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Generalized Stability of the Quadratic Type λ-Functional Equation with $3k$-Variables in Non-Archimedean Banach Space and Non-Archimedean Random Normed Space

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Abstract

In this paper, we study to solve the quadratic type λ-functional equation with $3k$ variables. First, we investigated in non-Archimedean Banach spaces with a fixed point method, next, we investigated in non-Archimedean Banach spaces with a direct method and finally we do research in non-Archimedean random spaces. I will show that the solutions of the quadratic type λ-functional equation are quadratic type mappings. These are the main results of this paper.

Subject Areas

Mathematics

Keywords

Quadratic λ-Functional Equation, Non-Archimedean Normed Space, Non-Archimedean Banach Space, Fixed Point Method, Direct Method, Hyers-Ulam Stability, Random Normed Spaces, Non-Archimedean Random Normed Space

1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ be a normed spaces on the same field $\mathbb{K}$, and $f:\mathbf{X}\to\mathbf{Y}$. We use the notation $\|\cdot\|$ for all the norm on both $\mathbf{X}$ and $\mathbf{Y}$. In this paper, I study and expand the λ-function equation from non-Archimedean normed space to non-Archimedean random normed space.

In fact, when $\mathbf{X}$ is non-Archimedean normed space and $\mathbf{Y}$ is non-Archimedean Banach spaces.

Or $\mathbf{X}$ is a vector over field $\mathbb{K}$ and $(\mathbf{Y},\Gamma,T)$ be a non-Archimedean random Banach space over field $\mathbb{K}$. We solve and prove the Hyers-Ulam-Rassisa type stability of forllowing quadratic λ-functional equation.

$$\begin{aligned}&2\sum_{j=1}^{k} f(z_j)+2\sum_{j=1}^{k} f(x_j+y_j)\\&=f\left(\sum_{j=1}^{k} x+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j\right)+\lambda^{-2m} f\left(\lambda^m\left(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j-\sum_{j=1}^{k} z_j\right)\right)\end{aligned} \tag{1}$$

where: Let $|2k|\neq 1$, λ is a fixed non-Archimedean number with $\lambda^{-2m}\neq 4k-1$ and k,m is a positive integer. The notions of non-Archimedean normed space and non-Archimedean Banach spaces and non-Archimedean random Banach space over field $\mathbb{K}$ will remind in the next section. The study the stability of generalized stability of the quadratic type λ-functional equation with variables in non-Archimedean Banach space and non-Archimedean Random normed space originated from a question of S.M. Ulam [1], concerning the stability of group homomorphisms. Let $(\mathbf{G},*)$ be a group and let $(\mathbf{G}',\circ,d)$ be a metric group with metric $d(\cdot,\cdot)$. Geven $\varepsilon>0$, does there exist a $\delta>0$ such that if $f:\mathbf{G}\to\mathbf{G}'$ satisfies

$$d\left(f(x*y),f(x)\circ f(y)\right)<\delta,\forall x\in\mathbf{G}$$

then there is a homomorphism $h:\mathbf{G}\to\mathbf{G}'$ with

$$d\left(f(x),h(x)\right)<\varepsilon,\forall x\in\mathbf{G}$$

The Hyers [2] gave firts affirmative partial answer to the equation of Ulam in Banach spaces. After that, Hyers' Theorem was generalized by Aoki [3] additive mappings and by Rassias [4] for linear mappings considering an unbouned Cauchy difference. Gajda following the same approach as in Rassias gave an affirmative solution to this question for $p>1$. It was shown by Gajda [5], as well as by Rassias and Semr [6] that one cannot prove a Rassias, type theorem when $p=1$. The counterexamples of Gajda, as well as of Rassias and Semr have stimulated several matematicians to invent new definition of approximately additive or approximately linear mappings, was obtained by Găvruta [7].

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping.

The functonal equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic functional mapping.

The stability the quadratic functional equation was proved by Skof [8] for mappings $f:E_1\to E_2$, where E_1 is a normed space and E_2 is a Banach space.

Recently the author studied the Hyers-Ulam stability for the following α-functional equation.

$$2f(x)+2f(y)=f(x+y)+\alpha^{-2}f(\alpha(x-y))$$

in Non-Archimedean Banach spaces and non-Archimedean Random normed space.

In this paper, we solve and proved the Hyers-Ulam stability for λ-functional Equation (1.1), *i.e.* the λ-functional equation with $3k$-variables. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, we will prove that the mappings satisfying the λ-functional Equation (1.1). Thus, the results in this paper are generalization of those in [9] for λ-functional equation with $3k$-variables.

In this paper, based on the work of world mathematicians [1]-[33], I introduce a new generalized quadratic function equation with $3k$-variables to improve the classical form, which is a new breakthrough for the development of this field functional equation.

The paper is organized as followns: In section preliminarier we remind some basic notations in [10] [11] [12] [13] [14] such as non-Archimedean field, Non-Archimedean normed space and non-Archimedean Banach space, Random normed spaces, Non-Archimedean random normed space.

Section 3: Establishing the solution for (1.1) by the fixed point method in Non-Archimedean Banach space.

+ Condition for existence of solutions for Equation (1.1)

+ Constructing a solution for (1.1).

Section 4: Establishing the solution for (1.1) by the direct method in Non-Archimedean Banach space

Section 5: Construct a solution for (1.1) on non-Archimedean Random normed space.

2. Preliminaries

2.1. Non-Archimedean Normed and Banach Spaces

A valuation is a function $|\cdot|$ from a field $\mathbb{K}$ into $[0,\infty)$ such that 0 is the unique element having the 0 valuation,

$$|r\cdot s|=|r|\cdot|s|,\forall r,s\in\mathbb{K}$$

and the triangle inequality holds, *i.e.*;

$$|r+s|\le|r|+|s|,\forall r,s\in\mathbb{K}$$

A field $\mathbb{K}$ is called a valued filed if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuation. Let us consider a vavluation which satisfies a stronger condition than the triangle inaquality. If the tri triangle inequality is replaced by

$$|r+s|\le\max\{|r|,|s|\},\forall r,s\in\mathbb{K}$$

then the function $|\cdot|$ is called a norm-Archimedean valuational, and filed. Clearly $|1|=|-1|=1$ and $|n|\le 1,\forall n\in N$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ talking everything except for 0 into 1 and $|0|=0$ this paper, we assume that the base field is a non-Archimedean filed, hence call it simply a filed. Let be a vecter space over a filed $\mathbb{K}$ with a non-Archimedean $|\cdot|$. A function $\|\cdot\|: X \to [0,\infty)$ is said a non-Archimedean norm if it satisfies the follwing conditions:

1) $\|x\|=0$ if and only if $x=0$;

2) $\|rx\|=|r|\|x\|(r \in \mathbb{K}, x \in X)$;

3) the strong triangle inequlity

$$\|x+y\| \le \max\{\|x\|,\|y\|\}, x, y \in X$$

hold. Then $(X,\|\cdot\|)$ is called a norm-Archimedean norm space.

1) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space *X*. Then sequence $\{x_n\}$ is called cauchy if for a given $\varepsilon > 0$ there a positive integer N such that

$$\|x_n - x\| \le \varepsilon$$

for all $n,m \ge N$

2) Let $\{x_n\}$ be a sequence in a norm-Archimedean normed space *X*. Then sequence $\{x_n\}$ is called cauchy if for a given $\varepsilon > 0$ there a positive integer N such that

$$\|x_n - x\| \le \varepsilon$$

for all $n,m \ge N$. The we call $x \in X$ a limit of sequence x_n and denote $\lim_{n\to\infty} x_n = x$.

3) If every sequence Cauchy in *X* converger, then the norm-Archimedean normed space *X* is called a norm-Archimedean Bnanch space.

2.2. Random Normed Spaces

A random normed space is triple $(\mathbf{X},\Gamma,T)$, where $\mathbf{X}$ is a vector space, *T* is a is a continuous t-norm, and Γ is a mapping from $\mathbf{X}$ into $\mathbf{D}^+$ such that, the following conditions hold:

1) (RN$_1$) $\Gamma_x(t)$ for all $t>0$ if and only if $x=0$;

2) (RN$_2$) $\Gamma_{\alpha x}(t)=\Gamma_x\left(\frac{t}{|\alpha|}\right)$ for all $x \in \mathbf{X}$, $\alpha \ne 0$;

3) (RN$_3$) $\Gamma_{x+y}(t+s) \ge T(\Gamma_x(t),\Gamma_y(s))$ for all $x, y \in \mathbf{X}$, $t, s \ge 0$;

Note: If $(\mathbf{X},\Gamma,T)$ is a random normed space an $\{x_n\}$ is a sequence such that $x_n \to x$ then $\lim_{n\to\infty}\Gamma_{x_n}(t)=\Gamma_x(t)$ almost everywhere.

2.3. Non-Archimedean Random Normed Space

A non-Archimedean random normed space is triple $(\mathbf{X},\Gamma,T)$, where $\mathbf{X}$ is a linear space over a non-Archimedean filed $\mathbb{K}$, *T* is a is a continuous t-norm, and Γ is a mapping from $\mathbf{X}$ into $\mathbf{D}^+$ such that, the following conditions hold:

1) (NA-RN$_1$) $\Gamma_x(t)=\varepsilon_0(t)$ for all $t>0$ if and only if $x=0$;

2) (NA-RN$_2$) $\Gamma_{\alpha x}(t)=\Gamma_x\left(\frac{t}{|\alpha|}\right)$ for all $x\in\mathbf{X}$, $t>0$, $\alpha\neq 0$;

3) (NA-RN$_3$) $\Gamma_{x+y}(\max\{t,s\})\geq T(\Gamma_x(t),\Gamma_y(s))$ for all $x,y\in\mathbf{X}$, $t,s\geq 0$;

It is easy to see that if (NA-RN$_3$) hold then so is (RN$_3$)
$\Gamma_{x+y}(\max\{t,s\})\geq T(\Gamma_x(t),\Gamma_y(s))$

Let $(\mathbf{X},\Gamma,T)$ is a non-Archimedean random normed space. Suppose $\{x_n\}$ is a sequence in $\mathbf{X}$. Then $\{x_n\}$ is said to be convergent if there exists $x\in\mathbf{X}$ such that

$$\lim_{n\to\infty}\Gamma_{x_n-x}(t)=1$$

for all $t>0$. In that case, x is called the limit of sequence $\{x_n\}$

Theorem 1. Let (X,d) be a complete generalized metric space and let $J:X\to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in X$, either

$$d(J^n,J^{n+1})=\infty$$

for all nonegative integers n or there exists a positive integer n_0 such that

1) $d(J^n,J^{n+1})<\infty$, $\forall n\geq n_0$;

2) The sequence $\{J^n x\}$ converges to a fixed point y^* of J;

3) y^* is the unique fixed point of J in the set $Y=\{y\in X\mid d(J^n,J^{n+1})<\infty\}$;

4) $d(y,y^*)\leq\frac{1}{1-l}d(y,Jy)$ $\forall y\in Y$

2.4. Solutions of the Equation

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the cauchuy equation. In particular, every solution of the cauchuy equation is said to be an *additive mapping*.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called the quadratic functional equation In particular, every solution of the quadratic functional equation is said to be an *quadratic mapping*.

The functional equation

$$2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=f(x)+f(y)$$

is called a *Jensen* type the quadratic functional equation

3. Establishing the of (1.1) in Non-Archimedean Banach Space

3.1. Condition for Existence of Solutions for Equation (1.1)

Note that for Quadratic λ-functional equation, $\mathbb{X}$ and $\mathbb{Y}$ is be vector space.

Lemma 2. Suppose $\mathbb{X}$ and $\mathbb{Y}$ be vector space. If mapping $f:\mathbb{X}\to\mathbb{Y}$ satisfying

$$
\begin{aligned}
&2\sum_{j=1}^{k} f\left(z_j\right)+2\sum_{j=1}^{k} f\left(x_j+y_j\right) \\
&= f\left(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j\right)+\lambda^{-2m} f\left(\lambda^{m}\left(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j-\sum_{j=1}^{k} z_j\right)\right)
\end{aligned} \tag{2}
$$

for all $x_j, y_j, z_j \in \mathbb{X}$ for all $j = 1 \to k$ then $f : \mathbb{X} \to \mathbb{Y}$ is quadratic type

Proof. Assume that $f : \mathbb{X} \to \mathbb{Y}$ satisfies (2)

We replacing $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(0,\cdots,0,0,\cdots,0,0,\cdots,0\right)$ in (2), we get

$$
(4k-1) f(0) = \lambda^{-2m} f(0) \tag{3}
$$

So $f(0)=0$.

Next we replacing $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(x,0,\cdots,0,0,\cdots,0,0,\cdots,0\right)$ in (2), we have

$$
f(x) = \lambda^{-2m} f\left(\lambda^{m} x\right) \tag{4}
$$

and so $f\left(\lambda^m x\right)=\lambda^{2m} f(x)$ for all $x \in \mathbb{X}$. Thus from (2)

$$
\begin{aligned}
&2\sum_{j=1}^{k} f\left(z_j\right)+2\sum_{j=1}^{k} f\left(x_j+y_j\right) \\
&= f\left(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j\right)+\lambda^{-2m} f\left(\lambda^{m}\left(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j-\sum_{j=1}^{k} z_j\right)\right) \\
&= f\left(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j\right)+ f\left(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j-\sum_{j=1}^{k} z_j\right)
\end{aligned} \tag{5}
$$

for all $x_j, y_j, z_j \in \mathbb{X}$ for all $j = 1 \to k$

Next now we replacing $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(x,0,\cdots,0,0,\cdots,0,x,\cdots,0\right)$ in (2), we have

$$
f(2x) = 2^2 f(x) \tag{6}
$$

for all $v \in \mathbb{X}$.

Next we replace *x* by 2*x*, we get

$$
f\left(2^2 x\right) = 2^4 f(x) \tag{7}
$$

for all $x \in \mathbb{X}$. for all $x \in \mathbb{X}$, So from (6) and (7) we have the general case for every *m* being a positive integer, we have

$$
f\left(2^m x\right) = 2^{2m} f(x) \tag{8}
$$

for all $x \in \mathbb{X}$, So we get the desired result.

Notice now we replacing $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(x,\cdots,0,0,\cdots,0,y,\cdots,0\right)$ in (5) we have

$$
f(x+y)+f(x-y) = 2f(x)+2f(y)
$$

So, the function *f* is quadratic. □

3.2. Constructing a Solution for (1.1)

Now, we first study the solutions of (1.1). Note that for Quadratic λ-functional equation, $\mathbb{X}$ is a non-Archimedean normed space and $\mathbb{Y}$ is a non-Archimedean Banach spacebe then use fixed point method, we prove the Hyers-Ulam stability of the Quadratic λ-functional equation in Non-Archimedean Banach space. Under this setting, we can show that the mapping satisfying (1.1) is quadratic. These results are give in the following.

Theorem 3. Suppose $\varphi:\mathbb{X}^{3k}\to[0,\infty)$ be a function such that there exists an $0<L<1$ with

$$\begin{aligned}&\varphi\left(\frac{x_1}{2k},\frac{x_2}{2k},\cdots,\frac{x_k}{2k},\frac{y_1}{2k},\frac{y_2}{2k},\cdots,\frac{y_k}{2k},\frac{z_1}{2k},\frac{z_2}{2k},\cdots,\frac{z_k}{2k}\right)\\&\le\frac{L}{|4k|}\varphi\left(x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_k,z_1,z_2,\cdots,z_k\right)\end{aligned}\tag{9}$$

for all $x_j,y_j,z_j\in\mathbb{X}$, for all $j=1\to k$. Let $f:\mathbb{X}\to\mathbb{Y}$ be a mapping satisfying $f(0)=0$ and

$$\begin{aligned}&\left\|2\sum_{j=1}^{k}f(z_j)+2\sum_{j=1}^{k}f(x_j+y_j)-f\left(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j\right)\right.\\&\left.-\lambda^{-2m}f\left(\lambda^m\left(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j-\sum_{j=1}^{k}z_j\right)\right)\right\|\\&\le\varphi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)\end{aligned}\tag{10}$$

for all $x_j,y_j,z_j\in\mathbb{X}$, for all $j=1\to k$. Then there exists a unique quadratic type mapping $H:\mathbb{X}\to\mathbb{Y}$ such that

$$\|f(x)-H(x)\|\le\frac{L}{|4k|(1-L)}\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,x)\tag{11}$$

for all $x\in\mathbb{X}$.

Proof. We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,0,\cdots,0,x,\cdots,x)$ in (10), we get

$$\|f(2kx)-4kf(x)\|\le\varphi(x,\cdots,x,0,\cdots,0,x,\cdots,x)\tag{12}$$

for all $x\in\mathbb{X}$ for all $j=1\to k$.

Now we consider the set

$$\mathbb{M}:=\{h:\mathbb{X}\to\mathbb{Y},h(0)=0\}$$

and introduce the generalized metric on S as follows:

$$d(g,h):=\inf\left\{\beta\in\mathbb{R}:\|g(x)-h(x)\|\le\beta\varphi(x,\cdots,x,0,\cdots,0,x,\cdots,x),\forall x\in\mathbb{X}\right\},$$

where, as usual, $\inf\phi=+\infty$. That has been proven by mathematicians $(\mathbb{M},d)$ is complete see [14]

Now we cosider the linear mapping $T:\mathbb{M}\to\mathbb{M}$ such that

$$Tg(x):=4kg\left(\frac{x}{2k}\right)$$

for all $x \in \mathbb{X}$. Let $g, h \in \mathbb{M}$ be given such that $d(g,h) = \varepsilon$ then

$$\|g(x) - h(x)\| \le \varepsilon\varphi(x, \cdots, x, 0, \cdots, 0, x, \cdots, x)$$

for all $x \in \mathbb{X}$.

Hence

$$\begin{aligned}\|Tg(x) - Th(x)\| &= \left\|4kg\left(\frac{x}{2k}\right) - 4khf\left(\frac{x}{2k}\right)\right\| \\ &\le |4k|\varepsilon\varphi\left(\frac{x}{2k}, \frac{x}{2k}, \cdots, \frac{x}{2k}, 0, 0, \cdots, 0, \frac{x}{2k}, \frac{x}{2k}, \cdots, \frac{x}{2k}\right) \\ &\le |4k|\varepsilon\frac{L}{|4k|}\varphi(x, x, \cdots, x, 0, 0, \cdots, 0, x, x, \cdots, x) \\ &\le L\varepsilon\varphi(x, x, \cdots, x, 0, 0, \cdots, 0, x, x, \cdots, x)\end{aligned}$$

for all $x \in \mathbb{X}$. So $d(g,h) = \varepsilon$ implies that $d(Tg, Th) \le L \cdot \varepsilon$. This means that

$$d(Tg, Th) \le Ld(g,h)$$

for all $g, h \in \mathbb{M}$. It folows from (12) that

$$\begin{aligned}\left\|f(x) - 4kf\left(\frac{x}{2k}\right)\right\| &\le \varphi\left(\frac{x}{2k}, \frac{x}{2k}, \cdots, \frac{x}{2k}, 0, 0, \cdots, 0, \frac{x}{2k}, \frac{x}{2k}, \cdots, \frac{x}{2k}\right) \\ &\le \frac{L}{|4k|}\varphi(x, x, \cdots, x, 0, 0, \cdots, 0, x, x, \cdots, x)\end{aligned}$$

for all $x \in \mathbb{X}$. So $d(f, Tf) \le \frac{L}{|4k|}$ for all $x \in \mathbb{X}$ By Theorem 1.2, there exists a mapping $H : \mathbb{X} \to \mathbb{Y}$ satisfying the fllowing:

1) H is a fixed point of T, *i.e.*,

$$H(x) = 4kH\left(\frac{x}{2k}\right) \tag{13}$$

for all $x \in \mathbb{X}$. The mapping H is a unique fixed point T in the set

$$\mathbb{Q} = \{g \in \mathbb{M} : d(f,g) < \infty\}$$

This implies that H is a unique mapping satisfying (13) such that there exists a $\beta \in (0, \infty)$ satisfying

$$\|f(x) - H(x)\| \le \beta\varphi(x, x, \cdots, x, 0, 0, \cdots, 0, x, x, \cdots, x)$$

for all $x \in \mathbb{X}$

2) $d(T^l f, H) \to 0$ as $l \to \infty$. This implies equality

$$\lim_{l\to\infty}(4k)^n f\left(\frac{x}{(2k)^n}\right) = H(x)$$

for all $x \in \mathbb{X}$

3) $d(f, H) \le \frac{1}{1-L}d(f, Tf)$. Which implies

$$\|f(x) - H(x)\| \le \frac{L}{|4k|(1-L)}\varphi(x, x, \cdots, x, 0, 0, \cdots, 0, x, x, \cdots, x)$$

for all $x \in \mathbb{X}$. It follows (9) and (10) that

$$\begin{aligned}&\left\| 2\sum_{j=1}^{k} H\left(x_j + y_j\right) + 2\sum_{j=1}^{k} H\left(z_j\right) - H\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\right)\right.\\ &\left. -\lambda^{-2m} H\left(\lambda^m\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j - \sum_{j=1}^{k} z_j\right)\right)\right\|\\ &= \lim_{n\to\infty} |4k|^n \left\| 2\sum_{j=1}^{k} f\left(\frac{x_j + y_j}{(2k)^n}\right) + 2\sum_{j=1}^{k} f\left(\frac{z_j}{(2k)^n}\right)\right.\\ &\quad - f\left(\frac{\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j}{(2k)^n}\right)\\ &\quad \left. -\lambda^{-2m} f\left(\lambda^m\left(\frac{\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j}{(2k)^n}\right)\right)\right\|\\ &\le \lim_{n\to\infty} |4k|^n \varphi\left(\frac{x_1}{(2k)^n}, \frac{x_2}{(2k)^n}, \cdots, \frac{x_k}{(2k)^n}, \frac{y_1}{(2k)^n}, \frac{y_2}{(2k)^n}, \cdots, \frac{y_k}{(2k)^n},\right.\\ &\quad \left. \frac{z_1}{(2k)^n}, \frac{z_2}{(2k)^n}, \cdots, \frac{z_k}{(2k)^n}\right)\\ &= 0\end{aligned}$$

for all $x_j, y_j, z_j \in \mathbb{X}$ for all $j \to k$. So

$$\begin{aligned}&2\sum_{j=1}^{k} H\left(x_j + y_j\right) + 2\sum_{j=1}^{k} H\left(z_j\right) - H\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\right)\\ &-\lambda^{-2m} H\left(\lambda^m\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j - \sum_{j=1}^{k} z_j\right)\right) = 0\end{aligned}$$

for all $x_j, y_j, z_j \in \mathbb{X}$ for all $j = 1 \to k$. By Lemma 3.1, the mapping $H : \mathbb{X} \to \mathbb{Y}$ is quadratic type. □

Theorem 4. Suppose $\varphi : \mathbb{X}^{3k} \to [0, \infty)$ be a function such that there exists an $0 < L < 1$ with

$$\begin{aligned}&\varphi\left(x_1, x_2, \cdots, x_k, y_1, y_2, \cdots, y_k, z_1, z_2, \cdots, z_k\right)\\ &\le |4k| K \varphi\left(\frac{x_1}{2k}, \frac{x_2}{2k}, \cdots, \frac{x_k}{2k}, \frac{y_1}{2k}, \frac{y_2}{2k}, \cdots, \frac{y_k}{2k}, \frac{z_1}{2k}, \frac{z_2}{2k}, \cdots, \frac{z_k}{2k}\right)\end{aligned} \tag{14}$$

for all $x_j, y_j, z_j \in \mathbb{X}$, for all $j = 1 \to k$. Let $f : \mathbb{X} \to \mathbb{Y}$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned}&\left\| 2\sum_{j=1}^{k} f\left(z_j\right) + 2\sum_{j=1}^{k} f\left(x_j + y_j\right) - f\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\right)\right.\\ &\left. -\lambda^{-2m} f\left(\lambda^m\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j - \sum_{j=1}^{k} z_j\right)\right)\right\|\\ &\le \varphi\left(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k\right)\end{aligned} \tag{15}$$

for all $x_j, y_j, z_j \in \mathbb{X}$, for all $j = 1 \to k$. Then there exists a unique quadratic type mapping $H:\mathbb{X}\to\mathbb{Y}$ such that

$$\|f(x)-H(x)\|\le\frac{L}{|4k|(1-L)}\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,x) \tag{16}$$

for all $x\in\mathbb{X}$.

The rest of the proof is similar to the proof of theorem 3.2 with note that mapping $T:\mathbb{M}\to\mathbb{M}$, $Tg(x):=\frac{1}{4k}g(2kx)$.

Corollary 1. Let $r<2$ and θ be nonegative real numbers and let $f:\mathbb{X}\to\mathbb{Y}$ be a mapping satisfying $f(0)=0$ and

$$\begin{aligned}&\Bigg\|2\sum_{j=1}^{k}f(z_j)+2\sum_{j=1}^{k}f(x_j+y_j)-f\left(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j\right)\\&-\lambda^{-2m}f\left(\lambda^m\left(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j-\sum_{j=1}^{k}z_j\right)\right)\Bigg\|\\&\le\theta\left(\sum_{j=1}^{k}\|x_j\|^r+\sum_{j=1}^{k}\|y_j\|^r+\sum_{j=1}^{k}\|z_j\|^r\right)\end{aligned} \tag{17}$$

for all $x\in X$. Then there exists a unique quadratic type mapping $H:\mathbb{X}\to\mathbb{Y}$ such that

$$\|f(x)-H(x)\|\le\frac{2\theta}{|2k|^r-|4k|}\|x\|^r \tag{18}$$

for all $x\in\mathbb{X}$.

Corollary 2. Let $r>2$ and θ be nonegative real numbers and let $f:\mathbb{X}\to\mathbb{Y}$ be a mapping satisfying $f(0)=0$ and

$$\begin{aligned}&\Bigg\|2\sum_{j=1}^{k}f(z_j)+2\sum_{j=1}^{k}f(x_j+y_j)-f\left(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j\right)\\&-\lambda^{-2m}f\left(\lambda^m\left(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j-\sum_{j=1}^{k}z_j\right)\right)\Bigg\|\\&\le\theta\left(\sum_{j=1}^{k}\|x_j\|^r+\sum_{j=1}^{k}\|y_j\|^r+\sum_{j=1}^{k}\|z_j\|^r\right)\end{aligned} \tag{19}$$

for all $x\in\mathbb{X}$. Then there exists a unique quadratic type mapping $H:\mathbb{X}\to\mathbb{Y}$ such that

$$\|f(x)-H(x)\|\le\frac{2\theta}{|4k|-|2k|^r}\|x\|^r \tag{20}$$

for all $x\in\mathbb{X}$.

4. Establishing a Solution to the Quadratic λ-Functional Equation Using the Direct Methoduse in Non-Archimedean Banach Space

Next, we are going to study the solutions of (1.1) for Quadratic λ-functional equation use direct method, we prove the Hyers-Ulam stability of the Quadratic λ-functional equation, the $\mathbb{X}$ is a Non-Archimedean normed space and $\mathbb{Y}$ is a Non-Archimedean Banach space, and the field $\mathbb{K}$ satisfy $|2k| \neq 1, \lambda^{-2m} \neq 4k-1$. Under this setting, we can show that the mapping satisfying (1.1) is quadratic. These results are give in the following

Theorem 5. Let $\varphi : \mathbb{X}^{3k} \to [0,\infty)$ be a function and let $f : \mathbb{X} \to \mathbb{Y}$ be a mapping satisfying $f(0) = 0$ and

$$\lim_{j\to\infty} |4k|^j \varphi\left(\frac{x_1}{(2k)^j}, \frac{x_2}{(2k)^j}, \cdots, \frac{x_k}{(2k)^j}, \frac{y_1}{(2k)^j}, \frac{y_2}{(2k)^j}, \cdots, \frac{y_k}{(2k)^j}, \frac{z_1}{(2k)^j}, \frac{z_2}{(2k)^j}, \cdots, \frac{z_k}{(2k)^j}\right) = 0 \tag{21}$$

$$\begin{aligned} &\left\| 2\sum_{j=1}^{k} f(z_j) + 2\sum_{j=1}^{k} f(x_j + y_j) - f\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\right) - \lambda^{-2m} f\left(\lambda^m \left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j - \sum_{j=1}^{k} z_j\right)\right) \right\| \\ &\leq \varphi(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k) \end{aligned} \tag{22}$$

for all $x_j, y_j, z_j \in \mathbb{X}$ for all $j = 1 \to k$. Then there exists a unique quadratic type mapping $H : \mathbb{X} \to \mathbb{Y}$ such that

$$\|f(x) - H(x)\| \leq \sup_{j\in\mathbb{N}} \left\{ |4k|^{j-1} \varphi\left(\frac{x}{(2k)^j}, \cdots, \frac{x}{(2k)^j}, \frac{x}{(2k)^j}, \cdots, \frac{x}{(2k)^j}, \frac{x}{(2k)^j}, \cdots, \frac{x}{(2k)^j}\right) \right\} \tag{23}$$

for all $x \in \mathbb{X}$.

Proof. We replacing $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $(x, \cdots, x, 0, \cdots, 0, x, \cdots, x)$ in (22), we have

$$\|f(2kx) - 4kf(x)\| \leq \varphi(x, \cdots, x, 0, \cdots, 0, x, \cdots, x) \tag{24}$$

for all $x \in \mathbb{X}$. Therefore

$$\left\| f(x) - 4kf\left(\frac{x}{2k}\right) \right\| \leq \varphi\left(\frac{x}{2k}, \cdots, \frac{x}{2k}, 0, \cdots, 0, \frac{x}{2k}, \cdots, \frac{x}{2k}\right) \tag{25}$$

for all $x \in \mathbb{X}$.

Hence

$$\begin{aligned} &\left\| (4k)^l f\left(\frac{x}{(2k)^l}\right) - (4k)^m f\left(\frac{x}{(2k)^m}\right) \right\| \\ &\leq \max\left\{ \left\| (4k)^l f\left(\frac{x}{(2k)^l}\right) - (4k)^{l+1} f\left(\frac{x}{(2k)^{l+1}}\right) \right\|, \cdots, \left\| (4k)^{m-1} f\left(\frac{x}{(2k)^{m-1}}\right) - (4k)^m f\left(\frac{x}{(2k)^m}\right) \right\| \right\} \end{aligned}$$

$$\begin{aligned}
&\le \max\left\{|4k|^{l}\left\|f\left(\frac{x}{(2k)^{l}}\right)-4kf\left(\frac{x}{(2k)^{l+1}}\right)\right\|,\cdots,\right.\\
&\left.\quad |4k|^{m-1}\left\|f\left(\frac{x}{(2k)^{m-1}}\right)-4kf\left(\frac{x}{(2k)^{m}}\right)\right\|\right\}\\
&\le \sup_{j\in\{l,l+1,\cdots\}}\left\{|4k|^{j}\varphi\left(\frac{x_1}{(2k)^{j+1}},\cdots,\frac{x_k}{(2k)^{j+1}},\frac{y_1}{(2k)^{j+1}},\cdots,\right.\right.\\
&\left.\left.\quad \frac{y_k}{(2k)^{j+1}},\frac{z_1}{(2k)^{j+1}},\cdots,\frac{z_k}{(2k)^{j+1}}\right)\right\}
\end{aligned} \tag{26}$$

for all nonnegative integers m and l with $m>l$ and all $x\in\mathbf{X}$. It follows (26) that the sequence $\left\{(4k)^{n} f\left(\frac{x}{(2k)^{n}}\right)\right\}$ is a Cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{(4k)^{n} f\left(\frac{x}{(2k)^{n}}\right)\right\}$ converger so one can define the mapping $H:\mathbf{X}\to\mathbf{Y}$ by

$$H(x):=\lim_{n\to\infty}(4k)^{n} f\left(\frac{x}{(2k)^{n}}\right)$$

for all $x\in\mathbf{X}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (26), we get (23). It follows from (21) and (22) that

$$\begin{aligned}
&\left\|2\sum_{j=1}^{k}H(x_j+y_j)+2\sum_{j=1}^{k}H(z_j)-H\left(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j\right)\right.\\
&\left.-\lambda^{-2m}H\left(\lambda^{m}\left(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j-\sum_{j=1}^{k}z_j\right)\right)\right\|\\
&=\lim_{n\to\infty}|4k|^{n}\left\|2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{(2k)^{n}}\right)+2\sum_{j=1}^{k}f\left(\frac{z_j}{(2k)^{n}}\right)\right.\\
&\quad -f\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j}{(2k)^{n}}\right)\\
&\left.\quad -\lambda^{-2m}f\left(\lambda^{m}\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j}{(2k)^{n}}\right)\right)\right\|\\
&\le \lim_{j\to\infty}|4k|^{n}\varphi\left(\frac{x}{(2k)^{j}},\cdots,\frac{x}{(2k)^{j}},\frac{x}{(2k)^{j}},\cdots,\frac{x}{(2k)^{j}},\frac{x}{(2k)^{j}},\cdots,\frac{x}{(2k)^{j}}\right)\\
&=0
\end{aligned} \tag{27}$$

for all $x\in\mathbf{X}$.

$$2\sum_{j=1}^{k}H(x_j+y_j)+2\sum_{j=1}^{k}H(z_j)-H\left(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j\right)$$

$$-\lambda^{-2m}H\left(\lambda^{m}\left(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j-\sum_{j=1}^{k}z_j\right)\right)=0$$

for all $x\in\mathbf{X}$. By Lemma 3.1, the mapping $H:\mathbf{X}\to\mathbf{Y}$ is quadratic. Now, let $T:\mathbf{X}\to\mathbf{Y}$ be another quadratic mapping satisfying (23). Then we have

$$\begin{aligned}\|H(x)-T(x)\|&=\left\|(4k)^q H\left(\frac{x}{(2k)^q}\right)-(4k)^q T\left(\frac{x}{(2k)^q}\right)\right\|\\ &\le \max\left\{\left\|(4k)^q H\left(\frac{x}{(2k)^q}\right)-(4k)^q f\left(\frac{x}{(2k)^q}\right)\right\|,\right.\\ &\qquad\left.\left\|(4k)^q T\left(\frac{x}{(2k)^q}\right)-(4k)^q f\left(\frac{x}{(2k)^q}\right)\right\|\right\}\\ &\le \sup_{j\in\mathbb{N}}\left\{|4k|^{q+j-1}\varphi\left(\frac{x_1}{(2k)^{j+1}},\cdots,\frac{x_k}{(2k)^{j+1}},\frac{y_1}{2^{j+1}},\cdots,\right.\right.\\ &\qquad\left.\left.\frac{y_k}{(2k)^{j+1}},\frac{z_1}{(2k)^{j+1}},\cdots,\frac{z_k}{2^{j+1}}\right)\right\}\end{aligned}$$

which tends to zero as $q\to\infty$ for all $x\in\mathbf{X}$. So we can conclude that $H(x)=T(x)$ for all $x\in\mathbf{X}$. This proves the uniqueness of H. Thus the mapping $H:\mathbf{X}\to\mathbf{Y}$ is a unique quadratic mapping satisfying (23) □

Theorem 6. Let $\varphi:\mathbf{X}^{3k}\to[0,\infty)$ be a function and let $f:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfying $f(0)=0$ and

$$\begin{aligned}\lim_{j\to\infty}&\left\{\frac{1}{|4k|^j}\varphi\left((2k)^{j-1}x_1,\cdots,(2k)^{j-1}x_k,(2k)^{j-1}y_1,\cdots,\right.\right.\\ &\left.\left.(2k)^{j-1}y_k,(2k)^{j-1}z_1,\cdots,(2k)^{j-1}z_k\right)\right\}=0\end{aligned}\tag{28}$$

$$\begin{aligned}&\left\|2\sum_{j=1}^{k}f(z_j)+2\sum_{j=1}^{k}f(x_j+y_j)-f\left(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j\right)\right.\\ &\left.-\lambda^{-2m}f\left(\lambda^{m}\left(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j-\sum_{j=1}^{k}z_j\right)\right)\right\|\\ &\le\varphi(x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_k,z_1,z_2,\cdots,z_k)\end{aligned}\tag{29}$$

for all $x_j,y_j,z_j\in\mathbf{X}$ for all $j=1\to n$. Then there exists a unique quadratic type mapping $H:\mathbf{X}\to\mathbf{Y}$ such that

$$\begin{aligned}\|f(x)-H(x)\|\le\sup_{j\in\mathbb{N}}&\left\{\frac{1}{|4k|^{j-1}}\varphi\left((2k)^{j-1}x,\cdots,(2k)^{j-1}x,(2k)^{j-1}x,\cdots,\right.\right.\\ &\left.\left.(2k)^{j-1}x,(2k)^{j-1}x,\cdots,(2k)^{j-1}x\right)\right\}\end{aligned}\tag{30}$$

for all $x\in\mathbf{X}$.

The rest of the proof is similar to the proof of theorem 4.1.

Corollary 3. Let $r<2$ and θ be nonegative real numbers and let $f:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfying $f(0)=0$ and

$$\left\|2\sum_{j=1}^{k} f(z_j)+2\sum_{j=1}^{k} f(x_j+y_j)= f\left(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j\right)\right.$$
$$\left.+\lambda^{-2m} f\left(\lambda^{m}\left(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j-\sum_{j=1}^{k} z_j\right)\right)\right\| \tag{31}$$
$$\le\theta\left(\sum_{j=1}^{k}\|x_j\|^r+\sum_{j=1}^{k}\|y_j\|^r+\sum_{j=1}^{k}\|z_j\|^r\right)$$

for all $x\in\mathbf{X}$. Then there exists a unique quadratic mapping $H:\mathbf{X}\to\mathbf{Y}$ such that

$$\|f(x)-H(x)\|\le\frac{2k\theta}{|2k|^r}\|x\|^r$$

for all $x\in\mathbf{X}$.

Corollary 4. Let $r>2$, and θ be nonegative real numbers and let $f:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfying $f(0)=0$ and

$$\left\|2\sum_{j=1}^{k} f(z_j)+2\sum_{j=1}^{k} f(x_j+y_j)= f\left(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j\right)\right.$$
$$\left.+\lambda^{-2m} f\left(\lambda^{m}\left(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j-\sum_{j=1}^{k} z_j\right)\right)\right\| \tag{32}$$
$$\le\theta\left(\sum_{j=1}^{k}\|x_j\|^r+\sum_{j=1}^{k}\|y_j\|^r+\sum_{j=1}^{k}\|z_j\|^r\right)$$

for all $x\in\mathbf{X}$. Then there exists a unique quadratic mapping $H:\mathbf{X}\to\mathbf{Y}$ such that

$$\|f(x)-H(x)\|\le\frac{2k\theta}{|4k|}\|x\|^r$$

for all $x\in X$.

5. Construct a Solution for (1.1) on Non-Archimedean Random Normed Space

In this section, $\mathbf{K}$ be a non-Archimedean field, $\mathbf{X}$ is a vector space over $\mathbf{K}$ and let $(\mathbf{X},\Gamma,T)$ be a non-Archimedean random Banach space over $\mathbf{K}$

We investigate the stability of the quadratic functional equation

$$2\sum_{j=1}^{k} f(z_j)+2\sum_{j=1}^{k} f(x_j+y_j)$$
$$= f\left(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j\right)+\lambda^{-2m} f\left(\lambda^{m}\left(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j-\sum_{j=1}^{k} z_j\right)\right) \tag{33}$$

where $f:\mathbf{X}\to\mathbf{Y}$ and $f(0)=0$.

Next, we define a random approximately quadrtic function. Let

$\varphi: \mathbf{X}^{3k+1} \to [0,\infty)$ be a distribution function such that $\varphi(x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_k,z_1,z_2,\cdots,z)_k$ is symmetric, nondecreasing and

$$\varphi\left(x,\cdots,x,0,\cdots,0,x,\cdots,x,\frac{t}{|\lambda|}\right) \le \varphi(\lambda x,\cdots,\lambda x,0,\cdots,0,\lambda x,\cdots,\lambda x,t) \quad (34)$$

For $x \in \mathbf{X}$, $\lambda \ne 0$.

Next, we define:

A mapping $f: \mathbf{X} \to \mathbf{Y}$ is said to be φ-approximately quadratic mapping if

$$\begin{aligned}&\Gamma_{f\left(\sum_{j=1}^k x_j+\sum_{j=1}^k y_j+\sum_{j=1}^k z_j\right)+\lambda^{-2m} f\left(\lambda^m\left(\sum_{j=1}^k x_j+\sum_{j=1}^k y_j-\sum_{j=1}^k z_j\right)\right)-2\sum_{j=1}^k f(z_j)-2\sum_{j=1}^k f(x_j+y_j)}\\&\le \varphi(x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_k,z_1,z_2,\cdots,z_k,t)\end{aligned} \quad (35)$$

for all $x_j,y_j,z_j \in \mathbf{X}$, for all $j=1\to k$, $t>0$.

* Note: We assume that $2k \ne 0$ in $\mathbb{K}$

Theorem 7 For $f: \mathbf{X} \to \mathbf{Y}$ be a φ-approximately quadratic mapping if there exist an $\beta \in \mathbb{R}(\beta > 0)$ and an integer *h*, $h \ge 2$ with $\beta > \left|(2k)^h\right|$ and $|2k| \ne 0$ such that

$$\begin{aligned}&\varphi\left((2k)^{-h}x_1,\cdots,(2k)^{-h}x_k,(2k)^{-h}y_1,\cdots,(2k)^{-h}y_k,\cdots,(2k)^{-h}z_1,\cdots,(2k)^{-h}z_k,t\right)\\&\ge \varphi\left((2k)^{-h}x_1,\cdots,(2k)^{-h}x_k,(2k)^{-h}y_1,\cdots,(2k)^{-h}y_k,\cdots,(2k)^{-h}z_1,\cdots,(2k)^{-h}z_k,\beta t\right)\end{aligned} \quad (36)$$

for all $x_j,y_j,z_j \in \mathbf{X}$ for all $j=1\to k$, $t>0$ and

$$\lim_{n\to\infty} T_{j=n}^{\infty} M\left(x,\frac{\beta^j t}{\left|(2k)^{hj}\right|}\right)=1 \quad (37)$$

for all $x \in \mathbf{X}$ and $t>0$.

Then there exists a unique quadratic type mapping $Q: \mathbf{X} \to \mathbf{Y}$ such that

$$\Gamma_{f(x)-Q(x)}(t) \ge T_{i=1}^{\infty} M\left(x,\frac{\beta^{i+1}t}{\left|(2k)^{hi}\right|}\right)=1 \quad (38)$$

In there

$$\begin{aligned}M(x,t)=Q\big(&\varphi(x,\cdots,x,0,\cdots,0,x,\cdots,x,t),\varphi(2kx,\cdots,2kx,0,\cdots,0,2kx,\cdots,2kx,t),\\&\cdots,\varphi\left((2k)^{h-1}x,\cdots,(2k)^{h-1}x,0,\cdots,0,(2k)^{h-1}x,\cdots,(2k)^{h-1}x,t\right)\end{aligned} \quad (39)$$

for all $x \in \mathbf{X}$ and $\forall t>0$.

Proof. First, we show by induction on *j* that for each $x \in \mathbf{X}$, $t>0$ and $j \ge 1$,

$$\begin{aligned}&\Gamma_{f\left((2k)^j x\right)-(4k)^j f(x)}(t) \ge M_j(x,t)\\&:= T\left(\varphi(x,\cdots,x,0,\cdots,0,x,\cdots,x,t),\varphi(2kx,\cdots,2kx,0,\cdots,0,2kx,\cdots,2kx,t),\cdots,\right.\\&\quad\left.\varphi\left((2k)^{h-1}x,\cdots,(2k)^{h-1}x,0,\cdots,0,(2k)^{h-1}x,\cdots,(2k)^{h-1}x,t\right)\right)\end{aligned} \quad (40)$$

we replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k,t)$ by $(x,\cdots,x,0,\cdots,0,x,\cdots,x,t)$ in (35), we obtain

$$\Gamma_{f((2k)x)-(4k)f(x)}(t) \ge \varphi(x,\cdots,x,0,\cdots,0,x,\cdots,x,t) \quad (41)$$

$x\in\mathbf{X}$, $t>0$. This proves (40) for $j=1$. We now assume that (40) holds for some $j\ge 1$ Next we replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k,t)$ by $\left((2k)^j x,\cdots,(2k)^j x,0,\cdots,0,(2k)^j x,\cdots,(2k)^j x,t\right)$ in (35) we have

$$\Gamma_{f\left((2k)^{j+1}x\right)-(4k)f\left((2k)^j x\right)}(t)\ge\varphi\left((2k)^j x,\cdots,(2k)^j x,0,\cdots,0,(2k)^j x,\cdots,(2k)^j x,t\right) \quad (42)$$

Since $|4k|\le 1$

$$\begin{aligned}&\Gamma_{f\left((2k)^{j+1}x\right)-(4k)^{j+1}f(x)}(t)\\&\ge T\left(\Gamma_{f\left((2k)^{j+1}x\right)-(4k)f\left((2k)^j x\right)}(t),\Gamma_{(4k)f\left((2k)^j x\right)-(4k)^{j+1}f(x)}(t)\right)\\&=T\left(\Gamma_{f\left((2k)^{j+1}x\right)-(4k)f\left((2k)^j x\right)}(t),\Gamma_{f\left((2k)^j x\right)-(4k)^j f(x)}\left(\frac{t}{|4k|}\right)\right)\\&=T\left(\Gamma_{f\left((2k)^{j+1}x\right)-(4k)f\left((2k)^j x\right)}(t),\Gamma_{f\left((2k)^j x\right)-(4k)^j f(x)}(t)\right)\\&=T\left(\varphi\left((2k)^j x,\cdots,(2k)^j x,0,\cdots,0,(2k)^j x,\cdots,(2k)^j x,t\right),M_j(x,t)\right)\\&=M_{j+1}(x,t)\end{aligned} \quad (43)$$

for all $x\in\mathbf{X}$. So in (40) holds for all $j\ge 1$.

Other way

$$\Gamma_{f\left((2k)^h x\right)-(4k)^h f(x)}(t)\ge M(x,t),\forall x\in\mathbf{X},t>0. \quad (44)$$

Next we replacing x by $(2k)^{-(hn+h)}x$ in (44) and using inequality (36), we have

$$\begin{aligned}&\Gamma_{f\left(\frac{x}{(2k)^{hn}}\right)-(4k)^h f\left(\frac{x}{(2k)^{hn+h}}\right)}(t)\\&\ge M\left(\frac{x}{(2k)^{hn+h}},t\right)\ge M\left(x,\beta^{n+1}t\right),\forall x\in\mathbf{X},t>0,n\in\mathbb{N}.\end{aligned} \quad (45)$$

Then

$$\Gamma_{(4k)^{nh}f\left(\frac{x}{(2k)^{hn}}\right)-(4k)^{h+1}f\left(\frac{x}{(2k)^{hn+h}}\right)}(t)\ge M\left(x,\frac{\beta^{n+1}}{|4k|^{hn}}t\right),\forall x\in\mathbf{X},t>0,n\in\mathbb{N}. \quad (46)$$

Hence,

$$\begin{aligned}&\Gamma_{(4k)^{hn}f\left(\frac{x}{(2k)^{hn}}\right)-(4k)^{h(n+p)}f\left(\frac{x}{(2k)^{h(n+p)}}\right)}(t)\\&\ge T_{j=n}^{n+p}\left(\Gamma_{(4k)^{hj}f\left(\frac{x}{(2k)^{hj}}\right)-(4k)^{h(n+j)}f\left(\frac{x}{(2k)^{h(n+j)}}\right)}(t)\right)\\&\ge T_{j=n}^{n+p}M\left(x,\frac{\beta^{j+1}}{|4k|^{hj}}t\right)\\&\ge T_{j=n}^{n+p}M\left(x,\frac{\beta^{j+1}}{|4k|^{j}}t\right),\forall x\in\mathbf{X},t>0,n\in\mathbb{N}.\end{aligned} \quad (47)$$

Since

$$\lim_{n\to\infty} T_{j=n}^{n+p} M\left(x, \frac{\beta^{j+1}}{|4k|^{hj}} t\right) = 1, \forall x \in \mathbf{X}, t > 0, n \in \mathbb{N},$$

$\left\{(4k)^{hn} f\left(\frac{x}{(2k)^{hn}}\right)\right\}$ is a Cauchy sequence in the non-Archimedean random Banach space $(\mathbf{Y}, \Gamma, T)$. Hence, we can define a mapping $Q: \mathbf{X} \to \mathbf{Y}$ such that

$$\lim_{n\to\infty} \Gamma_{(4k)^{hn} f\left(\frac{x}{(2k)^{hn}}\right) - Q(x)}(t) = 1, \forall x \in \mathbf{X}, t > 0, \quad (48)$$

Next for each $n \ge 1$, $\forall x \in \mathbf{X}$ and $t > 0$.

$$\begin{aligned} \Gamma_{f(x)-(4k)^{hn} f\left(\frac{x}{(2k)^{hn}}\right)}(t) &= \Gamma_{\sum_{i=0}^{n-1}(4k)^{hi} f\left(\frac{x}{(2k)^{hi}}\right) - (4k)^{h(i+1)} f\left(\frac{x}{(2k)^{h(i+1)}}\right)}(t) \\ &\ge T_{i=0}^{n+p}\left(\Gamma_{\sum_{i=0}^{n-1}(4k)^{hi} f\left(\frac{x}{(2k)^{hi}}\right) - (4k)^{h(i+1)} f\left(\frac{x}{(2k)^{h(i+1)}}\right)}(t)\right) \\ &\ge T_{i=0}^{n-1} M\left(x, \frac{\beta^{i+1} t}{|4k|^{hi}}\right) \end{aligned} \quad (49)$$

Therefore,

$$\begin{aligned} \Gamma_{f(x)-Q(x)}(t) &\ge T\left(\Gamma_{f(x)-(4k)^{hn} f\left(\frac{x}{(2k)^{hn}}\right)}(t), \Gamma_{(4k)^{hn} f\left(\frac{x}{(2k)^{hn}}\right) - Q(x)}(t)\right) \\ &\ge T\left(T_{i=0}^{n-1} M\left(x, \frac{\beta^{i+1} t}{|4k|^{hi}}\right), \Gamma_{(4k)^{hn} f\left(\frac{x}{(2k)^{hn}}\right) - Q(x)}(t)\right) \end{aligned} \quad (50)$$

By letting $n \to \infty$, we obtain

$$\Gamma_{f(x)-Q(x)}(t) \ge T_{i=0}^{n-1} M\left(x, \frac{\beta^{i+1} t}{|4k|^{hi}}\right) \quad (51)$$

As T is continuous, from a well-known result in probabilistic metric space see [12].

Now we put

$$\begin{aligned} \Delta x &= 2(2k)^{hn} \sum_{j=1}^{k} f\left((2k)^{-hn} z_j\right) + 2(2k)^{hn} \sum_{j=1}^{k} f\left((2k)^{-hn}(x_j + y_j)\right) \\ &- (2k)^{hn} f\left((2k)^{hn}\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\right)\right) \\ &+ \lambda^{-2m}(2k)^{hn} f\left((2k)^{-hn} \lambda^m\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j - \sum_{j=1}^{k} z_j\right)\right) \end{aligned} \quad (52)$$

it follows that

$$\lim_{n\to\infty}\Gamma_{\Delta x}=\Gamma_{f\left(\sum_{j=1}^k x_j+\sum_{j=1}^k y_j+\sum_{j=1}^k z_j\right)+\lambda^{-2m}f\left(\lambda^m\left(\sum_{j=1}^k x_j+\sum_{j=1}^k y_j-\sum_{j=1}^k z_j\right)\right)-2\sum_{j=1}^k f(z_j)-2\sum_{j=1}^k f(x_j+y_j)}(t) \quad (53)$$

for almost all $t>0$, □

On the other hand, replacing x_j, y_j by $(2k)^{hn}x_j, (2k)^{hn}y_j$, respectively, in (35) and suing (NA-RN2) and (36), we have

$$\begin{aligned}\Gamma_{\Delta x}&\ge\varphi\left((2k)^{-hn}x_1,\cdots,(2k)^{-hn}x_k,0,\cdots,0,(2k)^{-hn}z_1,\cdots,(2k)^{-hn}z_k,\frac{t}{|2k|^{hn}}\right)\\&\ge\varphi\left(x_1,\cdots,x_k,0,\cdots,0,z_1,\cdots,z_k,\frac{\beta^n t}{|2k|^{hn}}\right)\end{aligned} \quad (54)$$

for all $x_j, y_j, z_j\in\mathbf{X}, j=1\to k$. Sence

$$\lim_{n\to\infty}\varphi\left(x_1,\cdots,x_k,0,\cdots,0,z_1,\cdots,z_k,\frac{\beta^n t}{|2k|^{hn}}\right)=1,$$

We infer that Q is a quadratic function.

Finally we have to prove that Q is a unique quadratic mapping.

Let $Q':\mathbf{X}\to\mathbf{Y}$ is another quadratic mapping such that

$$\Gamma_{Q'(x)-f(x)}(t)\ge M(x,t) \quad (55)$$

for all $x\in\mathbf{X}$ and $t>0$, then for each $n\in\mathbb{N}, x\in\mathbf{X}, t>0$

$$\Gamma_{Q(x)-Q'(x)}(t)\ge T\left(\Gamma_{Q(x)-(4k)^{hn}f\left(\frac{x}{(2k)^{hn}}\right)}(t),\Gamma_{(4k)^{hn}f\left(\frac{x}{(2k)^{hn}}\right)-Q'(x)}(t),t\right). \quad (56)$$

Form (48), we infer that $Q'=Q$.

From the theorem 5.1 we get the following corollary:

Corollary 5. For $f:\mathbf{X}\to\mathbf{Y}$ be a φ-approximately quadratic mapping if there exist an $\beta\in\mathbb{R}(\beta>0)$ and an integer $h, h\ge 2$ with $\beta>\left|(2k)^h\right|$ and $|2k|\ne 0$ such that

$$\begin{aligned}&\varphi\left((2k)^{-h}x_1,\cdots,(2k)^{-h}x_k,(2k)^{-h}y_1,\cdots,(2k)^{-h}y_k,\cdots,(2k)^{-h}z_1,\cdots,(2k)^{-h}z_k,t\right)\\&\ge\varphi\left((2k)^{-h}x_1,\cdots,(2k)^{-h}x_k,(2k)^{-h}y_1,\cdots,(2k)^{-h}y_k,\cdots,(2k)^{-h}z_1,\cdots,(2k)^{-h}z_k,\beta t\right)\end{aligned} \quad (57)$$

for all $x_j, y_j, z_j\in\mathbf{X}$ for all $j=1\to k$, $t>0$, then there exists a unique quadratic type mapping $Q:\mathbf{X}\to\mathbf{Y}$ such that

$$\Gamma_{f(x)-Q(x)}(t)\ge T_{i=1}^{\infty}M\left(x,\frac{\beta^{i+1}t}{\left|(2k)^{hi}\right|}\right) \quad (58)$$

for all $x\in\mathbf{X}$ and $\forall t>0$. In there

$$\begin{aligned}M(x,t)=Q\big(&\varphi(x,\cdots,x,0,\cdots,0,x,\cdots,x,t),\varphi(2kx,\cdots,2kx,0,\cdots,0,2kx,\cdots,2kx,t),\\&\cdots,\varphi\left((2k)^{h-1}x,\cdots,(2k)^{h-1}x,0,\cdots,0,(2k)^{h-1}x,\cdots,(2k)^{h-1}x,t\right)\end{aligned} \quad (59)$$

for all $x\in\mathbf{X}$ and $\forall t>0$.

Application Example: For $(\mathbf{X},\Gamma,T_M)$ non-Archimedean random normed space in which

$$\Gamma_x(t)=\frac{t}{t+\|t\|},\forall x\in\mathbf{X},t>0$$

and assuming that $(\mathbf{Y},\Gamma,T_M)$ complete non-Archimedean random normed space.

Now we define

$$\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k,t)=\frac{t}{1+t}.$$

It is easy to see that for $\beta=1$ then (36) holds, sence

$$M(x,t)=\frac{t}{1+t},$$

We have

$$\begin{aligned}\lim_{n\to\infty}T_{j=n}^{\infty}M\left(x,\frac{\beta^j}{|4k|^{hj}}t\right)&=\lim_{n\to\infty}\left(\lim_{m\to\infty}T_{j=n}^{m}M\left(x,\frac{t}{|4k|^{hj}}t\right)\right)\\&=\lim_{n\to\infty}\cdot\lim_{m\to\infty}\left(\frac{t}{t+\left|(4k)^h\right|^n}\right)=1\end{aligned},$$

$\forall x\in\mathbf{X}$, $t>0$.

6. Conclusion

In this paper, I have built the condition for existence of a solution for a functional equation of general form and then I have used two fixed point methods and a direct method to show their solutions on non-Archimedean space and finally establish their solution on the non-Archimedean Random normed space.

Conflicts of Interest

The author declares no conflicts of interest.

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