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Nuclear Binding Energy (B) Calculation without Quantum Mechanical Correction

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Abstract

A Yukawa-type potential was used to predict B for the current nuclides using a set of 23 nuclides as the source of the experimental values. The deviation between measurement and prediction was remarkably reduced by a special search on the average distance between nucleons. The best results were obtained for a 2-exponential model. The predictions of B inside and outside the range of A and Z used to do the fitting were carried out by fitting the obtained values of the average distance between nucleons to a 2-term power law that depends on A and Z. An equation was obtained that shows the potential existence of up to 3 exponential terms if the mass difference between the neutron and the proton is considered and if distinctions are made among the type of nucleon-nucleon interactions.

Subject Areas

Nuclear Physics, Nuclear Technology

Keywords

Nuclear Binding Energy, Yukawa Nuclear Potential, Nuclear Mass, Nuclear Radius

1. Introduction

The concept of nuclear binding energy was defined after the discovery of the mass defect by Aston [1] as referenced in Evans [2] and the so called by him "failure of the additive law with regards to mass". Ashton used the packing fraction equation to express the mass defect:

$$P = (M - A)/A$$

where,

P: Packing fraction;

A: Mass number;

M: Mass of the neutral atom.

At Ashton's publication time, the components of the atomic nucleus were not known. Later on, when the constituent of the atomic nucleus were in solid grounds, the concept of binding energy (B) was introduced [2]:

$$B = ZM_H + NM_N - M$$

where

Z: Atomic number;

M_H: Mass of the hydrogen atom;

N: Number of neutrons;

M_N: Mass of the neutron.

Note in equation above that the right hand side needs to be multiplied by c^2 to have the units of energy.

The prediction of B by a theory has been and still is a matter of great importance. Yukawa in 1934 [3] proposes an equation for B between the proton and the neutron. In 1935, Weizsacker [4] developed a model for the binding energy in similarity to a liquid drop tension. Later on, with the addition of some quantum mechanical correction, the liquid drop model improved significantly and is currently known as the semi-empirical mass formula [2].

The purpose of this work is to see if the binding energy of the nuclides could be predicted using Yukawa-type models with a discreet distribution of nucleons without using quantum mechanical corrections.

In this work, a Yukawa-type potential is developed to predict B for the current nuclides using a set of 23 nuclides as the source of the experimental values. The discrepancy between measurement and prediction is remarkably reduced by a special search on the average distance among nucleons. Up to this moment, the best results were obtained for a 2-exponential model.

2. Average Nuclear Binding Energy

The electrostatic potential of an assembly of n point charges can be written as [5]

$$U = G_e \frac{1}{2} \sum_{i=1}^{n} q_i \sum_{j=1, j\neq i}^{n} \frac{q_j}{R_{i.j}}$$

where,

G_e : Coulomb constant;

q_i : Point charge;

$R_{i.j}$: Distance between the charges.

The 1/2 factor in that equation is used to avoid double counting (*i.e.*, $q_i q_j$ and $q_j q_i$ appear both in the sums, therefore double counting the number of terms). Total number of terms: $N = \left(n^2 - n\right)/2$. Note that one isolated charge can be moved without any electrical inertia.

The electrostatic potential energy of the nucleus assuming point charge pro-

tons and using an average distance between any 2 protons, can then be expressed as

$$U_e \approx G_e e^2 \frac{z^2 - z.}{2R_z} \tag{1}$$

where,

e : Proton positive charge;

z : Atomic number (Number of protons in the nucleus);

R_z : Average distance among protons in the nucleus.

Note that Equation (1) tacitly assumes that the protons are at a fixed location (or slightly moving around an equilibrium point) inside the nucleus.

Using a similar approach to obtain Equation (1), the nucleo-static potential energy (using a Yukawa-type potential to consider the short range of the nuclear force and assuming the same mass for the protons and neutrons), can be written as

$$U_n \approx G_n m^2 \frac{\mathrm{e}^{-R_n/\lambda}}{2R_n}\left(A^2 - A\right) \tag{2}$$

where,

G_n : Nuclear force constant;

m : Nucleon mass;

R_n : Average distance among nucleons in the nucleus;

A : Number of nucleons in the nucleus.

The binding energy (B) that keeps the nucleons together forming the atomic nucleus can be written as

$$B = \left(Zm_H + \left(A - Z\right)m_n - M\left(Z, A\right)\right)c^2 \tag{3}$$

where,

m_H : Mass of the hydrogen atom;

$A - Z$: Number of neutrons inside the nucleus;

m_n : Neutron mass;

M : Mass of the neutral atom;

c : Speed of light in vacuum.

Equation (3) provides a big set of experimental values of B because M has been measured for up to $Z = 118$ and $A = 294$.

An expression for the prediction of B can be obtained from first principles, by adding Equation (1) and (2):

$$B_c = -G_e \mathrm{e}^2 \frac{z^2 - z.}{2R_z} + G_n m^2 \frac{\mathrm{e}^{-R_n/\lambda}}{2R_n}\left(A^2 - A\right) \tag{4}$$

The sign of the electric potential energy was taken negative to express the repulsive character of the electric force. Because of the lack of knowledge of the location of the protons and neutrons inside the nucleus, it will be assumed that $R_z = R_n = R = f\left(Z, A\right)$. Note that unlike here, the liquid drop model assumes $R = R_o A^{1/3}$, where R, is the nuclear radius.

The binding energy per nucleon (Average B) is usually used to better illustrate the stability of the nucleus. So from Equation (4):

$$\bar{B}_c = -G_e e^2 \frac{z^2 - z.}{2RA} + G_n m^2 \frac{e^{-R_n/\lambda}}{2R} \frac{A^2 - A}{A} \tag{5}$$

The unknowns in Equation (5) are G_n, λ and $R = f(z, A)$.

3. Computational Results and Analysis

As a first approximation, R is considered known, $R = A^{1/3}$.

The Least Square equation is used to obtain $G_n m^2$ and λ:

$$Q = \text{Minimum}\left(\sum_i (y_i - f_i)^2 = f_Q\right) \tag{6}$$

where,

$$y_i = B_i / A_i + G_e e^2 \frac{z_i^2 - z._i}{2R_i A_i} \quad \text{(From Equation (3) and (5))} \tag{7}$$

$$f_i = p_1 e^{p_2 R_i} \frac{A_i^2 - A_i}{2R_i A_i} \quad \text{(From Equation (5))} \tag{8}$$

p_1, p_2: Fitting parameters.

Table 1 shows f_i calculation for a set of 23 nuclides. The relative errors are between 1 and 11% (except for H2 and He4) and do not show a clear trend. The fitted parameters (FPs) are: $p_1 = 8.481846$ (Mev-fm) and $p_2 = -0.43684$ (fm^{-1}) which corresponds to $\lambda = 2.3$ fm. Some statistical parameters (SP) are:

$$Q = 15.3, \quad D = \sum_i |y_i - f_i| = 11.6, \quad R_*^2 = 1 - Q / \sum (y_i - \bar{y})^2 = 0.92$$

(Coefficient of Determination)

Even though R_* is close to 1, Q and D are still large (note the significant contribution of H2, He4 and Bk).

In order to improve D and therefore Q another exponential term was added making

$$f_i = p_1 e^{p_2 R_i} \frac{A_i^2 - A_i}{2R_i A_i} + p_3 e^{p_4 R_i} \frac{A_i^2 - A_i}{2R_i A_i} \tag{9}$$

The FPs are: $p_1 = 11.34174$, $p_3 = 1.496913$ and $p_2 = -0.70269$, $p_{4-} = -0.18515$ which corresponds to $\lambda_2 = 1.4$, and $\lambda_4 = 5.4$. The SPs are: Q = 10.6, D = 7.4, and $R_* = 0.95$. Even though the SPs were significantly improved, Table 1 shows that for light nuclides (H2, H3, He3, He4, and Li7) the deviations from experimental results are still large or larger. For $A > 7$ however the deviation decreased significantly (except for F19 and Ne23).

A third exponential term was briefly tried but no convergence was achieved.

Currently it is believed that when the nucleons are very close together a repulsion force arises, to consider that, the following equation was used:

$$f_i = p_1 e^{p_2 R_i} \frac{A_i^2 - A_i}{2(R_i + \text{Ln}R_i) A_i} \tag{10}$$

Table 1. Binding energy fitting for 3 exponential models.

Atom	Z	A	r (fm)	B (Mev)									
				Ref*	1 Exp	%	2Exp	%	1 Exp-Log	%	Ref**	LDMu	%
H (D)	1	2	1.260	1.113	1.941	−74.5	2.327	−109	2.261	−103	1.113	3.782	−239.9
H (T)	1	3	1.442	2.828	3.132	−10.7	3.649	−29	3.436	−22	2.828	5.086	−79.8
He	2	3	1.442	2.907	3.132	−7.8	3.649	−25.5	3.436	−18	2.849	5.086	−78.5
He	2	4	1.587	7.300	4.006	45.1	4.567	37.4	4.262	41.6	7.261	5.911	18.6
Li	3	7	1.913	5.929	5.768	2.7	6.285	−6	5.896	0.6	5.873	7.304	−24.4
Be	4	9	2.080	6.923	6.574	5	7.015	−1.3	6.644	4	6.843	7.85	−14.7
B	5	11	2.224	7.517	7.218	4	7.573	−0.7	7.244	3.6	7.414	8.255	−11.3
C	6	13	2.351	8.177	7.749	5.2	8.017	2	7.743	5.3	8.054	8.571	−6.4
N	7	15	2.466	8.517	8.197	3.8	8.381	1.6	8.167	4.1	8.375	8.829	−5.4
O	8	17	2.571	8.672	8.583	1	8.687	−0.2	8.533	1.6	8.512	9.044	−6.2
F	9	19	2.668	8.800	8.917	−1.3	8.946	−1.7	8.853	−0.6	8.623	9.228	−7
Ne	10	23	2.844	8.946	9.472	−5.9	9.367	−4.7	9.388	−5	8.774	9.529	−8.6
Mg	12	25	2.924	9.524	9.704	−1.9	9.539	−0.2	9.614	−0.9	9.299	9.654	−3.8
AL	13	27	3.000	9.719	9.912	−2	9.692	0.3	9.818	−1	9.478	9.767	−3
Si	14	29	3.072	9.921	10.099	−1.8	9.829	0.9	10.002	−0.8	9.665	9.869	−2.1
S	16	33	3.208	10.131	10.421	−2.9	10.063	0.7	10.321	−1.9	9.848	10.047	−2
Mn	25	55	3.803	10.825	11.435	−5.6	10.819	0.1	11.355	−4.9	10.466	10.682	−2.1
Cu	29	65	4.021	10.995	11.656	−6	11.011	−0.1	11.591	−5.4	10.606	10.867	−2.5
Pd	46	105	4.718	11.579	11.905	−2.8	11.43	1.3	11.905	−2.8	11.057	11.344	−2.6
I	53	127	5.027	11.550	11.829	−2.4	11.555	0	11.858	−2.7	11.01	11.514	−4.6
Xe	54	131	5.079	11.520	11.806	−2.5	11.573	−0.5	11.84	−2.8	10.982	11.54	−5.1
Pt	78	195	5.799	11.752	11.266	4.1	11.782	−0.3	11.357	3.4	11.088	11.858	−6.9
Bk	97	245	6.257	11.890	10.748	9.6	11.886	0	10.867	8.6	11.131	12.022	−8

*Coulomb discrete repulsion; **Coulomb homogeneous repulsion.

The FPs are: $p_1 = 11.84844$ and $p_2 = -0.44744$ ($\lambda = 2.2$). The SPs are: $Q =$ 14.1, $D =$ 11.3, and $R_* = 0.93$. From Table 1, it can be seen that Equation (10) does not significantly improve the results for light elements and get larger deviations for heavier nuclides. A 2nd exponential term was briefly tried but no convergence was achieved.

The Gauss-Newton method combined with the Levenverg-Marquardt method (GN-LM) was used to solve Equation (6).

Numerical derivatives were used in the gradient and Hessian calculations.

A Brute Force (BF) method that does not use derivatives was set up, in some cases, to obtain the initial values of the fitting parameters in the following way:

Step 1: Make an initial guess of the range where the value of each FP is

Step 2: Vary each FP (one at a time) from the smallest FP to the largest FP and determine the FP that yields the minimum f_Q.

Step 3: Repeat step 2 until convergence is considered achieved.

Table 1 also shows the results for a Liquid Drop Model without quantum mechanical corrections (LDMu). The equation for that model is:

$$y_i = B_i/A_i + p_c \frac{z_i^2 - z_{\cdot i}}{A_i^{4/3}} \tag{11}$$

$$f_i = p_v - p_s A_i^{-1/3}, \quad p_c = 0.595, \quad p_v = 14.1, \quad p_s = 13.0$$

All coefficients and functions were taken from Ref. [2].

The electric component in Equation (11) includes the z_i term which becomes important for light nuclides. Equation (11) assumes that the electrical and the nuclear "charge" are continuously and homogeneously distributed inside the nucleus unlike in this work where a discreet distribution of nucleons is used. Note in Table 1 that the LDMu yields results significantly larger than the model described by Equation (9) (except for He4)

It is believed that the lack of a precise knowledge of the distribution of the nucleons insides the nucleus is responsible for the significant deviations (shown in Table 1) of the calculated binding energies from the experimental values. That's why a double optimization of f_Q was attempted using the following algorithm:

Step 1: Set an initial set of values of R ($R_i = A_i^{1/3}$).

Step 2: For each nuclide *i*, make a R_i search that makes $\partial f_Q/\partial R \approx 0$.

Note that for each of the 23 nuclides a R_i search is made which involves as many Q computations as needed to achieve $\partial f_Q/\partial R \approx 0$. Once a search of R for a nuclide is made, R is not changed when making the search for the next nuclide.

The Newton-Raphlson method was used in this search (e.g.

$R_{i.k} = R_{i.k-1} - \left(\partial f_Q/\partial R_i\right)_{k-1} \Big/ \left(\partial^2 f_Q/\partial R_i^2\right)_{k-1}$)

Numerical derivatives were used in the R calculation. For some reason the GN-LM method did not converge, the BF however converged.

Table 2 shows the results for the R search.

The one-exponential model yields a relative deviation of 0.7% or less (the search did not converged for $A > 55$). The obtained FPs are: $p_1 = 8.482556$ and $p_2 = -0.43616$ which differ from the non-optimized R results only on the 3rd and 4th mantissa digits respectively. The SPs at the $A = 55$ search are: $Q = 2.2$, $D = 3.7$, and $R_* = 0.99$. It is remarkable that even though the FPs practically did not change, the SPs improve significantly just because of the R search.

The R search for the two-exponential model converged for all nuclides and yielded a relative deviation of 0.1% or less. The FPs are: $p_1 = 11.34169$ Mev-fm, $p_3 = 1.496876$ Mev-fm and $p_2 = -0.70274$ ($\lambda = 1.42$ fm), $p_{4-} = -0.18518$

($\lambda = 5.40$ fm).

The SPs are: $Q = 4.99\text{E}{-4}$, $D = 7.83\text{E}{-2}$, and $R_* = 0.99999749$. It is quite remarkable the huge improvement in SPs while the FPs are about the same.

The one-exponential-log model did not converged for the few attempts made.

Yukawa obtained an equation for the mass of the particle mediating the force between the proton and the neutron, using that equation regardless of the type of interaction, it is obtained:

$$m_2 = -p_2\bar{h}/c = 203 \text{ Mev}, \quad m_3 = -p_3\bar{h}/c = 53.6 \text{ Mev}$$

Value of $\lambda = 1.4$ has been referenced as a measured result in [6]. Values of $g^2(p_1, p_3)$ have been determined and referenced in Ref. [6]: $g^2 = 20,60$ Mev-fm.

Table 2. Binding energy fitting combined with R search.

Atom	Z	A	r (fm)		B (Mev)		r (fm)		B (Mev)	
			1Exp	Ref*	1Exp	%	2Exp	Ref*	2Exp	%
H (D)	1	2	1.764	1.113	1.114	−0.1	1.858	1.113	1.11	0
H (T)	1	3	1.534	2.828	2.831	−0.1	1.649	2.828	2.83	0
He	2	3	1.515	2.89	2.893	−0.1	1.638	2.867	2.87	0
He	2	4	1.074	7.409	7.419	−0.1	1.221	7.368	7.37	0
Li	3	7	1.882	5.934	5.95	−0.3	1.968	5.92	5.92	0
Be	4	9	2.019	6.938	6.968	−0.4	2.093	6.921	6.92	0
B	5	11	2.172	7.531	7.573	−0.6	2.232	7.515	7.51	0
C	6	13	2.283	8.198	8.238	−0.5	2.33	8.183	8.18	0
N	7	15	2.416	8.534	8.567	−0.4	2.448	8.523	8.53	0
O	8	17	2.553	8.679	8.727	−0.6	2.573	8.672	8.67	0
F	9	19	2.679	8.796	8.857	−0.7	2.688	8.793	8.79	0
Ne	10	23	2.916	8.921	8.966	−0.5	2.901	8.926	8.92	0.1
Mg	12	25	2.944	9.515	9.576	−0.6	2.927	9.523	9.52	0.1
AL	13	27	3.024	9.708	9.748	−0.4	2.997	9.72	9.71	0.1
Si	14	29	3.094	9.91	9.952	−0.4	3.061	9.926	9.92	0.1
S	16	33	3.25	10.11	10.119	−0.1	3.199	10.14	10.1	0
Mn	25	55	3.892	10.777	10.776	0	3.803	10.83	10.8	0.1
Cu	29	65					4.024	10.99	11	0.1
Pd	46	105					4.693	11.6	11.6	−0.1
I	53	127					5.025	11.55	11.6	−0.1
Xe	54	131					5.087	11.52	11.5	0
Pt	78	195					5.804	11.75	11.7	0
Bk	97	245					6.256	11.89	11.9	0

Table 3 shows the results of the binding energy calculation for the 2-exponential model with the electrostatics term incorporated. So the references now are the results from Equation (3) as usual and the model is Equation (9) plus the electrostatic repulsion term. Note in that table that He4 has a minimum on R and a local maximum on $\overline{B}_c$ as happens in the measured binding energy ($\overline{B}$) definition Figure 1 shows $\overline{B}_c$ vs. A. Note the local maximum of He4 that looks like an outlier but experimental results keep confirming this behavior of He4. Figure 2 shows the profile R vs. A.

Table 3. Binding energy fitting and R search with the repulsion term as part of the model. Here on the reference is the usual measured value.

Measured vs. 2Exp calculation						
Atom	Z	A	r (fm)		B (Mev)	
			2Exp	Meas.	2Exp	%
H (D)	1	2	1.858	1.113	1.112	0
H (T)	1	3	1.649	2.828	2.827	0
He	2	3	1.638	2.574	2.572	0
He	2	4	1.221	7.074	7.075	0
Li	3	7	1.968	5.606	5.607	0
Be	4	9	2.093	6.462	6.462	0
B	5	11	2.232	6.928	6.927	0
C	6	13	2.33	7.47	7.471	0
N	7	15	2.448	7.7	7.702	0
O	8	17	2.573	7.75	7.752	0
F	9	19	2.688	7.778	7.778	0
Ne	10	23	2.901	7.955	7.947	−0.1
Mg	12	25	2.927	8.224	8.216	−0.1
AL	13	27	2.997	8.332	8.324	−0.1
Si	14	29	3.061	8.45	8.444	−0.1
S	16	33	3.199	8.499	8.494	−0.1
Mn	25	55	3.803	8.76	8.754	−0.1
Cu	29	65	4.024	8.758	8.751	−0.1
Pd	46	105	4.693	8.57	8.578	0.1
I	53	127	5.025	8.441	8.45	0.1
Xe	54	131	5.087	8.423	8.426	0
Pt	78	195	5.804	7.928	7.926	0
Bk	97	245	6.256	7.517	7.517	0

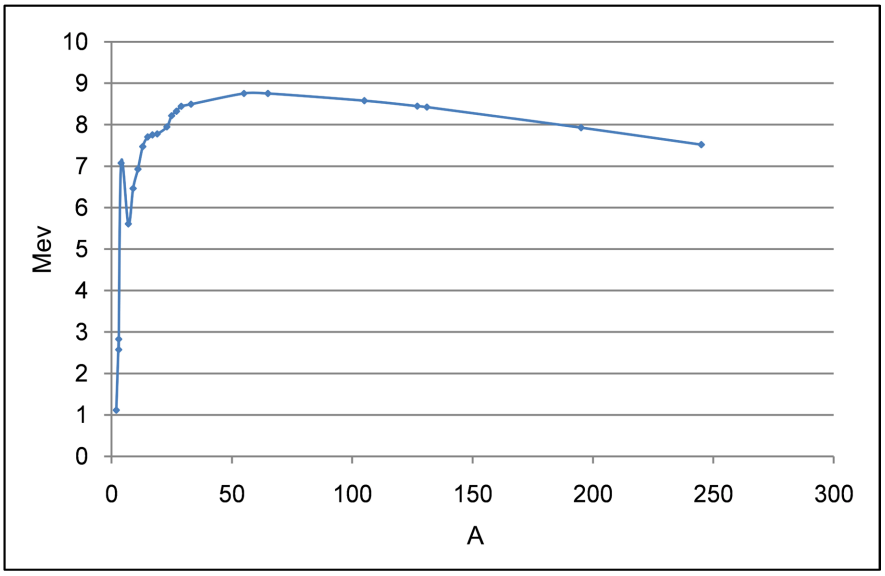

Figure 1. Calculated binding energy per nucleon (Mev) vs. the atomic mass number (A). The deviations from the measured values are very small due to the *R* search.

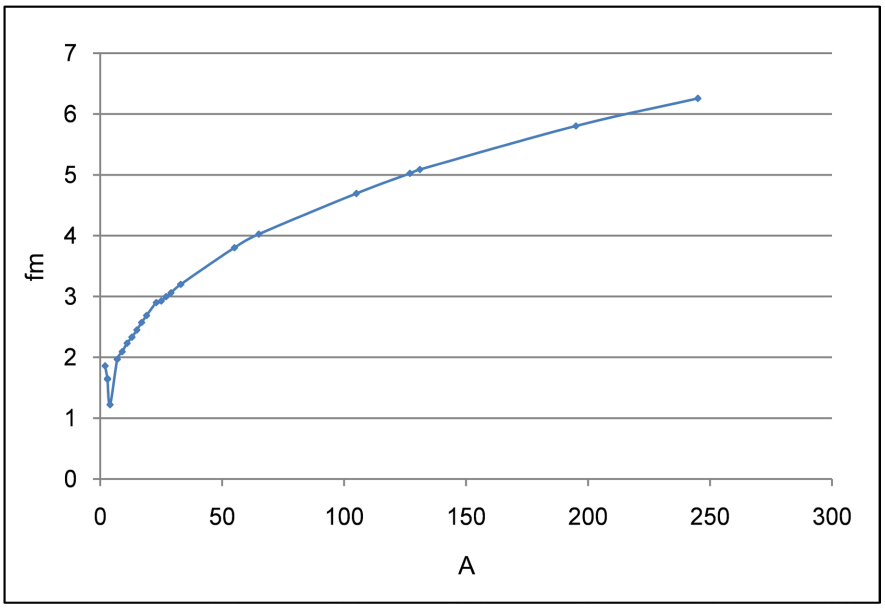

Figure 2. Average distance among nucleons (fm) vs. Atomic mass number (A).

In order to predict the binding energy of nuclides that were not used in **Table 3**, an equation for the average separation between nucleons (R) is needed. A two independent-variable spline interpolation scheme could be used to interpolate and extrapolate R with a potential for high precision. An algorithm for that interpolation scheme is not ready yet.

A fitting to the values of R from **Table 3** was performed for $A > 4$ (no equation that accurately describes R for all nuclides has been found yet) using the following equation:

$$f_i = p_1 A_i^{p_2} + p_3 z_i^{p_4} \tag{12}$$

The FPs are: $p_1 = 1.142888$, $p_2 = 0.366691$, $p_3 = -0.224360$, and $p_4 = 0.512880$. The SPs are $Q = 4.3\text{E}{-3}$, $D = 0.22$, and $R_* = 0.99996$. Note that the sign of p_3 is negative this could indicate that the 1st term over predicts the optimized value of R.

Table 4 shows the results of the R fitting, which does not reproduce exactly the R of Table 3 as expected in the case of an interpolation, but the relative error is less than 1.6%. That table also shows $\overline{B}$ (measured) and $\overline{B}_c$.

Note that the relative deviations between $\overline{B}$ and $\overline{B}_c$ appear to be an amplification by a factor of about 2 or greater of the R relative deviation. That indicates the importance of using an accurate value of R.

Table 4. r fit to the R searched of Table 3. $\overline{B}_c$ for the 2-Exp model using the fitted r.

Reproduction w/radius fit										
Atom	Z	A	r (fm)			B (Mev)				%
			Calc	Fit	%	Meas.	2Exp	%	LDMc	
Li	3	7	1.968	1.94	1.5	5.793	5.606	3.3	6.65	18.6
Be	4	9	2.093	2.1	−0.4	6.405	6.462	−0.9	7.234	12
B	5	11	2.232	2.24	−0.4	6.859	6.928	−1	7.611	9.9
C	6	13	2.33	2.37	−1.5	7.209	7.47	−3.5	7.875	5.4
N	7	15	2.448	2.48	−1.2	7.486	7.7	−2.8	8.069	4.8
O	8	17	2.573	2.58	−0.2	7.711	7.75	−0.5	8.216	6
F	9	19	2.688	2.67	0.6	7.895	7.778	1.5	8.331	7.1
Ne	10	23	2.901	2.88	0.8	8.117	7.955	2	8.387	5.4
Mg	12	25	2.927	2.92	0.3	8.283	8.224	0.7	8.549	4
AL	13	27	2.997	2.99	0.2	8.373	8.332	0.5	8.595	3.1
Si	14	29	3.061	3.06	0	8.45	8.45	0	8.631	2.1
S	16	33	3.199	3.19	0.3	8.568	8.499	0.8	8.68	2.1
Mn	25	55	3.803	3.8	0.1	8.783	8.76	0.3	8.818	0.7
Cu	29	65	4.024	4.02	0.1	8.777	8.758	0.2	8.798	0.5
Pd	46	105	4.693	4.7	−0.1	8.539	8.57	−0.4	8.567	0
I	53	127	5.025	5.03	−0.2	8.404	8.441	−0.4	8.425	−0.2
Xe	54	131	5.087	5.09	−0.1	8.384	8.423	−0.5	8.395	−0.3
Pt	78	195	5.804	5.81	0	7.917	7.928	−0.1	7.938	0.1
Bk	97	245	6.256	6.25	0.1	7.554	7.517	0.5	7.585	0.9

Table 4 also shows the results of the binding energy calculation for the LDMc with the usual corrections from Quantum Mechanics [2]:

$$\bar{B}_c = p_v - p_s A^{-1/3} - p_c \frac{z^2 - z.}{A^{4/3}} - f_a \left(1 - 2Z/A\right)^2 - f_p \frac{33.5}{A^{3/4} A} \tag{13}$$

$$p_c = 0.595\,, \quad p_v = 14.1\,, \quad p_s = 13.0\,, \quad f_a = 19.0\,, \quad f_p = \begin{pmatrix} 0 & \text{odd } A \\ 1 & \text{even } Z \text{ and } N \\ -1 & \text{odd } Z \text{ and } N \end{pmatrix},$$

$f_p = 0$ for the nuclides of Table 4.

Note in Table 4 that for $A < 105$ the LDMc does not perform as well as the 2 exponential-model.

All nuclear data (e.g. Atomic mass) used to calculate $\bar{B}$ were taken from Ref. [2].

Table 5 shows the results of $\bar{B}$ and $\bar{B}_c$ for some nuclides inside the range of Z and A used to do the R-fitting. Note that for Ca, for A < 37, the 2 exponential model out performs the LDMc; however, for A = 37 - 39, the LDMc performs better. The reason for these unexpected results has not yet been determined. The results of the LDMu show the need for some kind of correction for many nuclides.

Note that Table 5 includes some nuclides with the so called magic numbers in Z and N. It is curious that for Ca 40 (double magic number), the 2-exponential model and the LDMc yield relative error of just 1.7 % and that the LDMu is even better (1.1%).

All nuclear data (e.g. B) used to calculate $\bar{B}$ were taken from Ref. [7].

Table 6 shows the results of $\bar{B}$ and $\bar{B}_c$ for some nuclides outside the range of Z and A used to do the R-fitting.

The 2 exponential model in general shows better agreement with the measured values than what the LDMc does (except for A = 249). The LDMu however, shows a poor performance when compared to the other 2 methods.

All nuclear data (e.g. B) used to calculate $\bar{B}$ were taken from Ref. [7] and Ref. [8].

Considering the good accuracy of the R-optimized 2-exponential model, it is of interest to know if the reason for that good performance is just due the use of a good statistical model or there is a physic base for that model.

The R optimization has a physical explanation considering that R is not the radius of the nucleus but an average distance between nucleons.

A potential reason for the success of the two exponential terms is addressed next.

An equation for the binding energy of the attraction among the nucleons similar to the electrostatic case is written as:

$$U = G_n \frac{1}{2}\left(\sum_{i=1}^{Z} m_{p.i} + \sum_{i=Z+1}^{A} m_{n.i}\right)\left(\sum_{j=1, j\neq i}^{Z} \frac{m_{j.p} e^{-R_{i.j}/\lambda}}{R_{i.j}} + \sum_{j=Z+1, j\neq i}^{A} \frac{m_{n.j} e^{-R_{i.j}/\lambda}}{R_{i.j}}\right)$$

Carrying out the algebra, making $\lambda = \lambda_{i.j}$ (nucleon dependent interaction) and taking an average $R_{i.j} = R$, the following is obtained:

$$U = \frac{G_n}{2R}\left(m_p^2 z(z-1)\mathrm{e}^{-R/\lambda_{p.p}} + 2m_n m_p z(A-z)\mathrm{e}^{-R/\lambda_{n.p}} + m_n^2 (A-z)(A-z-1)\mathrm{e}^{-R/\lambda_{n.n}}\right) \quad (14)$$

Equation (14) shows the potential existence of up to 3 terms that could describe the interaction among nucleons.

The reviewer suggested adding Ref. [9] to the bibliography section. Two nuclides were identified common to the calculation shown in **Table 5** of this work: O16 and Ca40. For O16 Ref. [9] yields a deviation from measurement ~0.2 Mev, this work yields 0.4 Mev. For Ca40 Ref. [9] yields a deviation of 0.3 Mev, this work yields 0.1 Mev. Note that this work has the potential to reduce the deviations even further if 2 independent variable-spline interpolations are used to obtain the distance among nucleons.

Table 5. Calculation $\bar{B}_c$ for some nuclides inside the range of Z and A used to do the R fitting.

Atom	Z	A	r (fm)	B (Mev)						
				2Exp	Meas	%	LDMc	%	LDMu	%
C	6	13	2.365	7.209	7.47	−3.5	7.875	5.4	7.987	6.9
C	6	14	2.446	7.304	7.52	−2.9	8.12	8	8.177	8.7
N	7	15	2.476	7.486	7.7	−2.8	8.069	4.8	8.153	5.9
O	8	14	2.356	7.421	7.052	5.2	7.661	8.6	7.719	9.4
O	8	15	2.433	7.541	7.464	1	7.844	5.1	7.928	6.2
O	8	16	2.507	7.636	7.976	−4.3	8.376	5	8.115	1.7
Ca	20	37	3.253	8.642	8.004	8	8.24	2.9	8.365	4.5
Ca	20	38	3.295	8.665	8.24	5.2	8.468	2.8	8.463	2.7
Ca	20	39	3.337	8.683	8.37	3.7	8.545	2.1	8.557	2.2
Ca	20	40	3.378	8.698	8.551	1.7	8.699	1.7	8.646	1.1
Fe	26	56	3.808	8.785	8.793	−0.1	8.829	0.4	8.897	1.2
Ni	28	54	3.695	8.747	8.392	4.2	8.462	0.8	8.457	0.8
Sr	38	84	4.353	8.678	8.677	0	8.7	0.3	8.858	2.1
Cd	48	101	4.574	8.478	8.45	0.3	8.408	−0.5	8.455	0.1
La	57	128	4.987	8.308	8.338	−0.4	8.342	0	8.576	2.9
Er	68	149	5.206	8.039	8.074	−0.4	8.071	0	8.216	1.8
Pb	82	181	5.539	7.71	7.733	−0.3	7.774	0.5	7.942	2.7
Pb	82	182	5.554	7.718	7.756	−0.5	7.792	0.5	7.974	2.8
Th	90	232	6.166	7.706	7.615	1.2	7.689	1	8.641	13.5
U	92	233	6.154	7.653	7.604	0.6	7.673	0.9	8.513	12
U	92	235	6.181	7.661	7.591	0.9	7.664	1	8.558	12.7
U	92	238	6.22	7.672	7.57	1.3	7.649	1	8.625	13.9
Pu	94	239	6.208	7.618	7.56	0.8	7.633	1	8.498	12.4

Table 6. Calculation $\bar{B}_c$ for some nuclides outside the range of Z and A used to do the R fitting.

Atom	Z	A	r (fm)	B (Mev)						
				2Exp	Meas	%	LDMc	%	LDMu	%
Cm	96	249	6.312	7.6	7.486	1.5	7.574	1.2	8.57	14.5
Cf	98	252	6.325	7.554	7.465	1.2	7.552	1.2	8.488	13.7
Es	99	253	6.325	7.529	7.454	1.0	7.539	1.1	8.437	13.2
Fm	100	257	6.363	7.516	7.422	1.3	7.516	1.3	8.45	13.9
Md	101	258	6.363	7.491	7.41	1.1	7.502	1.2	8.399	13.4
Ed	113	284	6.535	7.239	7.174	0.9	7.294	1.7	8.088	12.7
FL	114	289	6.582	7.234	7.149	1.2	7.276	1.8	8.122	13.6
Ef	115	288	6.559	7.194	7.136	0.8	7.258	1.7	8.03	12.5
Lv	116	292	6.594	7.184	7.117	0.9	7.243	1.8	8.043	13.0
Ts	117	293	6.594	7.156	7.097	0.8	7.223	1.8	7.993	12.6
Ts	117	294	6.606	7.162			7.221		8.014	
Og	118	294	6.594	7.127	7.080	0.7	7.205	1.8	7.943	12.2

4. Summary and Conclusions

A Yukawa-type potential was developed to predict B for the current nuclides using a set of 23 nuclides as the source of the experimental values. The deviation between measurement and prediction is remarkably reduced by a special search on the average distance between nucleons. The best results were obtained for a 2-exponential model. The predictions of B inside and outside the range of A and Z used to do the fitting were carried out by fitting the obtained values of the average distance among nucleons to 2 terms power law that depends on A and Z.

It should be worthy to implement a 2 dimensional-independent variable spline interpolation with proper boundary conditions at the end of the range of A and Z of the experimental set.

It could be worthy also to make more attempts to use 1 and 3-exponential models to find out if better results can be obtained that proof (disproof) the physical existence of the terms of Equation (14), even its super generalization by allowing G_n to be dependent on the type of nucleon-nucleon interaction.

As long as the value(s) of G_n and $\lambda_{x.x}$ are not consistent with the results for H2, H3, He3, and He4, the matter cannot be considered settled even if perfect verification of B is achieved, because the noted nuclides cover all the nucleon-nucleon interaction in a unique way.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Aston, F.W. (1929) The Mass-Spectrum of Uranium Lead and the Atomic Weight of Protactinium. *Nature*, **123**, 313. https://doi.org/10.1038/123313a0

[2] Evans, R.D. (1955) The Atomic Nucleus. McGraw-Hill, New York.

[3] Yukawa, H. (1935) On the Interaction of Elementary Particles. I. *Progress of Theoretical Physics Supplement*, **1**, 1-10. https://academic.oup.com/ptps/article/doi/10.1143/PTPS.1.1/1878532?login=false

[4] von Weizsäcker, C.F. (1935) Zur Theorie der Kernmassen. *Zeitschrift für Physik*, **96**, 431-458. https://doi.org/10.1007/BF01337700

[5] Jackson, J.D. (1962) Classical Dynamics. John Wiley & Sons, Inc., Hoboken.

[6] Guthier, N. (1990) Yukawa Potential Approach to the Nuclear Binding Energy Formula. *American Journal of Physics*, **58**, 375-378. https://doi.org/10.1119/1.16176

[7] Audi, G. and Wapstra, A.H. (1993) The 1993 Atomic Mass Evaluation: (I) Atomic Mass table. *Nuclear Physics A*, **565**, 1-65. https://doi.org/10.1016/0375-9474(93)90024-R

[8] Dongfang, X.D. (2021) Nuclear Force Constants Mapped by Yukawa Potential. *Mathematics and Nature*, **1**, Article 009.

[9] Schaeffer, B. (2013) Pairing Effect on the Binding Energy Curve of N = Z Atomic Nuclei. *World Journal of Nuclear Science and Technology*, **3**, 79-82. https://doi.org/10.4236/wjnst.2013.33013