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Mode Analysis in a Spherical Resonator

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Abstract

The problem of spherical resonator for electromagnetic wave propagation is one of the hot topics; in this paper, we give spherical coordinate transformation and solution of Helmholtz equation, through spherical Bessel function and spherical Neumann function to analysis mode in spherical resonators. Finally, the distribution of electromagnetic wave in spherical resonator is given by numerical example.

Subject Areas

Electromagnetics

Keywords

Maxwell's Equations, Helmholtz Equation, Spherical Coordinate Transformation, Legendre Equation

1. Introduction

In 1864, James C. Maxwell first proposed the theory of electromagnetic fields, and more than 20 years later, the success of Heinrich R. Hertz's electromagnetic wave experiment inspired people to explore the way of using electromagnetic wave to realize wireless communication. Some famous scientists and mathematicians, such as A. Sommerfeld developed a basic theory of radio wave propagation along flat ground. Van der Pol and W. Wotson established the basic theory of radio wave propagation around a conducting sphere.

Electromagnetic wave can be used in mobile phone communications, satellite signals, navigation, remote control, positioning, home appliances (microwave ovens, induction cookers) infrared wave, industrial, medical equipment, etc., for example, radio wave is used for communication, microwave is used for microwave oven, satellite communication, infrared ray is used for remote control, thermal imager, infrared guided missile, etc. Ultraviolet light is used for medical disinfection, verification of counterfeit banknotes, distance measurement, engineering flaw detection, etc. X-rays are used for CT photography, and gamma rays are used for treatment, causing atomic transitions to produce new rays, etc. Therefore, it is of great significance to study the propagation of electromagnetic wave. A simple form is to study the propagation of electromagnetic waves in a resonant cavity.

An innovative filter configuration employing transversal coupled triple-mode spherical resonators while capable of realizing transmission zeros is studied in [1]. Spherical resonators inherently have very large quality factor and exhibit a further spurious response thus yielding bandpass filters with lower insertion loss and wider spurious-free window. The design methodology for transversal coupling is used for designing a third-order bandpass filter with TZs. Furthermore, a sixth-order bandpass filter using cascaded trisection is also presented in [1].

In reference [2], a modernization of the undergraduate physical chemistry laboratory experiment for determining the speed of sound in various gases from resonant frequencies in a spherical resonator is presented. Another innovation of this article is that the resonator (schematic IR = 7.5 cm) is constructed by 3D printing with eco-friendly poly (lactic acid), a commercially viable alternative to traditional construction methods.

The main contribution of literature [3] is that they develop the methodology to calculate the electromagnetic behavior of a statically magnetized sphere. An application demonstrating the optical properties of a magnetic micron-sized garnet particle is also included. An important piece of work is that they introduce a theoretical framework for studying the interaction of a dynamically-varying magnetization with light in spherical resonators [3]. The design of an additively manufactured V-band bandpass filter is proposed using spherical resonators with elliptical waveguides feeds in [4].

2. Preliminaries

In an ideal isotropic medium, Maxwell's equation is

$$\begin{cases} \nabla \times H = J + \dfrac{\partial D}{\partial t} \\ \nabla \times E = -\dfrac{\partial B}{\partial t} \\ \nabla \cdot B = 0 \\ \nabla \cdot D = \rho \end{cases} \tag{1}$$

where E is electric filed vectors, H is magnetic field vectors, J is volume current, ρ is charge density,

$$D = \varepsilon E, \quad B = \mu H, \tag{2}$$

with ε is permittivity, μ is permeability. Taking the curl of the first and second equations in the Maxwell's equations and apply vector differential identities, we get

$$\begin{cases} \nabla^2 E - \varepsilon\mu \dfrac{\partial^2 E}{\partial t^2} = \dfrac{1}{\varepsilon}\nabla\rho + \mu\dfrac{\partial J}{\partial t} \\ \nabla^2 H - \varepsilon\mu \dfrac{\partial^2 H}{\partial t^2} = -\nabla \times J \end{cases} \tag{3}$$

We note that for the passive region, both the free current density and the free charge density are zero, namely,

$$J = 0,\ \rho = 0. \tag{4}$$

Thus, we can obtain

$$\nabla^2 E - \varepsilon\mu \frac{\partial^2 E}{\partial t^2} = 0 \tag{5}$$

$$\nabla^2 H - \varepsilon\mu \frac{\partial^2 H}{\partial t^2} = 0 \tag{6}$$

In the Cartesian coordinate system, (5) and (6) contain three scalar wave equations. Each scalar wave equation will contain only one component of the field vector. Without loss of generality, we assume that ψ presents an arbitrary component, they can be represented collectively as

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \varepsilon\mu \frac{\partial^2 \psi}{\partial t^2}. \tag{7}$$

We combine the wave Equations (5)-(7) with the boundary conditions and initial conditions to form the definite solution problem in the case of time-varying field. In most cases, we do not analyze general time-varying electromagnetic problems. On the contrary, our research focuses on the so-called time-harmonic electromagnetic field problem, namely, the field quantity changes sinusoidal (cosine) with time. Under the circumstances, the components of an electromagnetic field can be represented by a vector, due to $\frac{\partial}{\partial t} = j\omega$, thus we can present Equations (5)-(7) as

$$\begin{cases} \nabla^2 E + k^2 E = 0 \\ \nabla^2 H + k^2 H = 0 \end{cases}, \tag{8}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0, \tag{9}$$

with $k = \omega\sqrt{\varepsilon\mu}$ is called wave number. (8) and (9) are electric field intensity, magnetic field intensity and the Helmholtz equation for each component.

3. Spherical Coordinate Transformation and Solution of Helmholtz Equation

For helmholtz equation

$$\Delta\phi + k^2\phi = 0 \tag{10}$$

specifically, we have

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}+k^2\phi=0 \tag{11}$$

$$\text{Define } \begin{cases} x=r\sin\theta\cos\varphi \\ y=r\sin\theta\sin\varphi \\ z=r\cos\theta \end{cases} \tag{12}$$

and using $r=\sqrt{x^2+y^2+z^2}$, $\theta=\arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$, $\varphi=\arctan\frac{y}{x}$, it yields that helmholtz equation can be presents as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right)+2\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2}+k^2\phi=0 \tag{13}$$

Let $\phi(r,\theta,\varphi)=R(r)\Theta(\theta)\Phi(\varphi)$, and substituting $\phi(r,\theta,\varphi)$ into (13), one has

$$\frac{\sin^2\theta}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{\sin\theta}{\Theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}+k^2r^2\sin^2\theta=0 \tag{14}$$

The penultimate term on the left side of the equation is only a function of φ, to make the above formula true for all values of r,θ,φ, there must be

$$\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}=-m^2, \tag{15}$$

with m as the separation constant.

In most cases, the angular function should also satisfy periodic boundary conditions, $\Phi(\varphi+2\pi)=\Phi(\varphi)$. This condition together with formula (15) constitutes the eigenvalue problem. It is clear that the solution of (15) is

$$\Phi_m(\varphi)=C_m\sin m\varphi+D_m\cos m\varphi, \tag{16}$$

with m being an arbitrary number. Then from (14), we have

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+k^2r^2+\frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)-\frac{m^2}{\sin^2\theta}=0 \tag{17}$$

We note that the first two terms in the above formula are only functions of r, and the last terms are only functions of θ, so we get the form of variable separation. We assume that the separation constant is λ, it can be seen that

$$\begin{cases} \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+k^2r^2=\lambda \\ \frac{m^2}{\sin^2\theta}-\frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\Theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)=\lambda \end{cases} \tag{18}$$

We can simplify the equation as

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+2r\frac{\mathrm{d}R}{\mathrm{d}r}+\left[k^2r^2-\lambda\right]R=0, \tag{19}$$

let $x=\cos\theta$ and bring x into the second equation of (18), Using the chain derivation rule we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(1-x^2\right)\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right]+\left(\lambda-\frac{m^2}{1-x^2}\right)\Theta=0\,. \tag{20}$$

Obviously, (20) is the associated Legendre equation, the solution involves the so-called natural boundary conditions and the corresponding eigenvalue problems. The eigenvalue problem is formed by combining the natural boundary conditions on the polar axis. As for parameter variable λ, it is required that λ meets

$$\lambda=n\left(n+1\right),n=0,1,2,\cdots \tag{21}$$

Its solution is a Legendre polynomial $P_n^m\left(\cos\theta\right)$ of order m of degree n. Therefore,

$$\Theta\left(\theta\right)=P_n^m\left(\cos\theta\right). \tag{22}$$

Furthermore, for (19) we can get

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+2r\frac{\mathrm{d}R}{\mathrm{d}r}+\left[k^2r^2-n\left(n+1\right)\right]R=0. \tag{23}$$

Let $x=kr$, $R\left(r\right)=\sqrt{\frac{\pi}{2x}}y\left(x\right)$. We can rewrite (23) as

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+\left[x^2-\left(n+\frac{1}{2}\right)^2\right]y=0\,, \tag{24}$$

this is Bessel equation of order $n+\frac{1}{2}$, thus the solution of the equation is

$$\left\{J_{n+\frac{1}{2}}\left(x\right),\ N_{n+\frac{1}{2}}\left(x\right)\right\}. \tag{25}$$

Hence, we define functions as follows:

$$j_n\left(x\right)=\sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}\left(x\right), \tag{26}$$

$$n_n\left(x\right)=\sqrt{\frac{\pi}{2x}}N_{n+\frac{1}{2}}\left(x\right). \tag{27}$$

These functions are spherical Bessel function and spherical Neumann function, respectively. Hence, the radial function can finally be expressed as

$$R\left(r\right)=A_nj_n\left(kr\right)+B_nn_n\left(kr\right). \tag{28}$$

We can get the expression of the general solution in the spherial coordinate system as

$$\phi\left(r,\theta,\varphi\right)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(C_m\sin m\varphi+D_m\cos m\varphi\right)P_n^m\left(\cos\theta\right)\left[A_nj_n\left(kr\right)+B_nn_n\left(kr\right)\right]. \tag{29}$$

Remark: In the general solution, spherical Bessel function and spherical Neumann function stand for standing wave. We also can express the expression of the general solution in the spherial coordinate system as

$$\phi(r,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(C_m\sin m\varphi+D_m\cos m\varphi\right)P_n^m\left(\cos\theta\right)\left[A_nh_n^{(1)}\left(kr\right)+B_nh_n^{(2)}\left(kr\right)\right], \quad (30)$$

with $h_n^{(1)}(kr)$, $h_n^{(2)}(kr)$ represents the first kind of spherical Hankel function and the second kind of Hankel function, respectively. At this moment, spherical Hankel function stands for traveling wave.

4. Mode Analysis in Spherical Resonators

In this paper, a spherical resonator with internal and external radii $b-\varepsilon$ and b is studied. It is common knowledge that we can use the Borgnis function to analyze the electromagnetic field distribution, resonant frequencies, and so on. For simplicity, we only consider the properties of spherical resonators.

$$F=\left(C_m\sin m\varphi+D_m\cos m\varphi\right)P_n^m\left(\cos\theta\right)\left[A_nj_n\left(kr\right)+B_nn_n\left(kr\right)\right], \quad (31)$$

$$F=\cos m\varphi P_n^m\left(\cos\theta\right)\left[A_nj_n\left(kr\right)+B_nn_n\left(kr\right)\right], \quad (32)$$

according to Borgnis function method corresponding to formula in [5], it follows that

$$\begin{aligned}H_\theta=&-\frac{\sin\theta}{r}\left(P_n^m\right)'\left(\cos\theta\right)\cos m\varphi\left\{A_n\left[j_n\left(kr\right)+krj_n'\left(kr\right)\right]\right.\\&\left.+B_n\left[n_n\left(kr\right)+krn_n'\left(kr\right)\right]\right\},\end{aligned} \quad (33)$$

$$\begin{aligned}H_\varphi=&-\frac{m}{r\sin\theta}P_n^m\left(\cos\theta\right)\sin m\varphi\left\{A_n\left[j_n\left(kr\right)+krj_n'\left(kr\right)\right]\right.\\&\left.+B_n\left[n_n\left(kr\right)+krn_n'\left(kr\right)\right]\right\},\end{aligned} \quad (34)$$

$$\begin{aligned}H_r=&P_n^m\left(\cos\theta\right)\cos m\varphi\left\{A_nk\left[krj_n''\left(kr\right)+krj_n\left(kr\right)+2j_n'\left(kr\right)\right]\right\}\\&+P_n^m\left(\cos\theta\right)\cos m\varphi\left\{B_nk\left[krn_n''\left(kr\right)+krn_n\left(kr\right)+2n_n'\left(kr\right)\right]\right\}\end{aligned} \quad (35)$$

$$E_\theta=j\omega\mu\frac{m}{\sin\theta}P_n^m\left(\cos\theta\right)\sin m\varphi\left[A_nj_n\left(kr\right)+B_nn_n\left(kr\right)\right], \quad (36)$$

$$E_\varphi=-j\omega\mu\sin\theta\left(P_n^m\right)'\left(\cos\theta\right)\cos m\varphi\left[A_nj_n\left(kr\right)+B_nn_n\left(kr\right)\right], \quad (37)$$

$$E_r=0, \quad (38)$$

We note that

$$\begin{cases}xj_n''+xj_n+2j_n'=\dfrac{n(n+1)}{x}j_n\\xn_n''+xn_n+2n_n'=\dfrac{n(n+1)}{x}n_n\end{cases}, \quad (39)$$

thus

$$H_r=P_n^m\left(\cos\theta\right)\cos m\varphi\frac{n(n+1)}{r}\left\{A_nj_n\left(kr\right)+B_nn_n\left(kr\right)\right\} \quad (40)$$

is the field distribution in a spherical cavity. Considering the boundary condition $r=b,b-\varepsilon$, then we have

$$\begin{cases} A_n j_n\left(k\left(b-\varepsilon\right)\right)+B_n n_n\left(k\left(b-\varepsilon\right)\right)=0 \\ A_n j_n\left(kb\right)+B_n n_n\left(kb\right)=0 \end{cases}. \tag{41}$$

Hence the characteristic equation is

$$\begin{vmatrix} j_n\left(k\left(b-\varepsilon\right)\right) & n_n\left(k\left(b-\varepsilon\right)\right) \\ j_n\left(kb\right) & n_n\left(kb\right) \end{vmatrix}=0, \tag{42}$$

which equal to the equation as

$$j_n\left(k\left(b-\varepsilon\right)\right)n_n\left(kb\right)-j_n\left(k\left(b-\varepsilon\right)\right)n_n\left(k\left(b-\varepsilon\right)\right)=0 \tag{43}$$

5. Simulation

The radius of the outer circle can be taken arbitrarily, but for simplicity, we take it as 1 m. When the radius of the outer circle is fixed, in theory, the radius of the inner circle should be less than the radius of the outer circle. However, in order to display better and give consideration to the requirement of making the complete waveform appear in the ring as much as possible, there is a relatively good inner circle radius, and researchers need to experiment several times to obtain it. Without losing generality, we suppose there is a spherical resonator with an inner radius of 0.2 m and an outer radius of 1 m, namely, $b=1$ (m) and $\varepsilon=0.8$ (m). Bring b and ε into (43), we can obtain that $k=2.2376\text{e}+08$ (Hz) then make use of (40), and the field distribution corresponding to this pattern can be further plotted (Figures 1-4).

From Figure 2, it is clear that the smaller the wave number of the incident wave, the smaller the value of the characteristic function. Combing Figure 1 and Figure 4, we find that the energy of the electromagnetic field is smaller at the edges of the inner circle and the outer circle, and the energy of the electromagnetic field increases significantly as it gets closer to the interior of the cavity region.

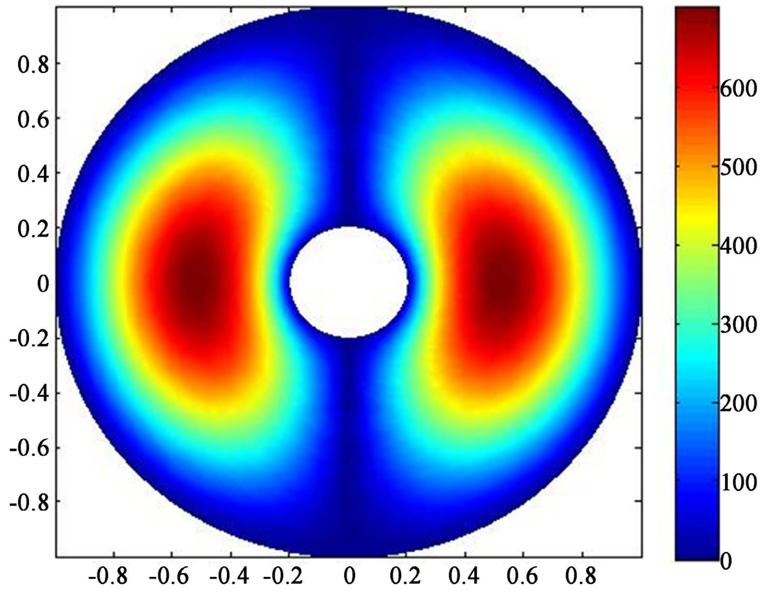

Figure 1. The field distribution in the longitudinal section.

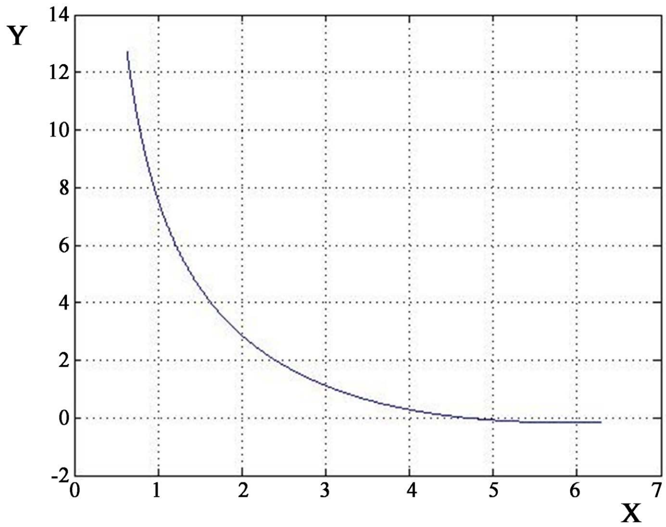

Figure 2. A curve in which the characteristic function varies with the number of waves.

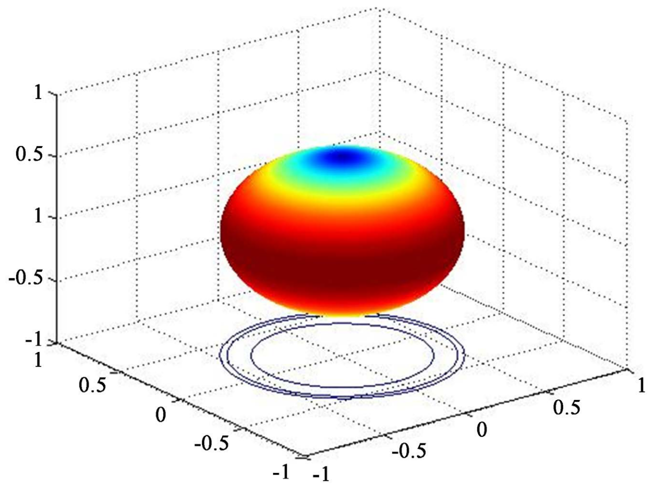

Figure 3. The field distribution on a sphere.

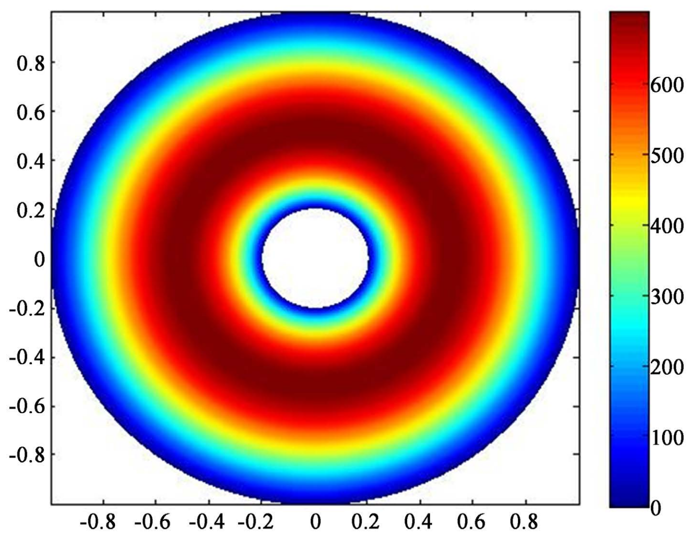

Figure 4. The field distribution in the equatorial plane.

If we only pay attention to the distribution of the electromagnetic field of a spherical field, the electromagnetic field energy of the two poles is less, and the energy is mostly in the equatorial region.

6. Conclusion

In this paper, we give spherical coordinate transformation and solution of Helmholtz equation, through spherical Bessel function and spherical Neumann function to analysis mode in spherical resonators. The distribution of electromagnetic wave in spherical resonator is given finally.

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Conflicts of Interest

The authors declare no conflicts of interest.

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