and only if the function is spherically symmetric itself. This is one of the notorious properties of the Fourier transform.

Let k be defined as $k = \left\{ (\xi_1, \xi_2, \xi_3) \in \Re^3 \middle| \xi_1^2 + \xi_2^2 + \xi_3^2 \le R, (R > 0) \right\}$. Also, let us consider

$$V = \iiint_K \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \tag{1}$$

In order to attain a solution of acoustic Helmholtz integral equation subject to the Sommerfeld radiation condition, we need a spherically symmetric ansatz.

In spherical coordinate system, we have

$$\begin{aligned} x = \xi_1 &= \rho \sin\phi \cos\theta \\ y = \xi_2 &= \rho \sin\phi \sin\theta \\ z = \xi_3 &= \rho \cos\phi \end{aligned} \tag{2}$$

Squaring and adding each of expressions in Equation (2), we get

$$\begin{aligned} \xi_1^2 + \xi_2^2 + \xi_3^2 &= \rho^2 \sin^2\phi \left(\cos^2\theta + \sin^2\theta \right) + \rho^2 \cos^2\phi \\ &= \rho^2 \sin^2\phi + \rho^2 \cos^2\phi \\ &= \rho^2 \left(\sin^2\phi + \cos^2\phi \right) \\ \therefore \ \xi_1^2 + \xi_2^2 + \xi_3^2 &= \rho^2 \end{aligned}$$

Because of $\left(\cos^2\theta + \sin^2\theta\right) = 1$ and $\left(\sin^2\phi + \cos^2\phi\right) = 1$, $\tan\theta = \frac{\xi_2}{\xi_1}$ and $\tan\phi = \frac{\sqrt{\xi_1^2 + \xi_2^2}}{\xi_3}$.

Now, the determinant of the Jacobian matrix of (ξ_1, ξ_2, ξ_3) with respect to (ρ, θ, ϕ) is defined as

$$J = \frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \frac{\partial \xi_1}{\partial \rho} & \frac{\partial \xi_1}{\partial \theta} & \frac{\partial \xi_1}{\partial \phi} \\ \frac{\partial \xi_2}{\partial \rho} & \frac{\partial \xi_2}{\partial \theta} & \frac{\partial \xi_2}{\partial \phi} \\ \frac{\partial \xi_3}{\partial \rho} & \frac{\partial \xi_3}{\partial \theta} & \frac{\partial \xi_3}{\partial \phi} \end{vmatrix} \tag{3}$$

Now, from Equation (21) we can write

$$V = \iiint \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 = \iiint J \frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(\rho, \theta, \phi)} \mathrm{d}\boldsymbol{\xi} \tag{4}$$

Using Equations (3) in (4) and the range of spherical coordinates (ρ, θ, ϕ) are as follows $\rho \ge 0$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$. We get

$$V = \int_{\rho=0}^{R} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} \mathrm{d}\rho \mathrm{d}\theta \mathrm{d}\phi$$

$$\Rightarrow V=\int_{\rho=0}^{R}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\Big[\sin\phi\cos\theta\left(-\rho^2\sin^2\phi\cos\theta\right)$$
$$+\rho\sin\phi\sin\theta\left(-\rho\sin^2\phi\sin\theta-\rho\cos^2\phi\sin\theta\right)$$
$$+\rho\cos\phi\cos\theta\left(-\rho\sin\phi\cos\phi\cos\theta\right)\Big]\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi$$

$$\Rightarrow V=\int_{\rho=0}^{R}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\Big[-\rho^2\sin^3\phi\left(\sin^2\theta+\cos^2\theta\right)$$
$$-\rho^2\cos^2\phi\sin\phi\left(\sin^2\theta+\cos^2\theta\right)\Big]\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi$$

$$\Rightarrow V=\int_{\rho=0}^{R}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\left[-\rho^2\sin^3\phi-\rho^2\cos^2\phi\sin\phi\right]\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi\quad\left[\because\sin^2\theta+\cos^2\theta=1\right]$$

$$\Rightarrow V=\int_{\rho=0}^{R}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\left[-\rho^2\sin\phi\right]\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi$$

$$\Rightarrow V=\int_{\rho=0}^{R}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\rho^2\sin\phi\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi\quad\left[\because|J|=\left|-\rho^2\sin\phi\right|=\rho^2\sin\phi\right]$$

$$\Rightarrow V=\int_{\rho=0}^{R}\int_{\theta=0}^{2\pi}\rho^2\left[-\cos\phi\right]_0^{\pi}\mathrm{d}\rho\mathrm{d}\theta$$

$$\Rightarrow V=2\int_{\rho=0}^{R}\rho^2\left[\theta\right]_0^{2\pi}\mathrm{d}\rho$$

Therefore, $V=4\pi\left[\dfrac{\rho^3}{3}\right]_0^R=\dfrac{4}{3}\pi R^3$

Since V does not depends on θ and ϕ but it is a function of R only. Hence, we can summarize that V is spherically symmetric. We can write

$$G(\boldsymbol{x})=\frac{1}{4\pi^2}\int_0^{\infty}\frac{\xi^2}{\xi^2-k^2}\exp(i\boldsymbol{\xi}\cdot\boldsymbol{x})\mathrm{d}\xi\int_0^{\pi}\sin\phi\mathrm{d}\phi$$

$$\Rightarrow G(\boldsymbol{x})=\frac{1}{4\pi^2}\int_0^{\infty}\frac{\xi^2}{\xi^2-k^2}\mathrm{d}\xi\int_0^{\pi}\exp(i\boldsymbol{\xi}\cdot\boldsymbol{x})\sin\phi\mathrm{d}\phi$$

$$\Rightarrow G(\boldsymbol{x})=\frac{1}{4\pi^2}\int_0^{\infty}\frac{\xi^2}{\xi^2-k^2}\mathrm{d}\xi\int_0^{\pi}\mathrm{e}^{i\xi r\cos\phi}\sin\phi\mathrm{d}\phi\tag{5}$$

[since $\mathrm{e}^{i\xi x}=\mathrm{e}^{i\xi r\cos\phi}$]

To solve $\int_0^{\pi}\mathrm{e}^{i\xi r\cos\phi}\sin\phi\mathrm{d}\phi$ let us consider $z=i\xi r\cos\phi$, then $\mathrm{d}z=-i\xi r\sin\phi\mathrm{d}\phi$ and the limit of z takes the following form.

When, $\phi=0$ then $z=i\xi r\cos 0=i\xi r$ and

when, $\phi=\pi$ then $z=i\xi r\cos\pi=-i\xi r$

$$\int_0^{\pi}\mathrm{e}^{i\xi r\cos\phi}\sin\phi\mathrm{d}\phi=\int_{i\xi r}^{-i\xi r}\mathrm{e}^{z}\frac{\mathrm{d}z}{-i\xi r}$$
$$=-\frac{1}{i\xi r}\int_{i\xi r}^{-i\xi r}\mathrm{e}^{z}\mathrm{d}z$$
$$=-\frac{1}{i\xi r}\left[\mathrm{e}^{z}\right]_{i\xi r}^{-i\xi r}$$
$$=-\frac{1}{i\xi r}\left[\mathrm{e}^{-i\xi r}-\mathrm{e}^{i\xi r}\right]$$

Therefore,

$$\int_0^{\pi}\mathrm{e}^{i\xi r\cos\phi}\sin\phi\mathrm{d}\phi=\frac{1}{i\xi r}\left[\mathrm{e}^{i\xi r}-\mathrm{e}^{-i\xi r}\right]\tag{6}$$

Using this in Equation (25), we can write

$$G(\boldsymbol{x}) = \frac{1}{4\pi^2}\int_0^\infty \frac{\xi^2}{\xi^2 - k^2}\frac{1}{i\xi r}\left[e^{i\xi r} - e^{-i\xi r}\right]d\xi \tag{7}$$

$$\text{or, } G(\boldsymbol{x}) = \frac{1}{4\pi^2}\int_0^\infty \frac{\xi^2}{\xi^2 - k^2} G(\xi)\, d\xi \tag{7a}$$

where, $G(\xi) = \frac{1}{i\xi r}\left[e^{i\xi r} - e^{-i\xi r}\right]$.

B. Even Function Test

The function $G(\boldsymbol{x})$ of the above can be considered as a correct form if $G(\xi)$ satisfy even function property. *i.e.* $G(\xi) = G(-\xi)$.

Now, let us consider,

$$G(\xi) = \frac{1}{i\xi r}\left[e^{i\xi r} - e^{-i\xi r}\right] \tag{8}$$

If we replace ξ by $-\xi$, then we can write

$$G(-\xi) = \frac{1}{-i\xi r}\left[e^{-i\xi r} - e^{i\xi r}\right]$$

$$\Rightarrow G(-\xi) = -\frac{1}{-i\xi r}\left[e^{i\xi r} - e^{-i\xi r}\right]$$

$$\Rightarrow G(-\xi) = \frac{1}{i\xi r}\left[e^{i\xi r} - e^{-i\xi r}\right]$$

Since $G(-\xi) = G(\xi)$. So, we can say that $G(\xi) = \frac{1}{i\xi r}\left[e^{i\xi r} - e^{-i\xi r}\right]$ is an even function.

C. Contour Integration (Jordan Lemma)

$$\text{We have, } G(\boldsymbol{x}) = \frac{1}{4\pi^2}\int_0^\infty \frac{\xi^2}{\xi^2 - k^2}\frac{1}{i\xi r}\left[e^{i\xi r} - e^{-i\xi r}\right]d\xi$$

$$\Rightarrow G(\boldsymbol{x}) = \frac{1}{4\pi^2}\frac{1}{2}\int_{-\infty}^\infty \frac{\xi^2}{\xi^2 - k^2}\frac{1}{i\xi r}\left[e^{i\xi r} - e^{-i\xi r}\right]d\xi$$

Therefore,

$$G(\boldsymbol{x}) = \frac{1}{8\pi^2 ir}\int_{-\infty}^\infty \frac{\xi}{\xi^2 - k^2}\left[e^{i\xi r} - e^{-i\xi r}\right]d\xi \tag{9}$$

From the above equation, $\xi^2 - k^2 = 0$. Therefore, $\xi = \pm k$. So, here we observe two pole points for which $G(\xi)$ are not analytic. To remove this, we need contour integration.

According to Cauchy's residue theorem, we know

$$\int_\Gamma G(\xi)\, d\xi = 2\pi i \times \text{sum of the residues} \tag{10}$$

In equation (29), there has two parts. One is $\int_{-\infty}^\infty \frac{\xi e^{i\xi r}}{\xi^2 - k^2} d\xi$ and another part of the form $\int_{-\infty}^{\infty} \frac{-\xi e^{-i\xi r}}{\xi^2 - k^2} d\xi$.

Case I: First, Consider $\int_{-\infty}^{\infty} \frac{\xi e^{i\xi r}}{\xi^2 - k^2} d\xi$

$$\text{Now, } \int_{-\infty}^{\infty} \frac{\xi e^{i\xi r}}{\xi^2 - k^2} d\xi = \lim_{R\to\infty} \int_{-R}^{R} + \int_{\Gamma} \frac{\xi e^{i\xi r}}{\xi^2 - k^2} d\xi \tag{11}$$

In the above, Γ is the semi-circular arc of radius R. The poles have obtained by using the equation. The integrand has simple poles $\xi = \pm k$. If $I_m(\xi) > 0$, then we can only see the location of the pole $\xi = +k$. Because $\xi = +k$ only lies inside the contour C. On the other hand, $\xi = -k$ lies outside the contour C. As a result, we cannot see the location of the pole $\xi = -k$.

Now, according to Cauchy residue theorem, we have

$$\begin{aligned} I_1 &= 2\pi i \times (\text{Residue of upper-half plane at } \xi = +k) \\ &= 2\pi i \lim_{\xi\to k} \left\{ (\xi - k) \frac{\xi e^{i\xi r}}{\xi^2 - k^2} \right\} \\ &= 2\pi i \lim_{\xi\to k} \left\{ \frac{1}{2} (\xi - k) \left(\frac{1}{\xi + k} + \frac{1}{\xi - k} \right) e^{i\xi r} \right\} \\ &= 2\pi i \lim_{\xi\to k} \left\{ \frac{1}{2} (\xi - k) \left(\frac{1}{\xi - k} \right) e^{i\xi r} \right\} \\ &= \pi i \lim_{\xi\to k} e^{i\xi r} = \pi i e^{ikr} \end{aligned}$$

Taking $\lim_{R\to\infty}$ and using the property of contour integral

$\lim_{R\to\infty} \int_{\Gamma} e^{i\xi r} F(\xi) d\xi = 0$.

$$\textit{i.e.} \quad \lim_{R\to\infty} \int_{\Gamma} \frac{\xi e^{i\xi r}}{\xi^2 - k^2} d\xi = 0$$

$$\text{Therefore, } I_1 = \pi i e^{ikr} \tag{12}$$

$$\text{Second, Consider } \int_{-\infty}^{\infty} \frac{-\xi e^{-i\xi r}}{\xi^2 - k^2} d\xi \tag{13}$$

In the above integral, a negative sign can be measured because with a positive parameter ξ, then Jordan's lemma states the following upper bound for the contour integral: where equal sign is when $G(\xi)$ vanishes everywhere. An analogous proclamation for a semicircular contour in the lower half-plane holds when $\xi < 0$.

$$\text{Now, } \int_{-\infty}^{\infty} \frac{-\xi e^{-i\xi r}}{\xi^2 - k^2} d\xi = \lim_{R\to\infty} \int_{-R}^{R} + \int_{\Gamma'} \frac{-\xi e^{-i\xi r}}{\xi^2 - k^2} d\xi \tag{14}$$

where Γ' is the semi-circular arc of radius *R*. In the similar process of the above, the integrand has simple poles at $\xi = \pm k$. Here, $\xi = -k$ only lies inside the contour C. On the other hand, $\xi = k$ lies outside the contour C. As a result, we cannot see the location of the pole $\xi = k$.

According to Cauchy residue theorem, we know

$$I_2 = -2\pi i \times \left(\text{Residue of lower half-plane at } \xi = -k\right)$$
$$= -2\pi i \lim_{\xi \to -k} \left\{ (\xi + k) \frac{-\xi e^{-i\xi r}}{\xi^2 - k^2} \right\}$$
$$= -2\pi i \lim_{\xi \to -k} \left\{ \left(-\frac{1}{2}\right) (\xi + k) \left(\frac{1}{\xi + k} + \frac{1}{\xi - k} \right) e^{-i\xi r} \right\}$$
$$= -2\pi i \lim_{\xi \to -k} \left\{ \left(-\frac{1}{2}\right) (\xi + k) \left(\frac{1}{\xi + k} \right) e^{-i\xi r} \right\}$$
$$= \pi i \lim_{\xi \to -k} e^{-i\xi r} = \pi i e^{-ikr}$$

In the similar process of the above, we can write

$$I_2 = \pi i e^{ikr} = \pi i e^{ikr} \tag{15}$$

We have

$$I = I_1 + I_2 = \pi i e^{ikr} + \pi i e^{ikr} = 2\pi i e^{ikr}$$

Using this value in Equation (9), we get

$$G(\boldsymbol{x}) = \frac{1}{8\pi^2 ir} 2\pi i e^{ikr}$$
$$\Rightarrow G(\boldsymbol{x}) = \frac{1}{4\pi} \frac{e^{ikr}}{r}$$

Therefore,

$$G(\boldsymbol{x}) = \frac{1}{4\pi} \left(\frac{\cos kr + i \sin kr}{r} \right) \tag{16}$$

Case **II:**

First, Consider $\int_{-\infty}^{\infty} \frac{\xi e^{i\xi r}}{\xi^2 - k^2} \mathrm{d}\xi$

If $I_m(\xi) < 0$, then we get only the location of the pole $\xi = -k$. In case of upper half-plane, $\xi = -k$ only lies inside the contour C. On the other hand, $\xi = +k$ lies outside the contour C. As a result, we cannot see the location of the pole $\xi = +k$.

Now, according to Cauchy residue theorem, we have

$$I_1 = 2\pi i \times \left(\text{Residue of upper-half plane at } \xi = -k\right)$$
$$= 2\pi i \lim_{\xi \to -k} \left\{ (\xi + k) \frac{\xi e^{i\xi r}}{\xi^2 - k^2} \right\}$$
$$= 2\pi i \lim_{\xi \to -k} \left\{ \frac{1}{2} (\xi + k) \left(\frac{1}{\xi + k} + \frac{1}{\xi - k} \right) e^{i\xi r} \right\}$$
$$= 2\pi i \lim_{\xi \to -k} \left\{ \frac{1}{2} (\xi + k) \left(\frac{1}{\xi + k} \right) e^{i\xi r} \right\}$$
$$= \pi i \lim_{\xi \to -k} e^{i\xi r} = \pi i e^{-ikr}$$

Therefore, we get

$$I_1 = \pi i e^{-ikr} \tag{17}$$

In this alike process, according to Cauchy residue theorem, we know

$$\begin{aligned} I_2 &= -2\pi i \times \left(\text{Residue of lower-half plane at } \xi = k\right) \\ &= 2\pi i \lim_{\xi \to k} \left\{ (\xi - k) \frac{-\xi \mathrm{e}^{-i\xi r}}{\xi^2 - k^2} \right\} \\ &= 2\pi i \lim_{\xi \to k} \left\{ \left(-\frac{1}{2}\right)(\xi - k)\left(\frac{1}{\xi + k} + \frac{1}{\xi - k}\right)\mathrm{e}^{-i\xi r} \right\} \\ &= \pi i \lim_{\xi \to k} \left\{ (\xi - k)\left(\frac{1}{\xi - k}\right)\mathrm{e}^{-i\xi r} \right\} \\ &= \pi i \lim_{\xi \to k} \mathrm{e}^{-i\xi r} = \pi i \mathrm{e}^{-ikr} \end{aligned}$$

Therefore, we have

$$I_2 = \pi i \mathrm{e}^{-ikr} \tag{18}$$

$$\text{Now} \quad I = I_1 + I_2 = \pi i \mathrm{e}^{-ikr} + \pi i \mathrm{e}^{-ikr} = 2\pi i \mathrm{e}^{-ikr}$$

Using this value, we have

$$G(\boldsymbol{x}) = \frac{1}{8\pi^2 ir} 2\pi i \mathrm{e}^{-ikr}$$

$$\Rightarrow G(\boldsymbol{x}) = \frac{1}{4\pi} \frac{\mathrm{e}^{-ikr}}{r}$$

Therefore,

$$G(\boldsymbol{x}) = \frac{1}{4\pi}\left(\frac{\cos kr - i \sin kr}{r}\right) \tag{19}$$

So, we can write from Equations (36) and (39)

$$G(\boldsymbol{x}) = \begin{cases} \dfrac{1}{4\pi}\left(\dfrac{\cos kr + i \sin kr}{r}\right); & I_m(k) > 0 \\ \dfrac{1}{4\pi}\left(\dfrac{\cos kr - i \sin kr}{r}\right); & I_m(k) < 0 \end{cases} \tag{20}$$