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Extension of Homomorphisms-Isomorphisms and Derivatives on Quasi-Banach Algebra Based on the General Additive Cauchy-Jensen Equation

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Abstract

In this paper, I establish homomorphisms, isomorphisms, and derivatives of quasi-algebras based on the general additive equation Cauchy-Jensen with 3k variables. First, I establish the homomorphisms for Equation (1.1); second, I establish the isomorphisms for Equation (1.2); and finally, I develop the derivative for Equation (1.3). These are the main results of this paper.

Subject Areas

Mathematics

Keywords

Generalized Additive Equation Cauchy-Jensen Additive, Homomorphisms, Isomorphism and Derivatives on Quasi-Banach Algebra, Quasi-Normed Algebras, *p*-Banach Algebras

1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ are two linear spaces on the same field $\mathbb{K}$, and $f:\mathbf{X}\to\mathbf{Y}$ be a linear mapping. I use the notation $\|\cdot\|_{\mathbf{X}}$ ($\|\cdot\|_{\mathbf{Y}}$) for corresponding the norms on $\mathbf{X}$ and $\mathbf{Y}$. In this paper, I investigate the stability of generalized homomorphisms-isomorphism and derivatives when $\mathbf{X}$ is a quasi-normed algebras with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a *p*-Banach algebras with *p*-norm $\|\cdot\|_{\mathbf{Y}}$.

In fact, when $\mathbf{X}$ is a quasi-normed algebras with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a *p*-Banach algebras with *p*-norm $\|\cdot\|_{\mathbf{Y}}$, I solve and prove the Hyers-Ulam-Rassias type stability of generalized Homomorphisms-isomorphism and derivatives on quasi-Banach algebra, associated to the following generalized Cauchy-Jensen additive functional equations

$$k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}\phi(x_i)+2k\sum_{i=1}^{k}\phi(z_i) \quad (1)$$

$$k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}\phi(y_i) \quad (2)$$

$$2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}\phi(x_i)+\sum_{i=1}^{k}\phi(y_i)+2k\sum_{i=1}^{k}\phi(z_i) \quad (3)$$

The study the stability of generalized homomorphisms-isomorphism and derivatives in quasi-Banach algebras originated from a question of S. M. Ulam [1], concerning the stability of group homomorphisms.

Let $(\mathbf{G},*)$ be a group and let $(\mathbf{G}',\circ,d)$ be a metric group with metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that if $f:\mathbf{G}\to\mathbf{G}'$ satisfies inequality

$$d\left(f(x*y),f(x)\circ f(y)\right)<\delta,\forall x\in\mathbf{G},$$

then there is a homomorphism $h:\mathbf{G}\to\mathbf{G}'$ with

$$d\left(f(x),h(x)\right)<\varepsilon,\forall x\in\mathbf{G}$$

If the answer is affirmative, I would say that the equation of homomorphism

$$H(x*y)=H(x)\circ H(y),\forall x\in\mathbf{G}$$

is stable. The concept of stability for a functional equation arises when I replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation? In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces.

Let X and Y be Banach space. Assume that $f:X\to Y$ satisfies

$$f(x+y)-f(x)-f(y)\le\varepsilon \quad (4)$$

for all $x,y\in X$ and some $\varepsilon>0$. Then there exists a unique additive mapping $T:X\to Y$ such that

$$\left\|f(x)-T(x)\right\|\le\varepsilon \quad (5)$$

Next 1978 Th. M. Rassias [3] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded.

J. M. Rassias [4] [5] [6] built no continuity conditions are required for this result, but if $f(tx)$ is continuous in the real variable t for each fixed $x\in E$, then L is linear, and if f is continuous at a single point of E then $L:E\to E'$ is also continuous. J. M. Rassias assumed the following weaker inequality

$$\left\|f(x+y)-f(x)-f(y)\right\|=\theta\cdot\|x\|^{p}\cdot\|y\|^{q} \quad (6)$$

$\forall x, y \in E$; involving a product of different powers of norms, where $\theta > 0$ and real p, q such that $r = p + q \neq 1$, and retained the condition of continuity $f(tx)$ in t for fixed *x*. Analogous results could be investigated with additive type equations involving a product of powers of norms.

Next 1994 Găvruta [7] Generalized the Rassias' result. There are also many mathematicians who have built many results for this topic as [1]-[24].

Recently, the authors studied the Hyers-Ulam-Rassias type stability for the following functional equations (see [8] [9] [10] [11] [12])

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)=f(x)+2f(z) \tag{7}$$

$$f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x-y}{2}+z\right)=f(y) \tag{8}$$

$$2f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)=f(x)+f(y)+2f(z) \tag{9}$$

Next

$$f\left(\sum_{j=1}^{k}x_j+\frac{1}{k}\sum_{j=1}^{k}x_{k+j}\right)=\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) \tag{10}$$

and

$$2kf\left(\frac{1}{2k}\sum_{j=1}^{k}x_j+\frac{1}{2k^2}\sum_{j=1}^{k}x_{k+j}\right)=\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) \tag{11}$$

Final

$$mf\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}x_{k+j}}{m}\right)=\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(x_{k+j}) \tag{12}$$

In this paper, I have built a general problem about homomorphisms-isomorphism and derivatives on quasi-Banach algebra based on the general additive Cauchy-Jensen equation. To write these problems, I follow the ideas of mathematicians around the world see [1]-[24]. In order to provide researchers in Mathematics when building problems, there is no restriction on variables for the problem. This is what I consider an open dream problem or a bright horizon for the field of functional equations in quasi-Banach algebras.

In this paper, I solve and proved the Hyers-Ulam-Rassias type stability for functional Equations (1.1), (1.2) and (1.3), *i.e.*, the functional equations with 3k variables. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, I will prove that the mappings satisfying the functional Equation (1.1), (1.2) and (1.3).

Thus, the results in this paper are generalization of those in [8] [9] [10] [11] [12] [24] for functional equations with 3k variables. The paper is organized as follows:

In section preliminarier I remind some basic notations in such as Quasi-normed space—Quasi-Banach algebras. Some theorems $\mathbb{R}$-linear mapping and Solutions of the equations see ([3] [4] [5] [6]).

Section 3: Constructing homomorphisms on quasi-Banach algebras for (1.1).

Section 4: Constructing isomorphisms on quasi-Banach algebras for (1.2).

Section 5: Constructing derivatives on quasi-Banach algebras for (1.3).

2. Preliminaries

2.1. Quasi-Normed Space—Quasi-Banach Algebras

Let $\mathbf{X}$ be a real linear space. A quasi-norm is a real-valued function on $\mathbf{X}$ satisfying the following:

1) $\|x\| \ge 0$ for all $x \in \mathbf{X}$ and $\|x\| = 0$ if and only if $x = 0$.

2) $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbf{R}$ and all $x \in \mathbf{X}$.

3) There is a constant $K \ge 1$ such that

$$\|x + y\| \le K\left(\|x\| + \|y\|\right), \forall x, y \in \mathbf{X}.$$

The pair $(\mathbf{X}, \|\cdot\|)$ is called a quasi-normed space if $\|\cdot\|$ is a quasi-norm on $\mathbf{X}$.

The smallest possible K is called the modulus of concavity of $\|\cdot\|$.

A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a p-norm ($0 < p \le 1$) if

$$\|x + y\|^p \le \|x\|^p + \|y\|^p \ \forall x, y \in \mathbf{X}.$$

In this case, a quasi-Banach space is called a p-Banach space.

Note: Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz Theorem [13] (see also [14]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms, henceforth I restrict my attention mainly to p-norms.

Let $(\mathbf{X}, \|\cdot\|)$ be a quasi-normed space. The quasi-normed space $(\mathbf{X}, \|\cdot\|)$ is called a quasi-normed algebras if $\mathbf{X}$ is an algebras and there is a constant $K > 0$ such that

$$\|x \cdot y\| \le K\|x\|\|y\|$$

A quasi-Banach algebras is a complete quasi-normed algebras.

If the quasi-norm $\|\cdot\|$ is a p-norm, quasi-Banach is called p-Banach algebras.

2.2. Some Theorems $\mathbb{R}$-Linear Mapping

Theorem 1. Th. M. Rassias: Let $f : \mathbf{E}_1 \to \mathbf{E}_2$ be a mapping from a normed vector space $\mathbf{E}_1$ into a Banach space $\mathbf{E}_2$ subject to the inequality

$$\left\|f(x+y) - f(x) - f(y)\right\| \le \varepsilon\left(\|x\|^p + \|y\|^p\right), \tag{13}$$

for all $x, y \in \mathbf{E}_1$, where ε and p are constants with $p < 1$ and $\varepsilon \ge 0$. then the limit

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{14}$$

exists for all $x \in \mathbf{E}_1$ and that $T : \mathbf{E}_1 \to \mathbf{E}_2$ is the unique additive mapping satisfying

$$\|f(x)-T(x)\| \le \frac{2\theta}{2-2^p}\|x\|^p, \forall x \in \mathbf{E}_1. \tag{15}$$

If $p<0$ then (2.1) holds for $x, y \neq 0$ and (2.2) for $x \neq 0$. Also, if for each $x \in \mathbf{E}_1$ the function If $f(tx)$ is continuous in $t \in \mathbb{R}$, then T is linear.

Theorem 2. Let $\mathbf{E}$ be real normed linear space and $\mathbf{E}'$ a real complete normed linear space. Assume that $f: \mathbf{E} \to \mathbf{E}'$ is an approximately additive mapping for which there exists constants $\theta \ge 0$ and $p \in \mathbb{R}-\{1\}$ such that $f(x)$ satisfy inequality

$$\|f(x+y)-f(x)-f(y)\| \le \theta\|x\|^{\frac{p}{2}}\|y\|^{\frac{p}{2}}, \forall x, y \in \mathbb{E}. \tag{16}$$

then there exists a unique additive mapping linear $T: \mathbf{E} \to \mathbf{E}'$ satisfies

$$\|f(x)-L(x)\| \le \frac{\theta}{2^p-2}\|x\|^p, x \in \mathbf{E}. \tag{17}$$

If, in addition $f: \mathbf{E} \to \mathbf{E}'$ is a transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbf{E}$, then T is an $\mathbf{R}$-linear mapping.

Theorem 3. Let $\mathbf{E}$ be real normed linear space and $\mathbf{E}'$ a real complete normed linear space. Assume that $f: \mathbf{E} \to \mathbf{E}'$ is an approximately additive mapping for which there exists constants $\theta \ge 0$ such that $f(x)$ satisfy inequality

$$\left\| f\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_i) \right\| \le \theta K(x_1, x_2, \cdots, x_n). \tag{18}$$

$(x_1, x_2, \cdots, x_n) \in \mathbf{E}$ and $K: \mathbf{E}^n \to \mathbf{R}^+ - \{0\}$ is a non-negative real-valued function such that

$$R_n(x) = \sum_{i=1}^{n} \frac{1}{n^j} K\left(n^j x_1, n^j x_2, \cdots, n^j x_n\right) < \infty \tag{19}$$

is a non-negative function of x, and the condition

$$\lim_{m\to\infty} \frac{1}{n^m} K\left(n^m x_1, n^m x_2, \cdots, n^m x_n\right) = 0 \tag{20}$$

holds then there exists a unique additive mapping $T_n: \mathbf{E} \to \mathbf{E}'$ satisfies

$$\|f(x)-T_n(x)\| \le \frac{\theta}{n} R_n(x), x \in \mathbb{E}. \tag{21}$$

If, in addition $f: \mathbf{E} \to \mathbf{E}'$ is a transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbf{E}$, then T is an $\mathbb{R}$-linear mapping.

Theorem 4. Let $\mathbf{E}$ be real normed linear space and $\mathbf{E}'$ a real complete normed linear space. Assume that $f: \mathbf{E} \to \mathbf{E}'$ is an approximately additive mapping for which there exists constants $\theta \ge 0$ and $p, q \in \mathbb{R}$ such that $p+q \neq 1$ and f satisfy inequality

$$\|f(x+y)-f(x)-f(y)\| \le \theta\|x\|^p\|y\|^q, \forall x, y \in \mathbb{E}. \tag{22}$$

then there exists a unique additive mapping linear $T: \mathbf{E} \to \mathbf{E}'$ satisfies

$$\|f(x)-L(x)\| \leq \frac{\theta}{2^p-2}\|x\|^p, x \in \mathbb{E}. \tag{23}$$

If, in addition $f:\mathbf{E}\to\mathbf{E}'$ is a transformation $t\to f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in\mathbf{E}$, then T is an $\mathbb{R}$-linear mapping.

2.3. Solutions of the Equation

$$f(x+y)=f(x)+f(y) \tag{24}$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an *additive mapping*.

The functional equation

$$f\left(\frac{x+y}{2}\right)=\frac{1}{2}f(x)+\frac{1}{2}f(y) \tag{25}$$

is called the Jensen equation. In particular, every solution of the Jensen equation is said to be a Jensen *additive mapping*.

The functional equation

$$2f\left(\frac{x+y}{2}+z\right)=f(x)+f(y)+2f(z) \tag{26}$$

is called the Cauchy-Jensen equation. In particular, every solution of the Cauchy-Jensen equation is said to be a Jensen-Cauchy *additive mapping*.

3. Constructing Homomorphisms in Quasi-Banach Algebras

Now I construct a homomorphism for (1.1) Note that: (1.2) and (1.3) are also built exactly the same.

Here I assume that, $\mathbb{A}$ is a quasi-normed with norm $\|\cdot\|_{\mathbb{A}}$ and that $\mathbb{B}$ is a p-Banach algebra with norm $\|\cdot\|_{\mathbb{B}}$. Let $\mathbf{K}$ be the modulus of concavity of $\|\cdot\|_{\mathbb{B}}$. Under this setting, I can show that the mappings satisfying (1.1) is homomorphisms.

Theorem 5. Let $r>q$ with $q\geq 2$ and θ be positive real numbers, and $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that

$$\begin{aligned}&\left\|kf\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+kf\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}f(x_i)-2k\sum_{i=1}^{k}f(z_i)\right\|_{\mathbf{Y}}\\&\leq\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbf{X}}^r+\sum_{i=1}^{k}\|y_i\|_{\mathbf{X}}^r+\sum_{i=1}^{k}\|z_i\|_{\mathbf{X}}^r\right)\end{aligned} \tag{27}$$

$$\left\|f\left(\prod_{i=1}^{n}x_iy_i\right)-\prod_{i=1}^{n}f(x_i)\prod_{i=1}^{n}f(y_i)\right\|_{\mathbf{Y}}\leq\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbf{X}}^r+\sum_{i=1}^{k}\|y_i\|_{\mathbf{X}}^r\right) \tag{28}$$

for all $x_i,y_i,z_i\in\mathbf{X}$, for all $i=1\to k$. If $f(tx)$ is continuous in $t\in\mathbb{R}$ each fixed $x\in\mathbf{X}$, then there exists a unique homomorphism $H:\mathbf{X}\to\mathbf{Y}$ such that

$$\|f(x)-H(x)\|_{\mathbf{Y}}\leq\frac{\left(2+\frac{1}{k}\right)\theta}{\left(2^{pr}-2\right)^{\frac{1}{p}}}\|x\|_{\mathbf{X}}^r, \forall x\in\mathbf{X}. \tag{29}$$

Proof. I replace $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,x,\cdots,0)$ in (27), I have

$$\left\|kf(2x)-2kf(x)\right\|_{\mathbf{Y}}\le(2k+1)\theta\|x\|_{\mathbf{X}}^{r} \tag{30}$$

for all $x\in X$. So

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\|_{\mathbf{Y}}\le\left(2+\frac{1}{k}\right)\frac{\theta}{2^{r}}\|x\|_{\mathbf{X}}^{r} \tag{31}$$

for all $x\in X$. From $\mathbf{Y}$ is p-Banach algebra so I have

$$\begin{aligned}\left\|2^{l}f\left(\frac{x}{2^{l}}\right)-2^{m}f\left(\frac{x}{2^{m}}\right)\right\|_{\mathbf{Y}}^{p}&\le\sum_{j=l}^{m-1}\left\|2^{j}f\left(\frac{x}{2^{j}}\right)-2^{j+1}f\left(\frac{x}{2^{j+1}}\right)\right\|_{\mathbf{Y}}^{p}\\&\le\left(2+\frac{1}{k}\right)^{p}\frac{\theta^{p}}{2^{pr}}\sum_{j=l+1}^{m-1}\frac{2^{pj}}{2^{prj}}\|x\|_{\mathbf{X}}^{pr}\end{aligned} \tag{32}$$

for all nonnegative integers m and l with $m>l$ and for all $x\in\mathbf{X}$. It follows (32) that the sequence $\left\{2^{n}f\left(\frac{x}{2^{n}}\right)\right\}$ is a Cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{2^{n}f\left(\frac{x}{2^{n}}\right)\right\}$ converges. So one can define the mapping $H:\mathbf{X}\to\mathbf{Y}$ by

$$H(x)=\lim_{n\to\infty}2^{n}f\left(\frac{x}{2^{n}}\right), \tag{33}$$

for all $x\in X$. By (28) and (27),

$$\begin{aligned}&\left\|kH\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+kH\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}H(x_i)-2k\sum_{i=1}^{k}H(z_i)\right\|_{\mathbf{Y}}\\&=\lim_{n\to\infty}2^{n}\left\|kf\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2^{n+1}k}+\sum_{i=1}^{k}\frac{z_i}{2^{n}}\right)+kf\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2^{n+1}k}+\sum_{i=1}^{k}\frac{z_i}{2^{n}}\right)\right.\\&\quad\left.-\sum_{i=1}^{k}f\left(\frac{x_i}{2^{n}}\right)-2k\sum_{i=1}^{k}f\left(\frac{z_i}{2^{n}}\right)\right\|_{\mathbf{Y}}\\&\le\lim_{n\to\infty}\frac{2^{n}\theta}{2^{nr}}\left(\sum_{i=1}^{k}\|x_i\|_{\mathbf{X}}^{r}+\sum_{i=1}^{k}\|y_i\|_{\mathbf{X}}^{r}+\sum_{i=1}^{k}\|z_i\|_{\mathbf{X}}^{r}\right)=0\end{aligned} \tag{34}$$

for all $x_i,y_i,z_i\in\mathbf{X}$, for all $i=1\to k$.

So

$$kH\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+kH\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}H(x_i)+2k\sum_{i=1}^{k}H(z_i) \tag{35}$$

for all $x_i,y_i,z_i\in\mathbf{X}$, for all $i=1\to k$. By lemma 5 (see 24]), the mapping $H:\mathbf{X}\to\mathbf{Y}$ is Cauchy additive. (see the theorem of [3])

Then mapping $H:\mathbf{X}\to\mathbf{Y}$ is $\mathbb{R}$-linear. It follows from (28) that

$$\begin{aligned}&\left\|f\left(\prod_{i=1}^{k}x_iy_i\right)-\prod_{i=1}^{k}f(x_i)\prod_{i=1}^{k}f(y_i)\right\|_{\mathbf{Y}}\\&=\lim_{n\to\infty}2^{2nk}\left\|f\left(\prod_{i=1}^{k}\frac{x_i}{2^{n}}\cdot\frac{y_i}{2^{n}}\right)-\prod_{i=1}^{k}f\left(\frac{x_i}{2^{n}}\right)\cdot\prod_{i=1}^{k}f\left(\frac{y_i}{2^{n}}\right)\right\|_{\mathbf{Y}}\\&\le\lim_{n\to\infty}\frac{2^{2nk}\theta}{2^{nr}}\left(\sum_{i=1}^{k}\|x_i\|_{\mathbf{X}}^{r}+\sum_{i=1}^{k}\|y_i\|_{\mathbf{X}}^{r}\right)=0\end{aligned} \tag{36}$$

$\forall x, y \in \mathbf{X}$.

So

$$H\left(\prod_{i=1}^{k} x_i \cdot y_i\right) = \prod_{i=1}^{k} H(x_i) \prod_{i=1}^{k} H(y_i) \tag{37}$$

$\forall x, y \in \mathbf{X}$.

Now I prove the uniqueness of *H*. Assume that $H_1 : X \to Y$ is a Cauchy-Jensen additive mapping satisfying (29). Then I have

$$\begin{aligned} \|H(x) - H_1(x)\|_Y &= 2^n \left\| H\left(\frac{1}{2^n}x\right) + H_1\left(\frac{1}{2^n}x\right)\right\|_Y \\ &\le 2^n \mathbf{K}\left(\left\| H\left(\frac{1}{2^n}x\right) - f\left(\frac{1}{2^n}x\right)\right\|_Y + \left\| f\left(\frac{1}{2^n}x\right) + H_1\left(\frac{1}{2^n}x\right)\right\|_Y\right) \\ &\le 2\frac{\left(2+\frac{1}{k}\right)\mathbf{K}\theta}{\left(2^{pr}-2^{p}\right)^{\frac{1}{p}}} \cdot \frac{2^n}{2^{nr}}\|x\|_{\mathbf{X}}^r \end{aligned} \tag{38}$$

which tends to zero as $n \to \infty$ for all $x \in \mathbf{X}$. So I can conclude that $H(x) = H_1(x)$ for all $x \in \mathbf{X}$. This proves the uniqueness of *H*. Thus the mapping $H_1 : \mathbf{X} \to \mathbf{Y}$ is a unique homomorphism satisfying (29). □

Theorem 6. Let $r < q$ with $q \le 1$ and θ be positive real numbers, and $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that

$$\begin{aligned} &\left\| kf\left(\sum_{i=1}^{k} \frac{x_i + y_i}{2k} + \sum_{i=1}^{k} z_i\right) + kf\left(\sum_{i=1}^{k} \frac{x_i - y_i}{2k} + \sum_{i=1}^{k} z_i\right) - \sum_{i=1}^{k} f(x_i) - 2k\sum_{i=1}^{k} f(z_i)\right\|_{\mathbf{Y}} \\ &\le \theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbf{X}}^r + \sum_{i=1}^{k}\|y_i\|_{\mathbf{X}}^r + \sum_{i=1}^{k}\|z_i\|_{\mathbf{X}}^r\right) \end{aligned} \tag{39}$$

$$\left\| f\left(\prod_{i=1}^{k} x_i y_i\right) - \prod_{i=1}^{k} f(x_i) \prod_{i=1}^{k} f(y_i)\right\|_{\mathbf{Y}} \le \theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbf{X}}^r + \sum_{i=1}^{k}\|y_i\|_{\mathbf{X}}^r\right) \tag{40}$$

for all $x_i, y_i, z_i \in \mathbf{X}$, for all $i = 1 \to k$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ each fixed $x \in \mathbf{X}$, then there exists a unique homomorphism $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\|f(x) - H(x)\|_{\mathbf{Y}} \le \frac{\left(1+\frac{1}{2k}\right)\theta}{\left(2-2^{pr}\right)^{\frac{1}{p}}}\|x\|_{\mathbf{X}}^r, \forall x \in \mathbf{X}. \tag{41}$$

Proof. I replace $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $(x, \cdots, x, x, \cdots, x, x, \cdots, 0)$ in (39), I have

$$\|kf(2x) - 2kf(x)\|_{\mathbf{Y}} \le (2k+1)\theta\|x\|_{\mathbf{X}}^r \tag{42}$$

for all $x \in \mathbf{X}$. So

$$\left\| f(x) - \frac{1}{2}f(2x)\right\|_{\mathbf{Y}} \le \left(1+\frac{1}{2k}\right)\theta\|x\|_{\mathbf{X}}^r \tag{43}$$

for all $x \in X$. Since $\mathbf{Y}$ is a *p*-Banach algebra,

$$\left\|\frac{1}{2^l}f\left(2^l x\right)-\frac{1}{2^m}f\left(2^m x\right)\right\|_{\mathbf{Y}}^p \le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right)-\frac{1}{2^{j+1}}f\left(2^{j+1} x\right)\right\|_{\mathbf{Y}}^p \le \left(1+\frac{1}{2k}\right)^p \theta^p \sum_{j=l+1}^{m-1}\frac{2^{prj}}{2^{pj}}\|x\|_{\mathbf{X}}^{pr} \tag{44}$$

for all nonnegative integers m and l with $m>l$ and for all $x\in\mathbf{X}$. It follows (44) that the sequence $\left\{\frac{1}{2^n}f\left(2^n x\right)\right\}$ is a Cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{\frac{1}{2^n}f\left(2^n x\right)\right\}$ converges. So one can define the mapping $H:\mathbf{X}\to\mathbf{Y}$ by

$$H(x)=\lim_{n\to\infty}\frac{1}{2^n}f\left(2^n x\right)$$

for all $x\in X$.

Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (44) I get (41). The rest of the proof is similar to the proof of Theorem 5. □

Theorem 7. Let $r>k$, with $k\ge 1$ and θ be positive real numbers, and $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that

$$\left\|kf\left(\sum_{i=1}^k\frac{x_i+y_i}{2k}+\sum_{i=1}^k z_i\right)+kf\left(\sum_{i=1}^k\frac{x_i-y_i}{2k}+\sum_{i=1}^k z_i\right)-\sum_{i=1}^k f\left(x_i\right)-2k\sum_{i=1}^k f\left(z_i\right)\right\|_{\mathbf{Y}} \le \theta\cdot\prod_{i=1}^k\|x_i\|_{\mathbf{X}}^r\cdot\prod_{i=1}^k\|y_i\|_{\mathbf{X}}^r\cdot\|z_1\|_{\mathbf{X}}^{rk}\cdot\left(1+\prod_{i=2}^k\|z_i\|_{\mathbf{X}}^r\right) \tag{45}$$

$$\left\|f\left(\prod_{i=1}^k x_i y_i\right)-\prod_{i=1}^k f\left(x_i\right)\prod_{i=1}^k f\left(y_i\right)\right\|_{\mathbf{Y}} \le \theta\cdot\prod_{i=1}^k\|x_i\|_{\mathbf{X}}^r\cdot\prod_{i=1}^k\|y_i\|_{\mathbf{X}}^r \tag{46}$$

for all $x_i,y_i,z_i\in\mathbf{X}$, for all $i=1\to k$. If $f(tx)$ is continuous in $t\in\mathbb{R}$ each fixed $x\in\mathbf{X}$, then there exists a unique homomorphism $H:\mathbf{X}\to\mathbf{Y}$ such that

$$\|f(x)-H(x)\|_{\mathbf{Y}} \le \frac{\theta}{k\left(2^{3pkr}-2^p\right)^{\frac{1}{p}}}\|x\|_{\mathbf{X}}^{3kr}, \forall x\in\mathbf{X}. \tag{47}$$

Proof. I replace $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(x,\cdots,x,x,\cdots,x,x,\cdots,0\right)$ in (45), I have

$$\|kf(2x)-2kf(x)\|_{\mathbf{Y}} \le \theta\|x\|_{\mathbf{X}}^{3kr} \tag{48}$$

for all $x\in X$. So

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\|_{\mathbf{Y}} \le \frac{1}{k}\frac{\theta}{2^{3kr}}\|x\|_{\mathbf{X}}^{3kr} \tag{49}$$

for all $x\in X$. Since $\mathbf{Y}$ is a p-Banach algebras,

$$\left\|2^l f\left(\frac{x}{2^l}\right)-2^m f\left(\frac{x}{2^m}\right)\right\|_{\mathbf{Y}}^p \le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right)-2^{j+1}f\left(\frac{x}{2^{j+1}}\right)\right\|_{\mathbf{Y}}^p \le \frac{1}{k^p}\frac{\theta^p}{2^{3pkr}}\sum_{j=l+1}^{m-1}\frac{2^{pj}}{2^{3pkrj}}\|x\|_{\mathbf{X}}^{3pkr} \tag{50}$$

for all nonnegative integers m and l with $m > l$ and for all $x \in \mathbf{X}$. It follows (50) that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ converges. So one can define the mapping $H : \mathbf{X} \to \mathbf{Y}$ by

$$H(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right), \tag{51}$$

for all $x \in X$. By (46) and (45),

$$\begin{aligned}
&\left\| kH\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+kH\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}H(x_i)-2k\sum_{i=1}^{k}H(z_i)\right\|_{\mathbf{Y}} \\
&=\lim_{n\to\infty}2^n\left\| kf\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2^{n+1}k}+\sum_{i=1}^{k}\frac{z_i}{2^n}\right)+kf\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2^{n+1}k}+\sum_{i=1}^{k}\frac{z_i}{2^n}\right)\right. \\
&\quad\left.-\sum_{i=1}^{k}f\left(\frac{x_i}{2^n}\right)-2k\sum_{i=1}^{k}f\left(\frac{z_i}{2^n}\right)\right\|_{\mathbf{Y}} \\
&\le\lim_{n\to\infty}\frac{2^n\theta}{2^{3nkr}}\prod_{i=1}^{k}\|x_i\|_{\mathbf{X}}^r\cdot\prod_{i=1}^{k}\|y_i\|_{\mathbf{X}}^r\cdot\|z_1\|_{\mathbf{X}}^{rk}\cdot\left(1+\prod_{i=2}^{k}\|z_i\|_{\mathbf{X}}^r\right)=0
\end{aligned} \tag{52}$$

for all $x_i, y_i, z_i \in \mathbf{X}$, for all $i = 1 \to k$.

So

$$kH\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+kH\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}H(x_i)+2k\sum_{i=1}^{k}H(z_i) \tag{53}$$

for all $x_1, x_2, \cdots, x_{3k} \in X$. By lemma 5 (see [24]), the mapping $H : \mathbf{X} \to \mathbf{Y}$ is Cauchy additive. By proving as proof of the theorem of [3] the mapping $H : \mathbf{X} \to \mathbf{Y}$ is $\mathbb{R}$-linear. $\forall x, y \in \mathbf{X}$.

It follows from (46) that

$$\begin{aligned}
&\left\| f\left(\prod_{i=1}^{k}x_iy_i\right)-\prod_{i=1}^{k}f(x_i)\prod_{i=1}^{k}f(y_i)\right\|_{\mathbf{Y}} \\
&=\lim_{n\to\infty}2^{2nk}\left\| f\left(\prod_{i=1}^{k}\frac{x_i}{2^n}\cdot\frac{y_i}{2^n}\right)-\prod_{i=1}^{k}f\left(\frac{x_i}{2^n}\right)\cdot\prod_{i=1}^{k}f\left(\frac{y_i}{2^n}\right)\right\|_{\mathbf{Y}} \\
&\le\lim_{n\to\infty}\frac{2^{2nk}\theta}{2^{2nr}}\prod_{i=1}^{k}\|x_i\|_{\mathbf{X}}^r\cdot\prod_{i=1}^{k}\|y_i\|_{\mathbf{X}}^r=0
\end{aligned} \tag{54}$$

$\forall x, y \in \mathbf{X}$.

So

$$H\left(\prod_{i=1}^{k}x_i\cdot y_i\right)=\prod_{i=1}^{k}H(x_i)\prod_{i=1}^{k}H(y_i) \tag{55}$$

$\forall x, y \in \mathbf{X}$.

The rest of the proof is similar to the proof of Theorem 5 □

Theorem 8. Let $r < \frac{1}{q}$, with $q \ge 3$ and θ be positive real numbers, and $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that

$$\left\| kf\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+kf\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}f(x_i)-2k\sum_{i=1}^{k}f(z_i)\right\|_{\mathbf{Y}} \le \theta\cdot\prod_{i=1}^{k}\|x_i\|_{\mathbf{X}}^{r}\cdot\prod_{i=1}^{k}\|y_i\|_{\mathbf{X}}^{r}\cdot\|z_1\|_{\mathbf{X}}^{rk}\cdot\left(1+\prod_{i=2}^{k}\|z_i\|_{\mathbf{X}}^{r}\right) \quad (56)$$

$$\left\| f\left(\prod_{i=1}^{k}x_iy_i\right)-\prod_{i=1}^{k}f(x_i)\prod_{i=1}^{k}f(y_i)\right\|_{\mathbf{Y}} \le \theta\cdot\prod_{i=1}^{k}\|x_i\|_{\mathbf{X}}^{r}\cdot\prod_{i=1}^{k}\|y_i\|_{\mathbf{X}}^{r} \quad (57)$$

for all $x_i, y_i, z_i \in \mathbf{X}$, for all $i=1\to k$. If $f(tx)$ is continuous in $t\in\mathbb{R}$ each fixed $x\in\mathbf{X}$, then there exists a unique homomorphism $H:\mathbf{X}\to\mathbf{Y}$ such that

$$\|f(x)-H(x)\|_{\mathbf{Y}} \le \frac{\theta}{k\left(2^p-2^{3pkr}\right)^{\frac{1}{p}}}\|x\|_{\mathbf{X}}^{3kr}, \forall x\in\mathbf{X}. \quad (58)$$

Proof. I replace $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,x,\cdots,0)$ in (56), I have

$$\|kf(2x)-2kf(x)\|_{\mathbf{Y}} \le \theta\|x\|_{\mathbf{X}}^{3kr} \quad (59)$$

for all $x\in X$. So

$$\left\| f(x)-\frac{1}{2}f(2x)\right\|_{\mathbf{Y}} \le \frac{1}{2k}\theta\|x\|_{\mathbf{X}}^{3kr}$$

for all $x\in X$. Since $\mathbf{Y}$ is a p-Banach algebras,

$$\left\|\frac{1}{2^l}f\left(2^lx\right)-\frac{1}{2^m}f\left(2^mx\right)\right\|_{\mathbf{Y}}^{p} \le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^jx\right)-\frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\|_{\mathbf{Y}}^{p} \le \frac{\theta}{(2k)^p}\sum_{j=l+1}^{m-1}\frac{2^{3pkrj}}{2^{pj}}\|x\|_{\mathbf{X}}^{3pkr} \quad (60)$$

for all nonnegative integers m and l with $m>l$ and for all $x\in\mathbf{X}$. It follows (60) that the sequence $\left\{\frac{1}{2^n}f\left(2^nx\right)\right\}$ is a Cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{\frac{1}{2^n}f\left(2^nx\right)\right\}$ converges. So one can define the mapping $H:\mathbf{X}\to\mathbf{Y}$ by

$$H(x)=\lim_{n\to\infty}\frac{1}{2^n}f\left(2^nx\right)$$

for all $x\in X$. Moreover, letting $l=0$ and passing the $\lim_{m\to\infty}$ in (60) I have (56),

It follows from (57) that

$$\left\| f\left(\prod_{i=1}^{k}x_iy_i\right)-\prod_{i=1}^{k}f(x_i)\prod_{i=1}^{k}f(y_i)\right\|_{\mathbf{Y}} = \lim_{n\to\infty}\frac{1}{2^{2nk}}\left\| f\left(\prod_{i=1}^{k}2^nx_i\cdot2^ny_i\right)-\prod_{i=1}^{k}f\left(2^nx_i\right)\cdot\prod_{i=1}^{k}f\left(2^ny_i\right)\right\|_{\mathbf{Y}} \le \lim_{n\to\infty}\frac{\theta}{2^{2nkr}}\prod_{i=1}^{k}\|x_i\|_{\mathbf{X}}^{r}\cdot\prod_{i=1}^{k}\|y_i\|_{\mathbf{X}}^{r}=0$$

$\forall x, y \in \mathbf{X}$. So

$$H\left(\prod_{i=1}^{k} x_i \cdot y_i\right) = \prod_{i=1}^{k} H(x_i) \prod_{i=1}^{k} H(y_i)$$

$\forall x, y \in \mathbf{X}$.

The rest of the proof is similar to the proof of Theorem 5. □

4. Constructing Isomorphisms Based on Quasi-Banach Algebras

Now I construct isomorphisms for (1.2). Note that: (1.1) and (1.3) are also built exactly the same.

Here I assume that $\mathbb{A}$ is a quasi-Banach with norm $\|\cdot\|_{\mathbb{A}}$ and unit e and that $\mathbb{B}$ is a p-Banach algebra with norm $\|\cdot\|_{\mathbb{B}}$ and unit e'. Let $\mathbf{K}$ be the modulus of concavity of $\|\cdot\|_{\mathbb{B}}$. Under this setting, I can show that the mappings satisfying (1.2) is isomorphisms.

Theorem 9. Let $r > q$ with $q \ge 1$ and θ be positive real numbers, and $f: \mathbf{X} \to \mathbf{Y}$ be a mapping such that

$$\begin{aligned} &\left\| kf\left(\sum_{i=1}^{k} \frac{x_i + y_i}{2k} + \sum_{i=1}^{k} z_i\right) - kf\left(\sum_{i=1}^{k} \frac{x_i - y_i}{2k} + \sum_{i=1}^{k} z_i\right) - \sum_{i=1}^{k} f(x_i) \right\|_{\mathbf{Y}} \\ &\le \theta\left(\sum_{i=1}^{k} \|x_i\|_{\mathbf{X}}^r + \sum_{i=1}^{k} \|y_i\|_{\mathbf{X}}^r + \sum_{i=1}^{k} \|z_i\|_{\mathbf{X}}^r\right) \end{aligned} \tag{61}$$

$$f\left(\prod_{i=1}^{k} x_i y_i\right) = \prod_{i=1}^{k} f(x_i) \prod_{i=1}^{k} f(y_i) \tag{62}$$

for all $x_i, y_i, z_i \in \mathbf{X}$, for all $i = 1 \to k$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ each fixed $x \in \mathbf{X}$ and

$$\lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\right) = e' \tag{63}$$

then the mapping $f: \mathbf{X} \to \mathbf{Y}$ is an isomorphism.

Proof. I replace $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $(x, \cdots, x, x, \cdots, x, x, \cdots, 0)$ in (61), I have

$$\|kf(2x) - 2kf(x)\|_{\mathbf{Y}} \le (2k+1)\theta \|x\|_{\mathbf{X}}^r \tag{64}$$

for all $x \in X$. So

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\|_{\mathbf{Y}} \le \left(2 + \frac{1}{k}\right) \frac{\theta}{2^r} \|x\|_{\mathbf{X}}^r \tag{65}$$

for all $x \in X$. Since $\mathbf{Y}$ is a p-Banach algebra,

$$\begin{aligned} &\left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\|_{\mathbf{Y}}^p \\ &\le \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\|_{\mathbf{Y}}^p \le \left(2 + \frac{1}{k}\right)^p \frac{\theta^p}{2^{pr}} \sum_{j=l+1}^{m-1} \frac{2^{pj}}{2^{prj}} \|x\|_{\mathbf{X}}^{pr} \end{aligned} \tag{66}$$

for all nonnegative integers m and l with $m > l$ and for all $x \in \mathbf{X}$. It follows

(66) that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ converges. So one can define the mapping $H: \mathbf{X} \to \mathbf{Y}$ by

$$H(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right), \tag{67}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (66), I get

$$\left\| f(x) - H(x) \right\|_{\mathbf{Y}} \le \frac{\left(2 + \frac{1}{k}\right)\theta}{\left(2^{pr} - 2^p\right)^{\frac{1}{p}}} \|x\|_{\mathbf{X}}^r, \forall x \in \mathbf{X}. \tag{68}$$

It follows from (62) that

$$\begin{aligned} &\left\| kH\left(\sum_{i=1}^{k} \frac{x_i + y_i}{2k} + \sum_{i=1}^{k} z_i\right) - kH\left(\sum_{i=1}^{k} \frac{x_i - y_i}{2k} + \sum_{i=1}^{k} z_i\right) - \sum_{i=1}^{k} H(x_i) \right\|_{\mathbf{Y}} \\ &= \lim_{n\to\infty} 2^n \left\| kf\left(\sum_{i=1}^{k} \frac{x_i + y_i}{2^{n+1}k} + \sum_{i=1}^{k} \frac{z_i}{2^n}\right) + kf\left(\sum_{i=1}^{k} \frac{x_i - y_i}{2^{n+1}k} + \sum_{i=1}^{k} \frac{z_i}{2^n}\right) - \sum_{i=1}^{k} f\left(\frac{x_i}{2^n}\right) \right\|_{\mathbf{Y}} \\ &\le \lim_{n\to\infty} \frac{2^n \theta}{2^{nr}} \left(\sum_{i=1}^{k} \|x_i\|_{\mathbf{X}}^r + \sum_{i=1}^{k} \|y_i\|_{\mathbf{X}}^r + \sum_{i=1}^{k} \|z_i\|_{\mathbf{X}}^r \right) = 0 \end{aligned} \tag{69}$$

for all $x_i, y_i, z_i \in \mathbf{X}$, for all $i = 1 \to k$.

So

$$kH\left(\sum_{i=1}^{k} \frac{x_i + x_{k+i}}{2k} + \sum_{i=1}^{k} x_{2k+i}\right) - kH\left(\sum_{i=1}^{k} \frac{x_i - x_{k+i}}{2k} + \sum_{i=1}^{k} x_{2k+i}\right) = \sum_{i=1}^{k} H(x_i) \tag{70}$$

for all $x_i, y_i, z_i \in \mathbf{X}$, for all $i = 1 \to k$. By lemma 5 (see [24]), the mapping $H: \mathbf{X} \to \mathbf{Y}$ is Cauchy additive. See the theorem of [3].

The mapping $H: \mathbf{X} \to \mathbf{Y}$ is $\mathbb{R}$-linear. Since

$$f\left(\prod_{i=1}^{k} x_i y_i\right) = \prod_{i=1}^{k} f(x_i) \prod_{i=1}^{k} f(y_i) \tag{71}$$

for all $x_i, y_i, z_i \in \mathbf{X}$, for all $i = 1 \to k$.

$$\begin{aligned} H\left(\prod_{i=1}^{k} x_i y_i\right) &= \lim_{n\to\infty} 2^{2nk} f\left(\prod_{i=1}^{k} \frac{x_i}{2^n} \cdot \prod_{i=1}^{k} \frac{y_i}{2^n}\right) \\ &= \lim_{n\to\infty} 2^{nk} f\left(\prod_{i=1}^{k} \frac{x_i}{2^n}\right) \cdot 2^{nk} f\left(\prod_{i=1}^{k} \frac{y_i}{2^n}\right) \\ &= H\left(\prod_{i=1}^{k} x_i\right) \cdot H\left(\prod_{i=1}^{k} y_i\right) \end{aligned} \tag{72}$$

for all $x_i, y_i, z_i \in \mathbf{X}$, for all $i = 1 \to k$. So the mapping $H: \mathbf{X} \to \mathbf{Y}$ is homomorphism.

It follows from (62) that

$$\begin{aligned} H(x) = H(ex) &= \lim_{n\to\infty} 2^n f\left(\frac{ex}{2^n}\right) = \lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\cdot x\right) \\ &= \lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\right)\cdot f(x) = e'\cdot f(x) = f(x), \forall x\in \mathbf{X} \end{aligned} \tag{73}$$

$\forall x \in \mathbf{X}$. So the mapping $f:\mathbf{X}\to\mathbf{Y}$ is an isomorphism. □

Theorem 10. Let $r<q$ with $q\le 1$ and θ be positive real numbers, and $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that

$$\begin{aligned} &\left\| kf\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right) - kf\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right) - \sum_{i=1}^{k} f(x_i)\right\|_{\mathbf{Y}} \\ &\le \theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbf{X}}^r + \sum_{i=1}^{k}\|y_i\|_{\mathbf{X}}^r + \sum_{i=1}^{k}\|z_i\|_{\mathbf{X}}^r\right) \end{aligned} \tag{74}$$

$$f\left(\prod_{i=1}^{k} x_i y_i\right) = \prod_{i=1}^{k} f(x_i)\prod_{i=1}^{k} f(y_i) \tag{75}$$

for all $x_i, y_i, z_i \in \mathbf{X}$, for all $i=1\to k$. If $f(tx)$ is continuous in $t\in\mathbb{R}$ each fixed $x\in\mathbf{X}$ and

$$\lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\right) = e' \tag{76}$$

then the mapping $f:\mathbf{X}\to\mathbf{Y}$ is an isomorphism.

The rest of the proof is similar to the proof of Theorem 9.

Theorem 11. Let $r>\frac{1}{q}$ with $q\ge 3$ and θ be positive real numbers, and $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that

$$\begin{aligned} &\left\| kf\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right) - kf\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right) - \sum_{i=1}^{k} f(x_i)\right\|_{\mathbf{Y}} \\ &\le \theta\cdot\prod_{i=1}^{k}\|x_i\|_{\mathbf{X}}^r\cdot\prod_{i=1}^{k}\|y_i\|_{\mathbf{X}}^r\cdot\|z_1\|_{\mathbf{X}}^{rk}\cdot\left(1+\prod_{i=2}^{k}\|z_i\|_{\mathbf{X}}^r\right) \end{aligned} \tag{77}$$

$$f\left(\prod_{i=1}^{k} x_i y_i\right) = \prod_{i=1}^{k} f(x_i)\prod_{i=1}^{k} f(y_i) \tag{78}$$

for all $x_i, y_i, z_i \in \mathbf{X}$, for all $i=1\to k$. If $f(tx)$ is continuous in $t\in\mathbb{R}$ each fixed $x\in\mathbf{X}$ and

$$\lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\right) = e' \tag{79}$$

then the mapping $f:\mathbf{X}\to\mathbf{Y}$ is an isomorphism.

Proof. I replace $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,x,\cdots,0)$ in (77), I have

$$\|kf(2x)-2kf(x)\|_{\mathbf{Y}} \le \theta\|x\|_{\mathbf{X}}^{3kr} \tag{80}$$

for all $x\in X$. So

$$\left\| f(x)-2f\left(\frac{x}{2}\right)\right\|_{\mathbf{Y}} \le \frac{1}{k}\frac{\theta}{2^{3rk}}\|x\|_{\mathbf{X}}^{3kr} \tag{81}$$

for all $x \in X$. Hence

$$\begin{aligned} & \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\|_{\mathbf{Y}}^p \\ & \leq \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\|_{\mathbf{Y}}^p \leq \frac{1}{k^p} \frac{\theta^p}{2^{3krp}} \sum_{j=l+1}^{m-1} \frac{2^{pj}}{2^{pkrj}} \|x\|_{\mathbf{X}}^{3pkr} \end{aligned} \tag{82}$$

for all nonnegative integers m and l with $m > l$ and for all $x \in \mathbf{X}$. It follows (82) that the sequence $\left\{ 2^n f\left(\frac{x}{2^n}\right) \right\}$ is a Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{ 2^n f\left(\frac{x}{2^n}\right) \right\}$ converges. So one can define the mapping $H : \mathbf{X} \to \mathbf{Y}$ by

$$H(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right), \tag{83}$$

for all $x \in X$.

Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (82), I get

$$\| f(x) - H(x) \|_{\mathbf{Y}} \leq \frac{\theta}{k\left(2^{3pkr} - 2^p\right)^{\frac{1}{p}}} \|x\|_{\mathbf{X}}^{3kr}, \forall x \in \mathbf{X}. \tag{84}$$

The rest of the proof is similar to the proof of Theorems 7 and 9. □

Theorem 12. Let $r > \frac{1}{q}$ with $q \geq 3$ and θ be positive real numbers, and $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that

$$\begin{aligned} & \left\| kf\left(\sum_{i=1}^k \frac{x_i + y_i}{2k} + \sum_{i=1}^k z_i\right) - kf\left(\sum_{i=1}^k \frac{x_i - y_i}{2k} + \sum_{i=1}^k z_i\right) - \sum_{i=1}^k f(x_i) \right\|_{\mathbf{Y}} \\ & \leq \theta \cdot \prod_{i=1}^k \|x_i\|_{\mathbf{X}}^r \cdot \prod_{i=1}^k \|y_i\|_{\mathbf{X}}^r \cdot \|z_1\|_{\mathbf{X}}^{rk} \cdot \left(1 + \prod_{i=2}^k \|z_i\|_{\mathbf{X}}^r\right) \end{aligned} \tag{85}$$

$$f\left(\prod_{i=1}^k x_i y_i\right) = \prod_{i=1}^k f(x_i) \prod_{i=1}^k f(y_i) \tag{86}$$

for all $x_i, y_i, z_i \in \mathbf{X}$, for all $i = 1 \to k$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ each fixed $x \in \mathbf{X}$ and

$$\lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\right) = e' \tag{87}$$

then the mapping $f : \mathbf{X} \to \mathbf{Y}$ is an isomorphism.

The rest of the proof is similar to the proof of Theorems 8 and 9.

5. Constructing Derivatives on Quasi-Banach Algebras

Now I construct derivatives for (1.3). Note that: (1.1) and (1.2) are also built exactly the same.

Here I assume that, $\mathbb{A}$ is a p-Banach algebras with norm $\|\cdot\|_{\mathbb{A}}$ Let $\mathbf{K}$ be the modulus of concavity of $\|\cdot\|_{\mathbb{A}}$. Under this setting, I can show that the mappings satisfying (1.3) is generalized derivation.

A generalized derivations $\beta:\mathbf{X}\to\mathbf{X}$ is linear and fulfills the generalized Leibniz rue

$$\beta\left(\prod_{i=1}^{k}x_iy_iz_i\right)=\beta\left(\prod_{i=1}^{k}x_iy_i\right)\prod_{i=1}^{k}z_i-\prod_{i=1}^{k}x_i\beta\left(\prod_{i=1}^{k}y_i\right)\prod_{i=1}^{k}z_i+\prod_{i=1}^{k}x_i\beta\left(\prod_{i=1}^{k}y_iz_i\right) \tag{88}$$

for all $x_1,x_2,\cdots,x_{3k}\in\mathbf{X}$.

Theorem 13. Let $r>q$ with $q\geq 3$ and θ be positive real numbers, and $f:\mathbf{X}\to\mathbf{X}$ be a mapping such that

$$\begin{aligned}&\left\|2kf\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}f(x_i)-\sum_{i=1}^{k}f(y_i)-2k\sum_{i=1}^{k}f(z_i)\right\|_{\mathbf{Y}}\\&\leq\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbf{X}}^{r}+\sum_{i=1}^{k}\|y_i\|_{\mathbf{X}}^{r}+\sum_{i=1}^{k}\|z_i\|_{\mathbf{X}}^{r}\right)\end{aligned} \tag{89}$$

$$\begin{aligned}&\left\|\beta\left(\prod_{i=1}^{k}x_iy_iz_i\right)-\beta\left(\prod_{i=1}^{k}x_iy_i\right)\prod_{i=1}^{k}z_i+\prod_{i=1}^{k}x_i\beta\left(\prod_{i=1}^{k}y_i\right)\prod_{i=1}^{k}z_i-\prod_{i=1}^{k}x_i\beta\left(\prod_{i=1}^{k}y_iz_i\right)\right\|_{\mathbf{X}}\\&\leq\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbf{X}}^{r}+\sum_{i=1}^{k}\|y_i\|_{\mathbf{X}}^{r}+\sum_{i=1}^{k}\|z_i\|_{\mathbf{X}}^{r}\right)\end{aligned} \tag{90}$$

for all $x_i,y_i,z_i\in\mathbf{X}$, for all $i=1\to k$. If $f(tx)$ is continuous in $t\in\mathbb{R}$ each fixed $x\in\mathbf{X}$, then there exists a unique generalized derivation $\beta:\mathbf{X}\to\mathbf{X}$ such that

$$\|f(x)-\beta(x)\|_{\mathbf{X}}\leq\frac{\left(1+\frac{1}{2k}\right)\theta}{\left(2^{pr}-2^{p}\right)^{\frac{1}{p}}}\|x\|_{\mathbf{X}}^{r},\forall x\in\mathbf{X}. \tag{91}$$

Proof. I replace $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,x,\cdots,0)$ in (89), I have

$$\|2kf(2x)-4kf(x)\|_{\mathbf{X}}\leq(2k+1)\theta\|x\|_{\mathbf{X}}^{r} \tag{92}$$

for all $x\in X$. So

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\|_{\mathbf{X}}\leq\left(1+\frac{1}{2k}\right)\cdot\frac{\theta}{2^{r}}\|x\|_{\mathbf{X}}^{r}$$

for all $x\in X$. Since $\mathbf{X}$ is a p-Banach algebra,

$$\begin{aligned}&\left\|2^{l}f\left(\frac{x}{2^{l}}\right)-2^{m}f\left(\frac{x}{2^{m}}\right)\right\|_{\mathbf{Y}}^{p}\\&\leq\sum_{j=l}^{m-1}\left\|2^{j}f\left(\frac{x}{2^{j}}\right)-2^{j+1}f\left(\frac{x}{2^{j+1}}\right)\right\|_{\mathbf{Y}}^{p}\leq\left(1+\frac{1}{2k}\right)^{p}\cdot\frac{\theta^{p}}{2^{pr}}\sum_{j=l+1}^{m-1}\frac{2^{pj}}{2^{prj}}\|x\|_{\mathbf{X}}^{pr}\end{aligned} \tag{93}$$

for all nonnegative integers m and l with $m>l$ and for all $x\in\mathbf{X}$. It follows (93) that the sequence $\left\{2^{n}f\left(\frac{x}{2^{n}}\right)\right\}$ is a Cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{X}$ is complete, the sequence $\left\{2^{n}f\left(\frac{x}{2^{n}}\right)\right\}$ converges. So one can define the mapping $\beta:\mathbf{X}\to\mathbf{X}$ by

$$\beta(x)=\lim_{n\to\infty}2^n f\left(\frac{x}{2^n}\right)$$

for all $x\in\mathbf{X}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (93), I get (91). It follows from (89) that

$$\begin{aligned}&\left\|2k\beta\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\beta(x_i)-\sum_{i=1}^{k}\beta(y_i)-2k\sum_{i=1}^{k}\beta(z_i)\right\|_{\mathbf{X}}\\&=\lim_{n\to\infty}2^n\left\|2kf\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2^{n+1}k}+\sum_{i=1}^{k}\frac{z_i}{2^n}\right)-\sum_{i=1}^{k}f\left(\frac{x_i}{2^n}\right)-\sum_{i=1}^{k}f\left(\frac{y_i}{2^n}\right)-2k\sum_{i=1}^{k}f\left(\frac{z_i}{2^n}\right)\right\|_{\mathbf{X}}\\&\le\lim_{n\to\infty}\frac{2^n}{2^{nr}}\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbf{X}}^r+\sum_{i=1}^{k}\|y_i\|_{\mathbf{X}}^r+\sum_{i=1}^{k}\|z_i\|_{\mathbf{X}}^r\right)=0\end{aligned}$$

for all $x_i,y_i,z_i\in\mathbf{X}$, for all $i=1\to k$.

So

$$2k\beta\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\beta(x_i)-\sum_{i=1}^{k}\beta(y_i)-2k\sum_{i=1}^{k}\beta(z_i)=0$$

for all $x_i,y_i,z_i\in\mathbf{X}$, for all $i=1\to k$. By lemma 5 (see [24]), the mapping $\beta:\mathbf{X}\to\mathbf{X}$ is Cauchy additive. By the theorem of [3] the mapping $\beta:\mathbf{X}\to\mathbf{X}$ is $\mathbb{R}$-linear.

It follows from (90) that

$$\begin{aligned}&\left\|\beta\left(\prod_{i=1}^{k}x_iy_iz_i\right)-\beta\left(\prod_{i=1}^{k}x_iy_i\right)\prod_{i=1}^{k}z_i+\prod_{i=1}^{k}x_i\beta\left(\prod_{i=1}^{k}y_i\right)\prod_{i=1}^{k}z_i-\prod_{i=1}^{k}x_i\beta\left(\prod_{i=1}^{k}y_iz_i\right)\right\|_{\mathbf{X}}\\&=\lim_{n\to\infty}2^{3nk}\left\|f\left(\prod_{i=1}^{k}\frac{x_iy_iz_i}{2^{3nk}}\right)-f\left(\prod_{i=1}^{k}\frac{x_iy_i}{2^{2nk}}\right)\prod_{i=1}^{k}\frac{z_i}{2^{nk}}\right.\\&\left.+\prod_{i=1}^{k}\frac{x_i}{2^{nk}}f\left(\prod_{i=1}^{k}\frac{y_i}{2^{nk}}\right)\prod_{i=1}^{k}\frac{z_i}{2^{nk}}-\prod_{i=1}^{k}\frac{x_i}{2^{nk}}f\left(\prod_{i=1}^{k}\frac{x_{k+i}z_i}{2^{2nk}}\right)\right\|_{\mathbf{X}}\\&\le\lim_{n\to\infty}\theta\frac{2^{3nk}}{2^{nkr}}\left(\sum_{i=1}^{k}\|x_i\|_{\mathbf{X}}^r+\sum_{i=1}^{k}\|y_i\|_{\mathbf{X}}^r+\sum_{i=1}^{k}\|z_i\|_{\mathbf{X}}^r\right)=0\end{aligned}\tag{94}$$

for all $x_i,y_i,z_i\in\mathbf{X}$, for all $i=1\to k$. So

$$\beta\left(\prod_{i=1}^{k}x_iy_iz_i\right)=\beta\left(\prod_{i=1}^{k}x_iy_i\right)\prod_{i=1}^{k}z_i-\prod_{i=1}^{k}x_i\beta\left(\prod_{i=1}^{k}y_i\right)\prod_{i=1}^{k}z_i+\prod_{i=1}^{k}x_i\beta\left(\prod_{i=1}^{k}y_iz_i\right)\tag{95}$$

for all $x_i,y_i,z_i\in\mathbf{X}$, for all $i=1\to k$.

Now I prove the uniqueness of β. Assume that $\beta_1:\mathbf{X}\to\mathbf{X}$ is a Cauchy-Jensen additive mapping satisfying (91). Then I have

$$\begin{aligned}&\|\beta(x)-\beta_1(x)\|_{\mathbf{X}}=2^n\left\|\beta\left(\frac{1}{2^n}x\right)+\beta_1\left(\frac{1}{2^n}x\right)\right\|_{\mathbf{X}}\\&\le 2^n\mathbf{K}\left(\left\|\beta\left(\frac{1}{2^n}x\right)-f\left(\frac{1}{2^n}x\right)\right\|_{\mathbf{X}}+\left\|f\left(\frac{1}{2^n}x\right)+\beta_1\left(\frac{1}{2^n}x\right)\right\|_{\mathbf{X}}\right)\\&\le 2\frac{\left(1+\frac{1}{2k}\right)\mathbf{K}\theta}{\left(2^{pr}-2^p\right)^{\frac{1}{p}}}\cdot\frac{2^n}{2^{nr}}\|x\|_{\mathbf{X}}^r\end{aligned}$$

which tends to zero as $n\to\infty$ for all $x\in\mathbf{X}$. So I can conclude that $\beta(x)=\beta_1(x)$ for all $x\in\mathbf{X}$. This proves the uniqueness of β. Thus the mapping $\beta_1:\mathbf{X}\to\mathbf{X}$ is a unique generalized derivation satisfying (91). □

Theorem 14. Let $r<q$ with $q\le 1$ and θ be positive real numbers, and $f:\mathbf{X}\to\mathbf{X}$ be a mapping such that

$$\begin{aligned}&\left\|2kf\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}f(x_i)-\sum_{i=1}^{k}f(y_i)-2k\sum_{i=1}^{k}f(z_i)\right\|_{\mathbf{Y}}\\&\le\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbf{X}}^{r}+\sum_{i=1}^{k}\|y_i\|_{\mathbf{X}}^{r}+\sum_{i=1}^{k}\|z_i\|_{\mathbf{X}}^{r}\right)\end{aligned}\tag{96}$$

$$\begin{aligned}&\left\|\beta\left(\prod_{i=1}^{k}x_iy_iz_i\right)-\beta\left(\prod_{i=1}^{k}x_iy_i\right)\prod_{i=1}^{k}z_i+\prod_{i=1}^{k}x_i\beta\left(\prod_{i=1}^{k}y_i\right)\prod_{i=1}^{k}z_i-\prod_{i=1}^{k}x_i\beta\left(\prod_{i=1}^{k}y_iz_i\right)\right\|_{\mathbf{X}}\\&\le\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbf{X}}^{r}+\sum_{i=1}^{k}\|y_i\|_{\mathbf{X}}^{r}+\sum_{i=1}^{k}\|z_i\|_{\mathbf{X}}^{r}\right)\end{aligned}\tag{97}$$

for all $x_i,y_i,z_i\in\mathbf{X}$, for all $i=1\to k$. If $f(tx)$ is continuous in $t\in\mathbb{R}$ each fixed $x\in\mathbf{X}$, then there exists a unique generalized derivation $\beta:\mathbf{X}\to\mathbf{X}$ such that

$$\|f(x)-\beta(x)\|_{\mathbf{X}}\le\frac{\left(1+\frac{1}{2k}\right)\theta}{\left(2^{p}-2^{pr}\right)^{\frac{1}{p}}}\|x\|_{\mathbf{X}}^{r},\forall x\in\mathbf{X}.\tag{98}$$

The rest of the proof is similar to the proof of Theorem 13.

Theorem 15. Let $r>q$ with $q>1$ and θ be positive real numbers, and $f:\mathbf{X}\to\mathbf{X}$ be a mapping such that

$$\begin{aligned}&\left\|2kf\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}f(x_i)-\sum_{i=1}^{k}f(y_i)-2k\sum_{i=1}^{k}f(z_i)\right\|_{\mathbf{Y}}\\&\le\theta\cdot\prod_{i=1}^{k}\|x_i\|_{\mathbf{X}}^{r}\cdot\prod_{i=1}^{k}\|y_i\|_{\mathbf{X}}^{r}\cdot\|z_1\|_{\mathbf{X}}^{rk}\cdot\left(1+\prod_{i=2}^{k}\|z_i\|_{\mathbf{X}}^{r}\right)\end{aligned}\tag{99}$$

$$\begin{aligned}&\left\|\beta\left(\prod_{i=1}^{k}x_iy_iz_i\right)-\beta\left(\prod_{i=1}^{k}x_iy_i\right)\prod_{i=1}^{k}z_i+\prod_{i=1}^{k}x_i\beta\left(\prod_{i=1}^{k}y_i\right)\prod_{i=1}^{k}z_i-\prod_{i=1}^{k}x_i\beta\left(\prod_{i=1}^{k}y_iz_i\right)\right\|_{\mathbf{X}}\\&\le\theta\cdot\prod_{i=1}^{k}\|x_i\|_{\mathbf{X}}^{r}\cdot\prod_{i=1}^{k}\|y_i\|_{\mathbf{X}}^{r}\cdot\|z_1\|_{\mathbf{X}}^{rk}\cdot\left(1+\prod_{i=2}^{k}\|z_i\|_{\mathbf{X}}^{r}\right)\end{aligned}\tag{100}$$

for all $x_i,y_i,z_i\in\mathbf{X}$, for all $i=1\to k$. If $f(tx)$ is continuous in $t\in\mathbb{R}$ each fixed $x\in\mathbf{X}$, then there exists a unique generalized derivation $\beta:\mathbf{X}\to\mathbf{X}$ such that

$$\|f(x)-\beta(x)\|_{\mathbf{X}}\le\frac{\left(2+\frac{1}{k^{kr}}\right)\theta}{\left(2^{3pr}-2^{p}\right)^{\frac{1}{p}}}\|x\|_{\mathbf{X}}^{kr},\forall x\in\mathbf{X}.\tag{101}$$

Proof. I replace $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,x,\cdots,0)$ in (99), I have

$$\|2kf(2x)-4kf(x)\|_{\mathbf{X}}\le\theta\|x\|_{\mathbf{X}}^{3kr}\tag{102}$$

for all $x \in X$. So

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\|_{\mathbf{X}} \le \frac{1}{2k} \cdot \frac{\theta}{2^{3kr}} \|x\|_{\mathbf{X}}^{3kr}$$

for all $x \in X$. Since $\mathbf{X}$ is a p-Banach algebra,

$$\begin{aligned} &\left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\|_{\mathbf{Y}}^p \\ &\le \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\|_{\mathbf{Y}}^p \le \frac{1}{(2k)^p} \cdot \frac{\theta^p}{2^{3pkr}} \cdot \sum_{j=l+1}^{m-1} \frac{2^{pj}}{2^{3kprj}} \|x\|_{\mathbf{X}}^{3pkr} \end{aligned} \tag{103}$$

for all nonnegative integers m and l with $m > l$ and for all $x \in \mathbf{X}$. It follows (103) that the sequence $\left\{ 2^n f\left(\frac{x}{2^n}\right) \right\}$ is a Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{X}$ is complete, the sequence $\left\{ 2^n f\left(\frac{x}{2^n}\right) \right\}$ converges. So one can define the mapping $\beta : \mathbf{X} \to \mathbf{X}$ by

$$\beta(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in \mathbf{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (103), I get (101). It follows from (99) that

$$\begin{aligned} &\left\| 2k\beta\left(\sum_{i=1}^k \frac{x_i + y_i}{2k} + \sum_{i=1}^k z_i \right) - \sum_{i=1}^k \beta(x_i) - \sum_{i=1}^k \beta(y_i) - 2k \sum_{i=1}^k \beta(z_i) \right\|_{\mathbf{X}} \\ &= \lim_{n\to\infty} 2^n \left\| 2kf\left(\sum_{i=1}^k \frac{x_i + y_i}{2^{n+1}k} + \sum_{i=1}^k \frac{z_i}{2^n} \right) - \sum_{i=1}^k f\left(\frac{x_i}{2^n}\right) - \sum_{i=1}^k f\left(\frac{y_i}{2^n}\right) - 2k \sum_{i=1}^k f\left(\frac{z_i}{2^n}\right) \right\|_{\mathbf{X}} \\ &\le \lim_{n\to\infty} \theta \cdot \prod_{i=1}^k \|x_i\|_{\mathbf{X}}^r \cdot \prod_{i=1}^k \|y_i\|_{\mathbf{X}}^r \cdot \|z_1\|_{\mathbf{X}}^{rk} \cdot \left(1 + \prod_{i=2}^k \|z_i\|_{\mathbf{X}}^r\right) = 0 \end{aligned} \tag{104}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathbf{X}$.

So

$$2k\beta\left(\sum_{i=1}^k \frac{x_i + y_i}{2k} + \sum_{i=1}^k z_i \right) - \sum_{i=1}^k \beta(x_i) - \sum_{i=1}^k \beta(y_i) - 2k \sum_{i=1}^k \beta(z_i) = 0 \tag{105}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathbf{X}$. By lemma 5 (see [24]), the mapping $\beta : \mathbf{X} \to \mathbf{X}$ is Cauchy additive. By the theorem of [3] the mapping $\beta : \mathbf{X} \to \mathbf{X}$ is $\mathbb{R}$-linear.

It follows from (100) that

$$\begin{aligned} &\left\| \beta\left(\prod_{i=1}^k x_i y_i z_i \right) - \beta\left(\prod_{i=1}^k x_i y_i \right) \prod_{i=1}^k z_i + \prod_{i=1}^k x_i \beta\left(\prod_{i=1}^k y_i \right) \prod_{i=1}^k z_i - \prod_{i=1}^k x_i \beta\left(\prod_{i=1}^k y_i z_i \right) \right\|_{\mathbf{X}} \\ &= \lim_{n\to\infty} 2^{3nk} \left\| f\left(\prod_{i=1}^k \frac{x_i x_{k+i} x_{2k+i}}{2^{3nk}} \right) - f\left(\prod_{i=1}^k \frac{x_i x_{k+i}}{2^{2nk}} \right) \prod_{i=1}^k \frac{z_i}{2^{nk}} \right. \\ &\left. + \prod_{i=1}^k \frac{x_i}{2^{nk}} f\left(\prod_{i=1}^k \frac{y_i}{2^{nk}} \right) \prod_{i=1}^k \frac{z_i}{2^{nk}} - \prod_{i=1}^k \frac{x_i}{2^{nk}} f\left(\prod_{i=1}^k \frac{x_{k+i} x_{2k+i}}{2^{2nk}} \right) \right\|_{\mathbf{X}} \\ &\le \lim_{n\to\infty} 3k\theta \frac{2^{3nk}}{2^{nkr}} \left(\sum_{i=1}^k \|x_i\|_{\mathbf{X}}^r + \sum_{i=1}^k \|y_i\|_{\mathbf{X}}^r + \sum_{i=1}^k \|z_i\|_{\mathbf{X}}^r \right) = 0 \end{aligned} \tag{106}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{X}$. So

$$\beta\left(\prod_{i=1}^{k} x_i y_i z_i\right) = \beta\left(\prod_{i=1}^{k} x_i y_i\right)\prod_{i=1}^{k} z_i - \prod_{i=1}^{k} x_i \beta\left(\prod_{i=1}^{k} y_i\right)\prod_{i=1}^{k} z_i + \prod_{i=1}^{k} x_i \beta\left(\prod_{i=1}^{k} y_i z_i\right) \quad (107)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{X}$.

Now I prove the uniqueness of β. Assume that $\beta_1:\mathbf{X}\to\mathbf{X}$ is a Cauchy-Jensen additive mapping satisfying (101). Then I have

$$\begin{aligned}\left\|\beta(x)-\beta_1(x)\right\|_{\mathbf{X}} &= 2^n\left\|\beta\left(\frac{1}{2^n}x\right)+\beta_1\left(\frac{1}{2^n}x\right)\right\|_{\mathbf{X}}\\ &\le 2^n\mathbf{K}\left(\left\|\beta\left(\frac{1}{2^n}x\right)-f\left(\frac{1}{2^n}x\right)\right\|_{\mathbf{X}}+\left\|f\left(\frac{1}{2^n}x\right)+\beta_1\left(\frac{1}{2^n}x\right)\right\|_{\mathbf{X}}\right)\\ &\le 2\frac{\left(2k+\frac{1}{k^{r-1}}\right)\mathbf{K}\theta}{\left(2^{pr}-2^p\right)^{\frac{1}{p}}}\cdot\frac{2^n}{2^{nr}}\|x\|_{\mathbf{X}}^r\end{aligned} \quad (108)$$

which tends to zero as $n\to\infty$ for all $x\in\mathbf{X}$. So I can conclude that $\beta(x)=\beta_1(x)$ for all $x\in\mathbf{X}$. This proves the uniqueness of β. Thus the mapping $\beta_1:\mathbf{X}\to\mathbf{X}$ is a unique generalized derivation satisfying (101). □

6. Conclusion

In this paper, I construct extensions of homomorphisms, isomorphisms, and derivatives based on Banach algebra. The fundamental contribution here is the development of a general Cauchy-Jensen equation, which serves as the cornerstone for establishing mathematical links across various research areas in mathematics, without any restrictions on generality.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Ulam, S.M. (1960) A Collection of the Mathematical Problems. Interscience, New York.

[2] Hyers, D.H. (1941) On the Stability of the Linear Functional Equation. *Proceedings of the National Academy of Sciences of the United States of America*, **27**, 222-224. https://doi.org/10.1073/pnas.27.4.222

[3] Rassias, T.M. (1978) On the Stability of the Linear Mapping in Banach Spaces. *Proceedings of the American Mathematical Society*, **72**, 297-300. https://doi.org/10.1090/S0002-9939-1978-0507327-1

[4] Rassias, J.M. (1982) On Approximation of Approximately Linear Mappings by Linear Mappings. *Journal of Functional Analysis*, **46**, 126-130. https://doi.org/10.1016/0022-1236(82)90048-9

[5] Rassias, J.M. (1984) On Approximation of Approximately Linear Mappings by Linear Mappings. *Bulletin des Sciences Mathématiques*, **108**, 445-446.

[6] Rassias, J.M. (1989) Solution of a Problem of Ulam. *Journal of Approximation Theory*, **57**, 268-273. https://doi.org/10.1016/0021-9045(89)90041-5

[7] Găvruta, P. (1994) A Generalization of the Hyers-Ulam-Rassias Stability of Approximately Additive Mappings. *Journal of Mathematical Analysis and Applications*, **184**, 431-436. https://doi.org/10.1006/jmaa.1994.1211

[8] An, L.V. (2021) Generalized Hyers-Ulam-Rassias Type Stability of the Isometric Additive Mapping in Quasi-Banach Spaces. *International Journal of Mathematics Trends and Technology* (*IJMTT*), **67**, 31-45. https://doi.org/10.14445/22315373/IJMTT-V67I9P505

[9] An, L.V. (2021) Generalized Hyers-Ulam-Rassias Type Stability of the Homomrphisms in Quasi-Banach Spaces. *Bulletin of Mathematics and Statistics Research*, **9**, 29-43. http://bomsr.com/9.3.21/29-43%20LY%20VAN%20AN.pdf

[10] Baak, C. (2006) Cauchy-Rassias Stability of Cauchy-Jensen Additive Mappings in Banach Spaces. *Acta Mathematica Sinica*, **22**, 1789-1796. https://doi.org/10.1007/s10114-005-0697-z

[11] An, L.V. (2022) Generalized Approximation Hyers-Ulam-Rassias Type Stability of Generalized Homomorphisms in Quasi-Banach Algebras. *Asia Mathematika*, **6**, 7-19. https://www.asiamath.org/

[12] An, L.V. (2020) Generalized Hyers-Ulam Type Stability of the Additive Functional Equation Inequalities with 2n-Variables on an Approximate Group and Ring Homomorphism. *Asia Mathematika*, **4**, 161-175. http://www.asiamath.org/

[13] Rolewicz, S. (1984) Metric Linear Spaces. Polish Scientific Publishers, Dordrecht.

[14] Benyamini, Y. and Lindenstrauss, J. (2000) Geometric Nonlinear Functional Analysis: Volume 1. In: *Colloquium Publications*, Vol. 48, American Mathematical Society Colloquium Publications, Providence. https://doi.org/10.1090/coll/048

[15] Aoki, T. (1950) On the Stability of the Linear Transformation in Banach Space. *Journal of the Mathematical Society of Japan*, **2**, 64-66. https://doi.org/10.2969/jmsj/00210064

[16] Bahyrycz, A. and Piszczek, M. (2014) Hyperstability of the Jensen Functional Equation. *Acta Mathematica Hungarica*, **142**, 353-365. https://doi.org/10.1007/s10474-013-0347-3

[17] Boo, D.-H., Oh, S.-Q., Park, C.-G. and Park, J.-M. (2003) Generalized Jensen's Equations in Banach Modules over a C*-Algebra and Its Unitary Group. *Taiwanese Journal of Mathematics*, **7**, 641-655. https://doi.org/10.11650/twjm/1500407583

[18] Elhoucien, E. and Youssef, M. (2012) On the Paper by A. Najati and S.-M. Jung: The Hyers-Ulam Stability of Approximately Quadratic Mapping on Restricted Domains. *Journal of Nonlinear Analysis and Application*, **2012**, Article ID: Jnaa-00127. https://doi.org/10.5899/2012/jnaa-00127

[19] Park, C.-G. (2002) On the Stability of the Linear Mapping in Banach Modules. *Journal of Mathematical Analysis and Applications*, **275**, 711-720. https://doi.org/10.1016/S0022-247X(02)00386-4

[20] Park, C.-G. (2004) Lie *-Homomorphisms between Lie C*-Algebras and Lie *-Derivations on Lie C*-Algebras. *Journal of Mathematical Analysis and Applications*, **293**, 419-434. https://doi.org/10.1016/j.jmaa.2003.10.051

[21] Park, C.-G. (2005) Homomorphisms between Poisson JC*-Algebras. *Bulletin of the Brazilian Mathematical Society*, **36**, 79-97. https://doi.org/10.1007/s00574-005-0029-z

[22] Park, C.-G. (2006) Completion of Quasi-Normed Algebras and Quasi-Normed Modules. *Journal of the Chungcheong Mathematical Society*, **19**, 9-18.

[23] Park, C.-G. (2006) Hyers-Ulam-Rassias Stability of a Generalized Euler-Lagrange

Type Additive Mapping and Isomorphisms between C*-Algebras. *Bulletin of the Belgian Mathematical Society-Simon Stevin*, **13**, 619-631. https://doi.org/10.36045/bbms/1168957339

[24] An, L.V. (2023) Construc the General Jensen-Cauchy Equations in Banach Space and Using Fixed Point Method to Establish Homomorphisms in Banach Algebras. *Open Access Library Journal*, **10**, e9931. https://doi.org/10.4236/oalib.1109931