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Building Extended Homomorphism on Fuzzy Banach Algebra Based on Jensen Equation with 2*k*-Variables by Fixed Point Methods and Direct Methods

Ly Van An

Faculty of Mathematics Teacher Education, Tay Ninh University, Tay Ninh, Vietnam
Email: lyvanan145@gmail.com, lyvananvietnam@gmail.com

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Abstract

In this paper, I study to expand homomorphisms on fuzzy Banach algebra based on Jensen-type functional equation with 2*k*-variable. First, we study extended homomorphisms on fuzzy Banach algebra with the fixed point method. Next, we study extended homomorphism on fuzzy Banach algebra by direct method. These are the main results of this paper.

Subject Areas

Mathematics

Keywords

General Jensen-Type Additive Function Equation, Fuzzy-Banach Algebras, Fixed Point Method, Direct Method, Hyers-Ulam-Rassias Stability

1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ are two fuzzy normed vector spaces on the same field $\mathbb{K}$, and map $f:\mathbf{X}\to\mathbf{Y}$ be continuously on $\mathbf{X}$. We use the notation $N_{\mathbf{X}}$, N for corresponding the norms on $\mathbf{X}$ and $\mathbf{Y}$. In this paper, we investigate the stability of generalized Jensen-type additive function equation with 2*k*-variables when $\mathbf{X}$ is a fuzzy normed-algebras with norm $N_{\mathbf{X}}$ and $\mathbf{Y}$ is a fuzzy Banach algebras with norm *N*.

In fact, when $\mathbf{X}$ is a fuzzy normed algebras with norm $N_{\mathbf{X}}$ and $\mathbf{Y}$ is a fuzzy Banach algebras with norm *N*, we solve and prove the Hyers-Ulam-Rassias type stability of generalized Jensen-type additive function equation in fuzzy Banach algebras, associated to the Jensen type additive functional equation

$$mf\left(\frac{\alpha\sum_{j=1}^{k}x_j+\alpha\sum_{j=1}^{k}y_j}{m}\right)=\sum_{j=1}^{k}\alpha f\left(x_j\right)+\sum_{j=1}^{k}\alpha f\left(y_j\right) \tag{1}$$

The study of the stability of generalized Jensen-type additive function equation in fuzzy Banach algebras is originated from a question of S. M. Ulam [1], concerning the stability of group homomorphisms.

Let $(\mathbf{G},*)$ be a group and let $(\mathbf{G}',\circ,d)$ be a metric group with metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, there exists a $\delta>0$ such that if $f:\mathbf{G}\to\mathbf{G}'$ satisfies

$$d\left(f\left(x*y\right),f\left(x\right)\circ f\left(y\right)\right)<\delta,\forall x\in\mathbf{G}$$

then there is a homomorphism $h:\mathbf{G}\to\mathbf{G}'$ with

$$d\left(f\left(x\right),h\left(x\right)\right)<\varepsilon,\forall x\in\mathbf{G}$$

Since Hyers' answer to Ulam's question [2], many ideas have arisen from mathematicians who have built theories about space such as the Theory of fuzzy space. It has much progressed in developing the theory of randomness. Some mathematicians have defined fuzzy norms on a vector space from various points of view. Following Bag and Samanta [3] and Cheng and Mordeson [4] gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric was of Kramosil and Michalek type [5] and investigated some properties of fuzzy normed spaces. We use the definition of fuzzy normed spaces given in [3] [6] [7] [8] to investigate a fuzzy version of the Hyers-Ulam stability for the Jensen functional equation in the fuzzy normed algebra setting.

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called a quadratic functional equation. The Hyers-Ulam stability of the quadratic functional equation was proved by Skof [9] for mapping $f:\mathbf{X}\to\mathbf{Y}$, where $\mathbf{X}$ is a normed space and $\mathbf{Y}$ is a Banach space. Cholewa [10] noticed that the theorem of Skof is still true if the relevant domain $\mathbf{X}$ is replaced by an Abelian group. Czerwik [11] proved the Hyers-Ulam stability of the quadratic functional equation.

The stability problems for several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. Such as in 2008, Choonkil Park [12] have established and investigated the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras the following Jensen functional equation

$$2f\left(\frac{x+y}{2}\right)=f\left(x\right)+f\left(y\right)$$

And next in 2009, M. Éhaghi Gordji and M. Bavand Savadkouhi [13] have established and investigated the approximation of generalized stability of homomorphisms in quasi-Banach algebras the following Jensen functional equation

$$rf\left(\frac{x+y}{r}\right)=f\left(x\right)+f\left(y\right)$$

Next, in 2022 Ly Van An [14] have established and investigated the approximation of generalized stability of homomorphisms in quasi-Banach algebras the following Jensen type functional equation

$$mf\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}x_{k+j}}{m}\right)=\sum_{j=1}^{k}f\left(x_j\right)+\sum_{j=1}^{k}f\left(x_{k+j}\right) \quad (2)$$

Recently, Ly Van An continued to conduct extensive research (1.2) in the Hyers-Ulam-Rassias type on fuzzy Banach algebras for the following equation

$$mf\left(\frac{\alpha\sum_{j=1}^{k}x_j+\alpha\sum_{j=1}^{k}y_j}{m}\right)=\sum_{j=1}^{k}\alpha f\left(x_j\right)+\sum_{j=1}^{k}\alpha f\left(y_j\right)$$

i.e., the functional equation with $2k$-variables. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, we will prove that the mappings satisfying the functional (1). Thus, the results in this paper are generalization of those in [12] [13] [14] for functional equation with $2k$-variables.

In this paper, I build a general homomorphism based on Jensen equation with $2k$-variables on fuzzy Banach algebra. This is an extended problem for the field of homotopy research, exploiting unlimited problems of variables to build this problem based on the ideas of mathematicians around the world. See [1]-[30]. Allow me to express my deep gratitude to the mathematicians.

The paper is organized as follows:

In Section 2, we remind some basic notations in [3] [6] [7] [8] [16] [25] [30] such as Fuzzy normed spaces, extended metric space theorem and solutions of the Jensen function equation.

Section 3: Using the fixed point method, establish extended homomorphisms on fuzzy Banach algebra.

Section 4: Using the direct method, establish extended homomorphisms on fuzzy Banach algebra.

2. Preliminaries

2.1. Fuzzy Normed Spaces

Let X be a real vector space. A function $N: X\times R\to[0,1]$ is called a fuzzy norm on X if it stabilities the following conditions: for all $x,y\in X$ and $s,t\in\mathbb{R}$,

1) (N1) $N(x,t)=0$ for $t\le 0$;

2) (N2) $x=0$ if and only if $N(x,t)=1$ for all $t>0$;

3) (N3) $N(cx,t)=N\left(x,\frac{t}{|c|}\right)$ if $c\neq 0$

4) (N4) $N(x+y,s+t)\ge\min\{N(x,s),N(y,t)\}$;

5) (N5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$;

6) (N6) for $x\neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed vector space

1) Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x\in X$ such that

$\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.

2) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in N$ such that for all $n = n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space. We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X. Let X be an algebra and (X, N) a fuzzy normed space.

1) The fuzzy normed space (X, N) is called a fuzzy normed algebra if

$$N(xy, st) \ge N(x, s) \cdot N(y, t)$$

for all $x, y \in X$ and all positive real numbers s and t.

2) A complete fuzzy normed algebra is called a fuzzy Banach algebra.

EXAMPLE

Let $(X, \|\cdot\|)$ be a normed algebra. Let

$$N(x,t) = \begin{cases} \dfrac{t}{t + \|x\|} & t > 0 \\ 0 & t \le 0 \end{cases} \quad x \in X$$

Then $N(x,t)$ is a fuzzy norm on X and $(X, N(x,t))$ is a fuzzy normed algebra. Let (X, N_X) and (Y, N) be fuzzy normed algebras. Then a multiplicative $\mathbb{R}$-linear mapping $H : (X, N_X) \to (Y, N)$ is called a fuzzy algebra homomorphism.

2.2. Extended Metric Space Theorem

Theorem 1. Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n, J^{n+1}) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

1) $d(J^n, J^{n+1}) < \infty, \quad \forall n \ge n_0$;

2) The sequence $\{J^n x\}$ converges to a fixed point y^* of J;

3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^n, J^{n+1}) < \infty\}$;

4) $d(y, y^*) \le \dfrac{1}{1-l} d(y, Jy) \quad \forall y \in Y$

2.3. Solutions of the Equation

The functional equation

$$2f\left(\frac{x+y}{2}\right)=f(x)+f(y)$$

is called the Jensen equation. In particular, every solution of the Jensen equation is said to be a *Jensen-additive mapping*.

2.4. Complete Generalized Metric Space and Solutions of the Inequalities

Theorem 2. Let $(\mathbb{X},d)$ be a complete generalized metric space and let $J:\mathbb{X}\to\mathbb{X}$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in\mathbb{X}$, either

$$d\left(J^n,J^{n+1}\right)=\infty$$

for all nonnegative integers *n* or there exists a positive integer n_0 such that

1) $d\left(J^n,J^{n+1}\right)<\infty,\quad \forall n\geq n_0$;
2) The sequence $\left\{J^n x\right\}$ converges to a fixed point y^* of *J*;
3) y^* is the unique fixed point of *J* in the set $Y=\left\{y\in\mathbb{X}\,|\,d\left(J^n,J^{n+1}\right)<\infty\right\}$;
4) $d\left(y,y^*\right)\leq\frac{1}{1-l}d\left(y,Jy\right)\quad\forall y\in\mathbb{Y}$.

2.5. Solutions of the Inequalities

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping.

3. Using the Fixed Point Method, Establish Extended Homomorphisms on Fuzzy Banach Algebra

Now we study extended homomorphism by fixed point method.

When $\mathbf{X}$ is a fuzzy normed algebra with quasi-norm $N_{\mathbf{X}}$, $\mathbf{Y}$ is a fuzzy Banach algebras with norm *N*. Under this setting, we need to show that the mapping must satisfy (1). These results are given in the following.

Here we assume that $m\geq 2$ is a positive integer and $\alpha\in\mathbb{R}$.

Theorem 3. Suppose $\psi:\mathbf{X}^{2k}\to[0,\infty)$ be a function such that there exists an $L<\frac{1}{m}$

$$\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k\right)\leq\frac{L}{m}\left(mx_1,\cdots,mx_k,my_1,\cdots,my_k\right)\tag{3}$$

for all $x_j,y_j\in\mathbf{X}$ for $j=1\to k$. If $f:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfying $f(0)=0$ and

$$N\left(mf\left(\frac{\sum_{j=1}^{k}\alpha x_j+\sum_{j=1}^{k}\alpha y_j}{m}\right)-\sum_{j=1}^{k}\alpha f\left(x_j\right)-\sum_{j=1}^{k}\alpha f\left(y_j\right),t\right)\geq\frac{t}{t+\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k\right)}\tag{4}$$

$$N\left(f\left(\prod_{j=1}^{k} x_j \cdot y_j\right) - \prod_{j=1}^{k} f(x_j) \cdot \prod_{j=1}^{k} f(y_j), t\right) \geq \frac{t}{t + \psi(x_1, \cdots, x_k, y_1, \cdots, y_k)} \tag{5}$$

for all $x_j, y_j \in \mathbf{X}$ for $j = 1 \to k$, for all $t > 0$ and all $\alpha \in \mathbb{R}$. Then

$$A(x) = N - \lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right)$$

exists for each $x \in \mathbf{X}$ and defines a fuzzy algebras generalized homomorphism $A: \mathbf{X} \to \mathbf{Y}$ such that

$$N(f(x) - A(x), t) \geq \frac{(1-L)t}{(1-L)t + \psi(x_1, \cdots, x_k, y_1, \cdots, y_k)} \tag{6}$$

Proof. Putting $\alpha = 1$.

Replacing $(x_1, \cdots, x_k, y_1, \cdots, y_k)$ by $(x, \cdots, 0, 0, \cdots, 0)$ in hypothesis (4), we have

$$N\left(mf\left(\frac{x}{m}\right) - f(x), t\right) \geq \frac{t}{t + \psi(x, 0, \cdots, 0, 0, \cdots, 0)} \tag{7}$$

for all $x \in \mathbf{X}$.

Now we consider the set

$$\mathbb{M} := \{g : \mathbf{X} \to \mathbf{Y}\}$$

and introduce the generalized metric on $\mathbb{M}$ as follows:

$$d(g,h) := \inf\left\{\beta \in \mathbb{R}_+ : N(g(x) - h(x), \beta t) \geq \frac{t}{t + \varphi(x, 0, \cdots, 0, 0, \cdots, 0)}, \forall x \in \mathbf{X}, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. That has been proven by mathematicians $(\mathbb{M}, d)$ is complete [18] Now we consider the linear mapping $T : \mathbb{M} \to \mathbb{M}$ such that

$$Tg(x) := mg\left(\frac{x}{m}\right)$$

for all $x \in \mathbf{X}$. Let $g, h \in \mathbb{M}$ be given such that $d(g,h) = \varepsilon$ then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, 0, \cdots, 0, 0, \cdots, 0)}, \forall x \in \mathbf{X}, \forall t > 0.$$

Hence

$$\begin{aligned} N(g(x) - h(x), L\varepsilon t) &= N\left(mg\left(\frac{x}{m}\right) - mh\left(\frac{x}{m}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{m}\right) - h\left(\frac{x}{m}\right), \frac{L}{m}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{m}}{\frac{Lt}{m} + \varphi\left(\frac{x}{m}, 0, \cdots, 0, 0, \cdots, 0\right)} \\ &\geq \frac{\frac{Lt}{m}}{\frac{Lt}{m} + \frac{L}{m}\varphi(x, 0, \cdots, 0, 0, \cdots, 0)} \\ &= \frac{t}{t + \varphi(x, 0, \cdots, 0, 0, \cdots, 0)}, \forall x \in \mathbf{X}, \forall t > 0. \end{aligned} \tag{8}$$

So $d(g,h)=\varepsilon$ implies $d(Tg,Th)\le L\cdot\varepsilon$. This means that

$$d(Tg,Th)\le Ld(g,h)$$

for all $g,h\in\mathbb{M}$. On ther hand, (6) implies that $d(f,Tf)\le 1$.

By Theorem 2.5, there exists a mapping $A:\mathbf{X}\to\mathbf{Y}$ satisfying the following:

(1) A is a fixed point of T, *i.e.*,

$$A\left(\frac{x}{m}\right)=\frac{1}{m}A(x) \tag{9}$$

for all $x\in\mathbf{X}$. The mapping A is a unique fixed point T in the set

$$\mathbb{Q}=\{g\in\mathbb{M}:d(f,g)<\infty\}$$

This implies that A is a unique mapping satisfying (9) such that there exists a $\beta\in(0,\infty)$ satisfying.

$$N\left(f(x)-A(x),\beta t\right)\ge\frac{t}{t+\varphi(x,0,\cdots,0,0,\cdots,0)},\forall x\in\mathbf{X}.$$

(2) $d(T^n f,A)\to 0$ as $n\to\infty$. This implies equality

$$N-\lim_{n\to\infty}m^n f\left(\frac{x}{m^n}\right)=A(x)$$

for all $x\in\mathbf{X}$

(3) $d(f,A)\le\frac{1}{1-L}d(f,Tf)$,

which implies the inequality

$$d(f,A)\le\frac{1}{1-L}$$

This implies that the inequality (6)

By (4), I have

$$\begin{aligned}&N\left(m^{p+1}f\left(\frac{\sum_{j=1}^{k}\alpha x_j+\sum_{j=1}^{k}\alpha y_j}{m^{p+1}}\right)-m^n\sum_{j=1}^{k}\alpha f\left(\frac{x}{m^p}\right)-m^p\sum_{j=1}^{k}\alpha f\left(\frac{x}{m^p}\right),m^p t\right)\\&\ge\frac{t}{t+\psi\left(\frac{x_1}{m^p},\cdots,\frac{x_k}{m^p},\frac{y_1}{m^p},\cdots,\frac{y_k}{m^p}\right)}\end{aligned} \tag{10}$$

for all $(x_1,\dots,x_{k,}y_1,\dots,y_k)\in\mathbf{X}$, $\forall t>0$, $\alpha\in\mathbb{R}$. So

$$\begin{aligned}&N\left(m^{p+1}f\left(\frac{\sum_{j=1}^{k}\alpha x_j+\sum_{j=1}^{k}\alpha y_j}{m^{p+1}}\right)-m^n\sum_{j=1}^{k}\alpha f\left(\frac{x}{m^p}\right)-m^p\sum_{j=1}^{k}\alpha f\left(\frac{x}{m^p}\right),m^p t\right)\\&\ge\frac{\frac{t}{m^p}}{\frac{t}{m^p}+\frac{L^p}{m^p}\psi(x_1,\cdots,x_k,y_1,\cdots,y_k)}\end{aligned} \tag{11}$$

for all $(x_1,\dots,x_{k,}y_1,\dots,y_k)\in\mathbf{X}$, $\forall t>0$, $\alpha\in\mathbb{R}$. So

Since

$$\lim_{p\to\infty}\frac{\frac{t}{m^p}}{\frac{t}{m^p}+\frac{L^p}{m^p}\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k\right)}=1$$

for all $\left(x_1,\ldots,x_k,y_1,\ldots,y_k\right)\in\mathbf{X}$, $\forall t>0$, $m\in\mathbb{R}$. So

$$N\left(mA\left(\frac{\sum_{j=1}^{k}\alpha x_j+\sum_{j=1}^{k}\alpha y_j}{m}\right)-\sum_{j=1}^{k}\alpha A\left(x_j\right)-\sum_{j=1}^{k}\alpha A\left(y_j\right),t\right)=1 \tag{12}$$

for all $\left(x_1,\ldots,x_k,y_1,\ldots,y_k\right)\in\mathbf{X}$, $\forall t>0$, $\forall\alpha\in\mathbb{R}$. So we have

$$mA\left(\frac{\sum_{j=1}^{k}\alpha x_j+\sum_{j=1}^{k}\alpha y_j}{m}\right)-\sum_{j=1}^{k}\alpha A\left(x_j\right)-\sum_{j=1}^{k}\alpha A\left(y_j\right)=0$$

for all $\left(x_1,\ldots,x_k,y_1,\ldots,y_k\right)\in\mathbf{X}$, $\forall t>0$, $\alpha\in\mathbb{R}$. So the mapping $A:\mathbf{X}\to\mathbf{Y}$ is additive and $\mathbb{R}$-linear. From (5)

$$\begin{aligned}&N\left(m^{2k}f\left(\prod_{j=1}^{k}\frac{x_j}{m^k}\cdot\frac{y_j}{m^k}\right)-m^k\prod_{j=1}^{k}f\left(\frac{x_j}{m^k}\right)\cdot m^k\prod_{j=1}^{k}f\left(\frac{y_j}{m^k}\right),m^{2k}t\right)\\&\geq\frac{t}{t+\psi\left(\frac{x_1}{m^p},\cdots,\frac{x_k}{m^p},\frac{y_1}{m^p},\cdots,\frac{y_k}{m^p}\right)}\end{aligned} \tag{13}$$

for all $\left(x_1,\ldots,x_k,y_1,\ldots,y_k\right)\in\mathbf{X}$, and all $t>0$ .So

$$\begin{aligned}&N\left(m^{2k}f\left(\frac{\prod_{j=1}^{k}x_j\cdot y_j}{m^{2k}}\right)-m^k\prod_{j=1}^{k}f\left(\frac{x_j}{m^k}\right)\cdot m^k\prod_{j=1}^{k}f\left(\frac{y_j}{m^k}\right),m^{2k}t\right)\\&\geq\frac{\frac{t}{m^{2p}}}{\frac{t}{m^{2p}}+\frac{L^p}{m^p}\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k\right)}\end{aligned} \tag{14}$$

for all $\left(x_1,\ldots,x_k,y_1,\ldots,y_k\right)\in\mathbf{X}$, and all $t>0$. Since

$$\lim_{p\to\infty}\frac{\frac{t}{m^{2p}}}{\frac{t}{m^{2p}}+\frac{L^p}{m^p}\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k\right)}=1$$

for all $\left(x_1,\ldots,x_k,y_1,\ldots,y_k\right)\in\mathbf{X}$, and all $t>0$,

$$N\left(A\left(\prod_{j=1}^{k}x_j\cdot y_j\right)-\prod_{j=1}^{k}A\left(x_j\right)\cdot\prod_{j=1}^{k}A\left(y_j\right),t\right)=1. \tag{15}$$

Thus

$$A\left(\prod_{j=1}^{k}x_j\cdot y_j\right)-\prod_{j=1}^{k}A\left(x_j\right)\cdot\prod_{j=1}^{k}A\left(y_j\right)=0$$

for all $\left(x_1,\ldots,x_k,y_1,\ldots,y_k\right)\in\mathbf{X}$, and all $t>0$. So that mapping $A:\mathbf{X}\to\mathbf{Y}$ is a fuzzy algebra generalized homomorphism, as desired.

□

Theorem 4. Suppose $\psi:\mathbf{X}^{2k}\to[0,\infty)$ be a function such that there exists an $L<\frac{1}{m}$

$$\psi(x_1,\cdots,x_k,y_1,\cdots,y_k)\le mL\left(\frac{x_1}{m},\cdots,\frac{x_k}{m},\frac{y_1}{m},\cdots,\frac{y_k}{m}\right) \tag{16}$$

for all $(x_1,\ldots,x_k,y_1,\ldots,y_k)\in\mathbf{X}$, if $f:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfying $f(0)=0$ and

$$N\left(mf\left(\frac{\sum_{j=1}^{k}\alpha x_j+\sum_{j=1}^{k}\alpha y_j}{m}\right)-\sum_{j=1}^{k}\alpha f(x_j)-\sum_{j=1}^{k}\alpha f(y_j),t\right)\ge\frac{t}{t+\psi(x_1,\cdots,x_k,y_1,\cdots,y_k)} \tag{17}$$

$$N\left(f\left(\prod_{j=1}^{k}x_j\cdot y_j\right)-\prod_{j=1}^{k}f(x_j)\cdot\prod_{j=1}^{k}f(y_j),t\right)\ge\frac{t}{t+\psi(x_1,\cdots,x_k,y_1,\cdots,y_k)} \tag{18}$$

for all $(x_1,\ldots,x_k,y_1,\ldots,y_k)\in\mathbf{X}$, all $t>0$ and all $\alpha\in\mathbb{R}$. Then

$$A(x)=N-\lim_{p\to\infty}m^p f\left(\frac{x}{m^p}\right)$$

exists for each $x\in\mathbf{X}$ and defines a fuzzy algebras generalized homomorphism $A:\mathbf{X}\to\mathbf{Y}$ such that

$$N(f(x)-A(x),t)\ge\frac{(1-L)t}{(1-L)t+L\psi(x,\cdots,0,0,\cdots,0)} \tag{19}$$

for all $x\in\mathbf{X}$ and all $t>0$.

Proof. Let $(\mathbb{M},d)$ be the generalized metric space defined on the proof of Theorem 3. Now we consider the linear mapping $T:\mathbb{M}\to\mathbb{M}$ such that

$$Tg(x):=\frac{1}{m}mg(2x)$$

for all $x\in\mathbf{X}$.

Next putting $\alpha=1$.

Replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k)$ by $(x,\cdots,0,0,\cdots,0)$ in hypothesis (17), we have

$$N\left(mf\left(\frac{x}{m}\right)-f(x),t\right)\ge\frac{t}{t+\psi(x,0,\cdots,0,0,\cdots,0)} \tag{20}$$

for all $x\in\mathbf{X}$, all $t>0$. So

$$N\left(f(x)-\frac{1}{m}f(mx),\frac{t}{m}\right)\ge\frac{t}{t+\psi(mx,0,\cdots,0,0,\cdots,0)}\ge\frac{t}{t+mL\psi(x,0,\cdots,0,0,\cdots,0)} \tag{21}$$

for all $x\in\mathbf{X}$, all $t>0$.

Thus

$$d(f,Tf)\le L.$$

Hence

$$d(f,A) \le \frac{L}{1-L}.$$

which implies that the inequality (19) holds. The rest of the proof is similar to the proof of Theorem 3.

□

4. Using the Direct Method, Establish Extended Homomorphisms on Fuzzy Banach Algebra

Now we study extended homomorphism by direct method.

Where $\mathbf{X}$ is a fuzzy normed algebra with quasi-norm $N_{\mathbf{X}}$, $\mathbf{Y}$ is a fuzzy Banach algebras with norm *N*. Under this setting, we need to show that the mapping must satisfy (1). These results are given in the following.

Here we assume that $m \ge 2$ is a positive integer and $\alpha \in \mathbb{R}$.

Theorem 5. Suppose $\psi : \mathbf{X}^{2k} \to [0,\infty)$ be a function such that

$$\sum_{j=0}^{\infty} m^{2kj}\left(\frac{x_1}{m^j},\cdots,\frac{x_k}{m^j},\frac{y_1}{m^j},\cdots,\frac{y_k}{m^j}\right) < \infty \tag{22}$$

for all $(x_1,\ldots,x_k,y_1,\ldots,y_k) \in \mathbf{X}$, if $f : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfying $f(0) = 0$ and

$$\lim_{t\to\infty} N\left(mf\left(\frac{\sum_{j=1}^{k}\alpha x_j + \sum_{j=1}^{k}\alpha y_j}{m}\right) - \sum_{j=1}^{k}\alpha f(x_j) - \sum_{j=1}^{k}\alpha f(y_j), t\tilde{\psi}(x_1,\cdots,x_k,y_1,\cdots,y_k)\right) = 1 \tag{23}$$

uniformly on $\mathbf{X}^{2k}$ for each $\alpha \in \mathbb{R}$, and

$$\lim_{t\to\infty} N\left(f\left(\prod_{j=1}^{k} x_j \cdot y_j\right) - \prod_{j=1}^{k} f(x_j)\cdot\prod_{j=1}^{k} f(y_j), t\tilde{\psi}(x_1,\cdots,x_k,y_1,\cdots,y_k)\right) = 1 \tag{24}$$

uniformly on $\mathbf{X}^{2k}$, where

$$\tilde{\psi}(x_1,\cdots,x_k,y_1,\cdots,y_k) := \sum_{j=0}^{\infty} m^j \psi\left(\frac{x_1}{m^j},\cdots,\frac{x_k}{m^j},\frac{y_1}{m^j},\cdots,\frac{y_k}{m^j}\right) < \infty \tag{25}$$

for all $(x_1,\ldots,x_k,y_1,\ldots,y_k) \in \mathbf{X}$, then

$$A(x) = N - \lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right)$$

exists for each $x \in \mathbf{X}$ and defines a fuzzy algebras generalized homomorphism $A : \mathbf{X} \to \mathbf{Y}$ such that if for each $\theta > 0, \beta > 0$

$$N\left(mf\left(\frac{\sum_{j=1}^{k}\alpha x_j + \sum_{j=1}^{k}\alpha y_j}{m}\right) - \sum_{j=1}^{k}\alpha f(x_j) - \sum_{j=1}^{k}\alpha f(y_j), \theta\tilde{\psi}(x_1,\cdots,x_k,y_1,\cdots,y_k)\right) \ge \beta \tag{26}$$

for all $(x_1,\dots,x_k,y_1,\dots,y_k)\in\mathbf{X}$, then

$$N\left(f(x)-A(x),\theta\tilde{\psi}(x,0,\cdots,0,0,\cdots,0)\right)\ge\beta \tag{27}$$

for all $x\in\mathbf{X}$.

Furthermore, the fuzzy algebra generalized homomorphism $A:\mathbf{X}\to\mathbf{Y}$ is a unique mapping such that

$$\lim_{t\to\infty}N\left(f(x)-A(x),t\tilde{\psi}(x,0,\cdots,0,0,\cdots,0)\right)=1 \tag{28}$$

uniformly on $\mathbf{X}$.

Proof. We put $\alpha=1$ in (23).With $\varepsilon>0$, by (23), we can exist some $t>0$ such that

$$N\left(mf\left(\frac{\sum_{j=1}^{k}\alpha x_j+\sum_{j=1}^{k}\alpha y_j}{m}\right)-\sum_{j=1}^{k}\alpha f(x_j)-\sum_{j=1}^{k}\alpha f(y_j),t\right)\ge1-\varepsilon \tag{29}$$

for all $t\ge t_0$. Next we replace $(x_1,\cdots,x_k,y_1,\cdots,y_k)$ by $(x,\cdots,0,0,\cdots,0)$ in hypothesis (23), and we have

$$N\left(mf\left(\frac{x}{m}\right)-f(x),t\psi(x,0,\cdots,0,0,\cdots,0)\right)\ge1-\varepsilon \tag{30}$$

for all $x\in\mathbf{X}$. By induction on *n*, we will show that

$$N\left(f(x)-m^nf\left(\frac{x}{m^n}\right),t\sum_{p=0}^{n-1}m^p\psi\left(\frac{x}{m^p},0,\cdots,0,0,\cdots,0\right)\right)\ge1-\varepsilon \tag{31}$$

for all $t\ge t_0$, for all $x\in\mathbf{X}$, all $n\in\mathbb{N}$. It follows from (30) and (31) holds for $n=1$ We now assume that (31) satisfies all $n\in\mathbb{N}$. Then

$$\begin{aligned}&N\left(f(x)-m^nf\left(\frac{x}{m^n}\right),t\sum_{p=0}^{n-1}m^p\psi\left(\frac{x}{m^p},0,\cdots,0,0,\cdots,0\right)\right)\\&\ge\min\left\{N\left(f(x)-m^nf\left(\frac{x}{m^n}\right),t_0\sum_{p=0}^{n-1}m^p\psi\left(\frac{x}{m^p},0,\cdots,0,0,\cdots,0\right)\right),\right.\\&\left.N\left(m^nf\left(\frac{x}{m^n}\right)-m^{n+1}f\left(\frac{x}{m^{n+1}}\right),m^nt_0\psi\left(\frac{x}{m^p},0,\cdots,0,0,\cdots,0\right)\right)\right\}\\&\ge\{1-\varepsilon,1-\varepsilon\}=1-\varepsilon.\end{aligned} \tag{32}$$

This completes the induction argument. Letting $t=t_0$ and we replace *n* and x by q and $\frac{x}{m^n}$ in (31), respectively, we get

$$N\left(m^nf\left(\frac{x}{m^n}\right)-m^{n+q}f\left(\frac{x}{m^{n+q}}\right),m^nt_0\sum_{p=0}^{q-1}m^p\psi\left(\frac{x}{m^{q+p}},0,\cdots,0,0,\cdots,0\right)\right)\ge1-\varepsilon \tag{33}$$

for all $n\ge0,q>0$. It follows from (22) and the equality

$$\sum_{p=0}^{q-1}m^{n+p}\psi\left(\frac{x}{m^{n+p}},0,\cdots,0,0,\cdots,0\right)=\sum_{p=n}^{n+q-1}m^p\psi\left(\frac{x}{m^p},0,\cdots,0,0,\cdots,0\right) \tag{34}$$

That for a given $\theta>0$ there is an $n_0\in\mathbb{N}$ such that

$$t_0\sum_{p=n}^{n+q-1} m^p\psi\left(\frac{x}{m^p},0,\cdots,0,0,\cdots,0\right)<\theta \tag{35}$$

for all $n\ge n_0$ and $q>0$. Now we deuce since (31) that

$$\begin{aligned}&N\left(m^n f\left(\frac{x}{m^n}\right)-m^{n+q} f\left(\frac{x}{m^{n+q}}\right),\theta\right)\\ &\ge N\left(m^n f\left(\frac{x}{m^n}\right)-m^{n+q} f\left(\frac{x}{m^{n+q}}\right),m^n t_0\sum_{p=0}^{q-1} m^p\psi\left(\frac{x}{m^{n+p}},0,\cdots,0,0,\cdots,0\right)\right)\\ &\ge 1-\varepsilon.\end{aligned} \tag{36}$$

for all $n\ge n_0$, all $q>0$ and all $x\in\mathbb{X}$. It follows from (36) that the sequence $\left\{m^n f\left(\frac{x}{m^n}\right)\right\}$ is a Cauchy sequence for all $x\in\mathbb{X}$. Since $\mathbb{Y}$ is a fuzzy complete (fuzzy Banach space), the sequence $\left\{m^n f\left(\frac{x}{m^n}\right)\right\}$ converges. So one can define the mapping $A:\mathbb{X}\to\mathbb{Y}$ by

$$A(x):=N-\lim_{n\to\infty} m^n f\left(\frac{1}{m^n}x\right)\in\mathbb{Y} \tag{37}$$

In other words, for each $t\ge 0$ and $\forall x\in\mathbb{X}$

$$\lim_{n\to\infty} N\left(m^n f\left(\frac{1}{m^n}x\right)-A(x),t\right)=1 \tag{38}$$

Now we are fixed $t>0$ and $0<\varepsilon<1$. Since

$$\lim_{n\to\infty} m^n\psi\left(\frac{x_1}{m^n},\cdots,\frac{x_k}{m^n},\frac{y_1}{m^n},\cdots,\frac{y_k}{m^n}\right)=0,$$

there is an $n'>n_0$ such that

$$t_0 m^n\psi\left(\frac{x_1}{m^n},\cdots,\frac{x_k}{m^n},\frac{y_1}{m^n},\cdots,\frac{y_k}{m^n}\right)\le\frac{t}{m^{2k}},\forall n>n'.$$

Hence for each $p>n'$, we get

$$N\left(f(x)-m^n f\left(\frac{x}{m^n}\right),t\sum_{p=0}^{n-1} m^p\psi\left(\frac{x}{m^p},0,\cdots,0,0,\cdots,0\right)\right)\ge 1-\varepsilon \tag{39}$$

for all $t\ge t_0$, for all $x\in\mathbf{X}$ and for all $n\in\mathbb{N}$. It follows from (30) and (31) holds for $n=1$ We now assume that (31) satisfies all $n\in\mathbb{N}$. Then

$$\begin{aligned}&N\left(A\left(\frac{\sum_{j=1}^k x_j+\sum_{j=1}^k y_j}{m}\right)-\sum_{j=1}^k A(x_j)-\sum_{j=1}^k A(y_j),t\right)\\ &\ge\min\left\{N\left(A\left(\frac{\sum_{j=1}^k x_j+\sum_{j=1}^k y_j}{m}\right)-m^{p+1} f\left(\frac{\sum_{j=1}^k x_j+\sum_{j=1}^k y_j}{m^{p+1}}\right),\frac{t}{m^{2k}}\right),\right.\\ &\quad N\left(A(x_1)-m^n f\left(\frac{x}{m^n}\right),\frac{t}{m^{2k}}\right),\cdots,N\left(A(x_k)-m^n f\left(\frac{x_k}{m^n}\right),\frac{t}{m^{2k}}\right),\\ &\quad N\left(A(y_1)-m^n f\left(\frac{y_1}{m^n}\right),\frac{t}{m^{2k}}\right),\cdots,N\left(A(y_k)-m^n f\left(\frac{y_k}{m^n}\right),\frac{t}{m^{2k}}\right),\end{aligned}$$

$$N\left(m^{p+1}f\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j}{m^{p+1}}\right)-m^p\sum_{j=1}^{k}f\left(\frac{x}{m^p}\right)-m^p\sum_{j=1}^{k}f\left(\frac{x}{m^p}\right),\frac{t}{m^{2k}}\right)\Bigg\}$$

$$\geq N\left(m^{p+1}f\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j}{m^{p+1}}\right)-m^p\sum_{j=1}^{k}f\left(\frac{x}{m^p}\right)\right. \tag{40}$$

$$\left.-m^p\sum_{j=1}^{k}f\left(\frac{x}{m^p}\right),t_0m^n\psi\left(\frac{x_1}{m^n},\cdots,\frac{x_k}{m^n},\frac{y_1}{m^n},\cdots,\frac{y_k}{m^n}\right)\right)$$

$$\geq 1-\varepsilon.$$

for all $t\geq t_0$ and all $x\in\mathbf{X}$.

Thus

$$N\left(mA\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j}{m}\right)-\sum_{j=1}^{k}A(x_j)-\sum_{j=1}^{k}A(y_j),t\right)=1$$

for all $t>0$, by N_2,

$$mA\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j}{m}\right)-\sum_{j=1}^{k}A(x_j)-\sum_{j=1}^{k}A(y_j)=0,\forall x\in\mathbf{X}$$

Hence the mapping $A:\mathbf{X}\to\mathbf{Y}$ is additive.

Next we replace $(x_1,\cdots,x_k,y_1,\cdots,y_k)$ by $(x,\cdots,0,0,\cdots,0)$ in hypothesis (23). $\forall\varepsilon>0$, by (23), then exists $t_0>0$ such that

$$N\left(mf\left(\frac{\sum_{j=1}^{k}\alpha x_j}{m}\right)-\sum_{j=1}^{k}\alpha f(x_j),t\psi(x,\cdots,x,0,\cdots,0)\right)\geq 1-\varepsilon,\forall t\geq t_0. \tag{41}$$

It follows from (41), we have

$$A\left(\sum_{j=1}^{k}\alpha x_j\right)=mA\left(\frac{\sum_{j=1}^{k}\alpha x_j}{m}\right)=\alpha\sum_{j=1}^{k}A(x_j)$$

for all $\alpha\in\mathbb{R}$ and all $x\in\mathbb{X}$.

Similarly, it follows from (24) that

$$f\left(\prod_{j=1}^{k}x_j\cdot y_j\right)=\prod_{j=1}^{k}f(x_j)\cdot\prod_{j=1}^{k}f(y_j)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k\in\mathbf{X}$. We now assume that $\forall\theta>0$ and $\beta>0$ satisfied (26). Put

$$\psi_n(x_1,\cdots,x_k,y_1,\cdots,y_k)=\sum_{j=0}^{q-1}m^j\psi\left(\frac{x_1}{m^j},\cdots,\frac{x_k}{m^j},\frac{y_1}{m^j},\cdots,\frac{y_k}{m^j}\right) \tag{42}$$

for all $(x_1,\ldots,x_k,y_1,\ldots,y_k)\in\mathbf{X}$.

Suppose by the same reasoning as in the beginning of the proof, one can deuce from (26) that

$$N\left(f(x)-m^n f\left(\frac{x}{m^n}\right),\theta\sum_{p=0}^{n-1} m^{n-p},\theta\tilde{\psi}\left(\frac{x}{m^p},0,\cdots,0,0,\cdots,0\right)\right)\geq\beta \tag{43}$$

for all $(x_1,\ldots,x_k,y_1,\ldots,y_k)\in\mathbf{X}$, then for all positive integer n. Suppose $t>0$ we have

$$\begin{aligned}&N\left(f(x)-A(x),\theta\psi_n\left(\frac{x}{m^n},0,\cdots,0,0,\cdots,0\right)+t\right)\\&\geq\min\left\{N\left(f(x)-m^n f\left(\frac{x}{m^n}\right),\theta\psi_n\left(\frac{x}{m^n},0,\cdots,0,0,\cdots,0\right)\right),\right.\\&\left.N\left(m^n f\left(\frac{x}{m^n}\right)-A(x),t\right)\right\}\end{aligned} \tag{44}$$

Combining (43) and (44). If $f:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfying $f(0)=0$ and the fact that $\lim_{n\to\infty}N\left(m^n f\left(\frac{x}{m^n}\right)-A(x),t\right)=1$, we observe that

$$N\left(f(x)-A(x),\theta\psi_n(x,0,\cdots,0,0,\cdots,0)+t\right)\geq\alpha$$

For large enough $n\in\mathbb{N}$. Thanks to the continuity of the function

$$N\left(f(x)-A(x),\cdot\right),$$

we see that

$$N\left(f(x)-A(x),\theta\tilde{\psi}_n(x,0,\cdots,0,0,\cdots,0)+t\right)\geq\alpha$$

Now I give $t\to0$, we conclude that

$$N\left(f(x)-A(x),\theta\tilde{\psi}_n(x,0,\cdots,0,0,\cdots,0)+t\right)\geq\alpha$$

In the end I still have to prove the uniqueness. Suppose A' be another additive mapping satisfying (27) and (28). Fix $\eta>0$. Given $\varepsilon>0$, follow (28) for A, and A', then exist $t_0>0$ such that

$$N\left(f(x)-A(x),t\tilde{\psi}_n(x,0,\cdots,0,0,\cdots,0)\right)\geq1-\varepsilon$$

$$N\left(f(x)-A'(x),t\tilde{\psi}_n(x,0,\cdots,0,0,\cdots,0)\right)\geq1-\varepsilon$$

for all $x\in\mathbf{X}$ and $\forall t\geq t_0$. With fixed $x\in\mathbf{X}$ then exists $n_0\in\mathbb{N}$ such that

$$t_0\sum_{j=0}^{\infty}m^p\psi\left(\frac{x}{m^p},0,\cdots,0,0,\cdots,0\right)<\frac{\eta}{m}$$

for all $n\geq n_0$. From

$$\begin{aligned}&\sum_{j=0}^{\infty}m^p\psi\left(\frac{x}{m^p},0,\cdots,0,0,\cdots,0\right)\\&=m^n\sum_{j=0}^{\infty}m^{p-n}\psi\left(\frac{x}{m^{n-p}},0,\cdots,0,0,\cdots,0\right)\\&=m^n\sum_{j=0}^{\infty}m^i\psi\left(\frac{1}{m^i}\frac{x}{m^n},0,\cdots,0,0,\cdots,0\right)\\&=m^n\tilde{\psi}\left(m^{-i}\frac{x}{m^n},0,\cdots,0,0,\cdots,0\right)\end{aligned} \tag{45}$$

$$
\begin{aligned}
&N\left(A(x)-A'(x),\eta\right)\\
&\geq \min\left\{N\left(m^n f\left(\frac{x}{m^n}\right)-A(x),\frac{\eta}{m}\right),N\left(A'(x)-m^n f\left(\frac{x}{m^n}\right),\frac{\eta}{m}\right)\right\}\\
&=\min\left\{N\left(f\left(\frac{x}{m^n}\right)-A\left(\frac{x}{m^n}\right),\frac{\eta}{m^{n+1}}\right),N\left(A'\left(\frac{x}{m^{n+1}}\right)-f\left(\frac{x}{m^n}\right),\frac{\eta}{m}\right)\right\}\\
&\geq \min\left\{N\left(f(\frac{x}{m^n})-A\left(\frac{x}{m^n}\right),m^{-n}t_0\sum_{j=0}^{\infty}m^p\psi\left(\frac{x}{m^p},0,\cdots,0,0,\cdots,0\right)\right),\right.\\
&\quad\left.N\left(A\left(\frac{x}{m^n}\right)-f\left(\frac{x}{m^n}\right),m^{-n}t_0\sum_{j=0}^{\infty}m^p\psi\left(\frac{x}{m^p},0,\cdots,0,0,\cdots,0\right)\right)\right\},\\
&=\min\left\{N\left(f\left(\frac{x}{m^n}\right)-A\left(\frac{x}{m^n}\right),t_0\tilde{\psi}\left(\frac{x}{m^p},0,\cdots,0,0,\cdots,0\right)\right),\right.\\
&\quad\left.N\left(A'\left(\frac{x}{m^{n+1}}\right)-f\left(\frac{x}{m^n}\right),t_0\tilde{\psi}\left(\frac{x}{m^p},0,\cdots,0,0,\cdots,0\right)\right)\right\} \qquad (46)\\
&\geq 1-\varepsilon
\end{aligned}
$$

It follows that $N\left(A(x)-A'(x),\eta\right)=1$ for all $\eta>0$. So $A(x)=A'(x)$, $\forall x\in\mathbf{X}$. □

Theorem 6. Suppose $\psi:\mathbf{X}^{2k}\to[0,\infty)$ be a function such that

$$\sum_{j=0}^{\infty}m^{-2kj}\left(x_1m^jx_1,\cdots,m^jx_k,m^jy_1,\cdots,m^jy_k\right)<\infty \qquad (47)$$

for all $(x_1,\ldots,x_{k,}y_1,\ldots,y_k)\in\mathbf{X}$. If $f:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfying $f(0)=0$ and

$$
\begin{aligned}
&\lim_{t\to\infty}N\left(mf\left(\frac{\sum_{j=1}^{k}\alpha x_j+\sum_{j=1}^{k}\alpha y_j}{m}\right)\right.\\
&\left.-\sum_{j=1}^{k}\alpha f(x_j)-\sum_{j=1}^{k}\alpha f(y_j),t\tilde{\psi}(x_1,\cdots,x_k,y_1,\cdots,y_k)\right)=1
\end{aligned} \qquad (48)
$$

uniformly on $\mathbf{X}^{2k}$ for each $\alpha\in\mathbb{R}$, and

$$\lim_{t\to\infty}N\left(f\left(\prod_{j=1}^{k}x_j\cdot y_j\right)-\prod_{j=1}^{k}f(x_j)\cdot\prod_{j=1}^{k}f(y_j),t\tilde{\psi}(x_1,\cdots,x_k,y_1,\cdots,y_k)\right)=1 \qquad (49)$$

uniformly on $\mathbf{X}^{2k}$, where

$$\tilde{\psi}(x_1,\cdots,x_k,y_1,\cdots,y_k):=\sum_{j=0}^{\infty}m^{-j}\psi\left(m^jx_1,\cdots,m^jx_k,m^jy_1,\cdots,m^jy_k\right)<\infty \qquad (50)$$

for all $(x_1,\ldots,x_{k,}y_1,\ldots,y_k)\in\mathbf{X}$. Then

$$A(x)=N-\lim_{n\to\infty}m^{-n}f\left(m^jx\right)$$

exists for each $x\in\mathbf{X}$ and defines a fuzzy algebras generalized homomorphism $A:\mathbf{X}\to\mathbf{Y}$ such that if for each $\theta>0,\beta>0$

$$N\left(mf\left(\frac{\sum_{j=1}^{k}\alpha x_j+\sum_{j=1}^{k}\alpha y_j}{m}\right)-\sum_{j=1}^{k}\alpha f(x_j)-\sum_{j=1}^{k}\alpha f(y_j),\theta\tilde{\psi}(x_1,\cdots,x_k,y_1,\cdots,y_k)\right)\geq\beta \qquad (51)$$

for all $(x_1,\ldots,x_k,y_1,\ldots,y_k)\in\mathbf{X}$, then

$$N\left(f(x)-A(x),\theta\tilde{\psi}(x,0,\cdots,0,0,\cdots,0)\right)\geq\beta \tag{52}$$

for all $x\in\mathbf{X}$.

Furthermore, the fuzzy algebra generalized homomorphism $A:\mathbf{X}\to\mathbf{Y}$ is a unique mapping such that

$$\lim_{t\to\infty}N\left(f(x)-A(x),\theta\tilde{\psi}(x,0,\cdots,0,0,\cdots,0)\right)=1 \tag{53}$$

uniformly on $\mathbf{X}$.

5. Conclusion

In this paper, I built the existence of extended homomorphism on fuzzy Banach algebra based on Jensen equation $2k$ variables by two methods such as fixed point and direct method to check.

Conflicts of Interest

The author declares no conflicts of interest.

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