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Application and Popularization of Formal Calculation

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Abstract

Formal Calculation, formerly known as the Shape of Numbers, is suitable for calculating some nested sums. The formula has been obtained, and the calculation problem of various combinations of arithmetic sequences has been solved. This paper analysis the coefficients of the formulas and obtain some simplified identities. Furthermore, the Formal Calculation is extended from binomial coefficient to Gaussian coefficient, and the application is extended from two parameters forms to multiparameter forms.

Subject Areas

Discrete Mathematics

Keywords

Formal Calculation, Shape of Numbers, Calculation Formula, Combinatorics, Gaussian Coefficient, Stirling Number

1. Introduction

Peng, J. has introduced the Shape of Numbers and three forms of calculation in [1]: $K_i, D_i \in \text{Commutative Ring}$.

M series: $Serie_i = \{K_i, K_i + D_i, K_i + 2D_i, \cdots, K_i + (N-1)D_i\}$, $i \in [1, M]$

Use $PS = [K_1 : D_1, \cdots, K_M : D_M]$ to represent the series.

$[K_1 : 1, \cdots, K_M : 1]$ is abbreviated as $[K_1, \cdots, K_M]$.

$[K_1 : D, \cdots, K_M : D]$ is abbreviated as $[K_1, \cdots, K_M] : D$.

Use $PT = [T_1, T_2, \cdots, T_M]$ to indicate some items in M series (the Shape).

By default, the following uses:

$PS = [K_1 : D_1, K_2 : D_2, \cdots, K_M : D_M], PT = [T_1, T_2, \cdots, T_M]$

$PSA = [K_1 : D_1, \cdots, K_M : D_M, K_{M+1} : D_{M+1}] = [PS,\ K_{M+1} : D_{M+1}],$

$PTA = [PT,\ T_{M+1} = T_M + 2 - p]$

Recursive definie operator ∇^P, $p \in \mathbb{Z}$:

$\nabla^0 f(n) = f(n), \sum_{n=0}^{N-1} \nabla^1 f(n+1) = f(N), \sum_{n=0}^{N-1} f(n+1) = \nabla^{-1} f(N)$

Recursive define *SUN*(*N*, *PS*, *PT*), abbreviated as *SUM*(*N*):

$SUM(N, [K_1 : D_1], [T_1 = 1]) = \sum_{n=0}^{N-1} (K_1 + n \times D_1)$

$SUM(N, PSA, PTA) = \sum_{n=0}^{N-1} (K_{M+1} + n \times D_{M+1}) \times \nabla^P SUM(n+1)$

For example:

$SUM(N, PS, [1, 2, \cdots, M]) = \sum_{n=0}^{N-1} \prod_{i=1}^{M} (K_i + nD_i)$

$SUM(N, PS, [1, 3, \cdots, 2M-1])$

$= \sum_{n_M=0}^{N-1} (K_M + n_M D_M) \cdots \sum_{n_1=0}^{n_2} (K_2 + n_2 D_2) \sum_{n_1=0}^{n_2} (K_1 + n_1 D_1)$

$SUM(N, PS, [1, 2, 4]) = \sum_{n_3=0}^{N-1} (K_3 + n_3 D_3) \sum_{n=0}^{n_3} (K_2 + nD_1)(K_1 + nD_1)$

$SUM(N, PS, [1, 3, 4]) = \sum_{n=0}^{N-1} (K_3 + nD_3)(K_2 + nD_2) \sum_{n_1=0}^{n} (K_1 + n_1 D_1)$

$SUM(N, PS, [1, 4]) = \sum_{n=0}^{N-1} (K_2 + nD_2) \sum_{n_2=0}^{n} \sum_{n_1=0}^{n_2} (K_1 + n_1 D_1)$

The following use *K* to represent set $\{K_1, K_2, \cdots, K_M\}$, *T* to represent set $\{T_1, T_2, \cdots, T_M\}$.

Use the Form: $(T_1 + K_1)(T_2 + K_2) \cdots (T_M + K_M) = \sum \prod_{i=1}^{M} X_i$, $X_i = T_i$ or K_i

$X(T) =$ Count of $\{X_1, \cdots, X_M\} \in T$,

$X_{T-1} =$ Count of $\{X_1, \cdots, X_{i-1}\} \in T$, $X_{K-1} =$ Count of $\{X_1, \cdots, X_{i-1}\} \in K$.

Don't swap the factors, then each $\prod X_i$ corresponds to one expression in the *SUM*().

1.1) $H(q) = \sum_{\prod X_i \text{ with } X(T)=q} \prod_{i=1}^{M} B_i$, $q = X(T)$, $SUM(N) =$

$$\xrightarrow{\text{Form}_1} \sum_{q=0}^{M} H_1(q) \binom{N + T_M - M}{N - 1 - q} = \sum_{q=0}^{M} H_1(q) \binom{N + T_M - M}{T_M - M + 1 + q}$$

$$\xrightarrow{\text{Form}_2} \sum_{q=0}^{M} H_2(q) \binom{N + T_M - M + q}{N - 1} = \sum_{q=0}^{M} H_2(q) \binom{N + T_M - M + q}{T_M - M + 1 + q}$$

$$\xrightarrow{\text{Form}_3} \sum_{q=0}^{M} H_3(q) \binom{N + T_M - q}{N - 1 - q} = \sum_{q=0}^{M} H_3(q) \binom{N + T_M - q}{T_M + 1}$$

$$\xrightarrow{\text{Form}_1} B_i = \begin{cases} (T_i - X_{K-1}) D_i; X_i = T_i \\ K_i + X_{T-1} D_i; X_i = K_i \end{cases}$$

$$\xrightarrow{\text{Form}_2} B_i = \begin{cases} (T_i - X_{K-1}) D_i; X_i = T_i \\ K_i + (X_{K-1} - T_i) D_i; X_i = K_i \end{cases}$$

$$\xrightarrow{\text{Form}_3} B_i = \begin{cases} -K_i + (T_i - X_{T-1}) D_i; X_i = T_i \\ K_i + X_{T-1} D_i; X_i = K_i \end{cases}$$

$H_1(q)$, $H_2(q)$, $H_3(q)$, short for *H*(*q*, *PS*, *PT*), is also defined above.

Sometimes use *H*(*q*) to represent these three coefficients.

If $f(n) = \sum A_i \binom{N_i}{m_i}$, m_i is not changed with *n*, then

$$\nabla^p f(n) = \sum A_i \binom{N_i - p}{m_i - p}$$

Sometimes $\nabla SUM(N)$ and sometimes $SUM(N)$ are listed below,

The corresponding $SUM(N)$ and $\nabla SUM(N)$ are easily obtained.

In particular, $S_1()$, $S_2()$ is unsigned Stirling number:

1.2) $SUM\left(N,[1,2,\cdots,M],[1,3,\cdots,2M-1]\right)=S_1\left(N+M,N\right)$.

1.3) $SUM\left(N,[1,1,\cdots,1],[1,3,\cdots,2M-1]\right)=S_2\left(N+M,N\right)$.

1.4) $SUM\left(N,[1,1,\cdots,1],[1,2,\cdots,M]\right)=1^M+2^M+\cdots+N^M$

1.5) $\nabla SUM\left(N,PS,[1,2,\cdots,M]\right)=\prod_{i=1}^{M}\left(K_i+\left(N-1\right)D_i\right)=\prod_{i=1}^{M}\left(K_i+nD_i\right)$

1.6) In $SUM\left(N,[\cdots PS\cdots],[\cdots,T+1,T+2,\cdots,T+M,\cdots]\right)$, K_i can exchange order

1.7) $SUM\left(N,[L_1,\cdots,L_P,PS],[L_1,\cdots,L_P,PT]\right)=\prod_{i=1}^{P}L_i SUM\left(N,PS,PT\right)$

This indicates that T_1 can be greater than 1, T is defined in $\mathbb{N}$.

1.8) $SUM\left(N,PT,PT\right)=\prod_{i=1}^{M}T_i\binom{N+T_M}{T_M+1}$

1.9) $\sum_{q=0}^{M}H_1\left(q\right)\binom{A}{B-q}=\sum_{q=0}^{M}H_2\left(q\right)\binom{A+q}{B}=\sum_{q=0}^{M}H_3\left(q\right)\binom{A+M-q}{B-q}$

This indicates $\text{Form}_1 = \text{Form}_2 = \text{Form}_3$. If regardless of the actual meaning, PT's domain can be extended to $\mathbb{C}$.

The Shape of Numbers of [1] has nothing to do with triangle numbers, square numbers, etc. This paper calls them Formal Calculation.

2. Simplified Formula

$H\left(q\right)=\sum\prod X, X\in T$ or $K=\sum\left(\prod X\in T\right)\left(\prod X\in K\right)$

Sometimes simple expressions can be obtained.

Define

$F_q^{K=\{K_1,K_2,\cdots\}}=\sum\prod_{i=1}^{q}I_i$, $I_i\in K$ and $I_i\neq I_j$. $F_q^{\{1,2,\cdots,N\}}$ abbreviated as F_q^N

$E_q^{K=\{K_1,K_2,\cdots\}}=\sum\prod_{i=1}^{q}I_i$, $I_i\in K$, $E_q^{\{1,2,\cdots,N\}}$ abbreviated as E_q^N

$$F_q^N=\sum_{1\le\lambda_1<\lambda_2<\cdots<\lambda_q\le N}\prod\lambda=S_1\left(N+q,N\right)$$

$$E_q^N=\sum_{1\le\lambda_1<\lambda_2<\cdots<\lambda_q\le N}\prod\lambda=\sum_{\lambda_1+\lambda_2+\cdots+\lambda_N=q}1^{\lambda_1}2^{\lambda_2}\cdots N^{\lambda_N}=S_2\left(N+q,N\right)$$

$$[A:D]_q=A(A-D)(A-2D)\cdots\left(A-\left(q-1\right)D\right),[A:D]_0=1$$

$$[A:D]^q=A(A+D)(A+2D)\cdots\left(A+\left(q-1\right)D\right),[A:D]^0=1$$

$$H\left(q,T\right)=\left(\prod X\in T\right)\text{ of }H\left(q\right),H\left(q,K\right)=\left(\prod X\in K\right)\text{ of }H\left(q\right)$$

$$H\left(q,\sum T\right)=\sum H\left(q,T\right),H\left(q,\sum K\right)=\sum H\left(q,K\right)$$

$$PT=\left[T_1=T+1,\cdots,T_M=T+M\right]\rightarrow H_1\left(q,T\right)=H_2\left(q,T\right)=D^qT_1\cdots T_q$$

$$PT=\left[1,2,\cdots,M\right],D=1\rightarrow H_1\left(q,\sum K\right)=F_{M-q}^KE_0^q+F_{M-q-1}^KE_1^q+\cdots+F_0^KE_{M-q}^q$$

$$f\left(n\right)=a_0+a_1n+a_2n^2+\cdots+a_Mn^M=a_M\left(K_1+n\right)\left(K_2+n\right)\cdots\left(K_M+n\right)$$

$$F\left(N\right)=\sum_{n=0}^{N-1}f\left(n\right)=a_M SUM\left(N,[K_1,K_2,\cdots,K_M],[1,2,\cdots,M]\right)$$

According to Vieta's formulas, $F_i^K=\dfrac{a_{M-i}}{a_M}$

$$H_1(q)=q!\sum_{i=0}^{M-q}F_{M-q-i}^{K}E_i^q=q!\sum_{i=0}^{M-q}\frac{a_{q+i}}{a_M}E_i^q$$

$$a_M H_1(0)=0!a_0$$

$$a_M H_1(1)=1!(a_1+a_2+\cdots+a_M)$$

$$a_M H_1(2)=2!(a_2+3a_3+7a_4+\cdots)$$

$$=2!(S_2(2,2)a_2+S_2(3,2)a_3+S_2(4,2)a_4+\cdots)$$

2.1) $F(N)=\sum_{q=0}^{M}H_1(q)\binom{N}{q+1}$

$$a_M\begin{bmatrix}H_1(0)\\ \vdots\\ H_1(M)\end{bmatrix}=\begin{bmatrix}0!\\ \vdots\\ M!\end{bmatrix}\begin{bmatrix}S_2(0,0) & \cdots & S_2(M,0)\\ \vdots & \ddots & \vdots\\ S_2(0,M) & \cdots & S_2(M,M)\end{bmatrix}\begin{bmatrix}a_0\\ \vdots\\ a_M\end{bmatrix}$$

It's the same as: $F(N)=\sum_{q=0}^{M}a_q\sum_{n=0}^{N-1}n^q=\sum_{q=0}^{M}a_q\sum_{i=1}^{q}i!S_2(q,i)\binom{N}{i+1}$

2.1. *PT* = [*T* + 1, *T* + 2, ..., *T* + *M*], *PS* = [*P* − (*M* − 1)*D*, *P* − (*M* − 2)*D*, ..., *P*]:*D*

PS can exchange order $=[P,P-D,P-2D,\cdots]:D$

$H_1(q,K)=[P:D]_{M-q}$,

$$H_1(q,\sum K)=\binom{M}{M-q}H_1(q,K)\xrightarrow{D=1}\binom{M}{q}\binom{P}{M-q}(M-q)!$$

2.1.1) $\binom{n+A}{A}\binom{n+M+B}{M}$

① $=\sum_{q=0}^{M}\binom{A+q}{q}\binom{M+B}{M-q}\binom{n+A}{A+q}$,

② $=\sum_{q=0}^{M}(-1)^{M-q}\binom{A-B}{M-q}\binom{A+q}{q}\binom{n+A+q}{A+q}$,

③ $=\sum_{q=0}^{M}\binom{A-B}{q}\binom{M+B}{M-q}\binom{n+A+M-q}{A+M}$.

[Proof]

$$=\frac{1}{A!M!}\nabla SUM(N,[1,2,\cdots,A,B+1,B+2,\cdots,B+M],$$
$$[1,2,\cdots,A,A+1,\cdots,A+M])$$

$$=\frac{1}{M!}\nabla SUM(N,[B+1,B+2,\cdots,B+M],[A+1,A+2,\cdots,A+M])$$

$$=\frac{1}{M!}\nabla SUM(N,[B+M,\cdots,B+2,B+1],[A+1,A+2,\cdots,A+M])$$

$$\text{Form}_1=\frac{1}{M!}\nabla\sum_{q=0}^{M}H_1(q)\binom{N+A}{A+1+q}=\frac{1}{M!}\sum_{q=0}^{M}H_1(q)\binom{n+A}{A+q}$$

$$H_1(q)\xrightarrow{H_1(q,\sum K)=\binom{M}{M-q}\binom{M+B}{M-q}(M-q)!,H_1(q,T)=q!\binom{A+q}{q}}$$

$$q!\binom{A+q}{q}\binom{M}{q}\binom{M+B}{M-q}(M-q)!\to①$$

$$\mathrm{Form}_2 \xrightarrow{H_2(q,K)=[B-A]^{M-q}} \frac{1}{M!}\nabla\sum_{q=0}^{M}[B-A]^{M-q}[A+1]^q\binom{M}{q}\binom{N+A+q}{A+q+1}$$

$$=\frac{1}{M!}\sum_{q=0}^{M}(-1)^{M-q}\binom{A-B}{M-q}(M-q)!\binom{A+q}{q}q!\binom{M}{q}\binom{n+A+q}{A+q}\to ②$$

$$\mathrm{Form}_3 \xrightarrow{H_3(q,T)=[A-B]_q}$$

$$\frac{1}{M!}\sum_{q=0}^{M}[A-B]_q\binom{M}{q}\binom{M+B}{M-q}(M-q)!\binom{n+A+M-q}{A+M}\to ③$$

q.e.d.

$$A=M,B=0\to\binom{n+M}{M}^2\xrightarrow{\mathrm{Form}_1}\sum_{q=0}^{M}\binom{M+q}{q}\binom{M}{q}\binom{n+M}{M+q},$$

$$A=M,B=0\to\binom{n+M}{M}^2\xrightarrow{\mathrm{Form}_3}\sum_{q=0}^{M}\binom{M}{q}^2\binom{n+2M-q}{2M},$$ record at [2]: (6.32).

$$B=0\to\binom{n+A}{A}\binom{n+M}{M}\xrightarrow{\mathrm{Form}_3}\sum_{q=0}^{M}\binom{A}{q}\binom{M}{q}\binom{n+A+M-q}{A+M},$$ record at [2]: (6.21).

$$A=0\to\binom{x+y}{M}=\sum_{q=0}^{M}\binom{x}{M-q}\binom{y}{q}$$

if $0<B<M$, $PS=[0,-1,\cdots,-(B-1),A,A+1,\cdots,A+(M-B-1)]$

$H_1(q)\neq 0\to X_1,X_2,\cdots,X_B\in T\to H_1(q<B,\sum K)=0$

Method of 2.1.1) $\to H_1(q\geq B,\sum K)=\binom{M-B}{M-q}[A+(M-1)]_{M-q}$.

2.1.2) $\binom{n+X}{A}\binom{n+Y}{M}=\sum_{g=0}^{A}\binom{M+g}{g}\binom{M+X-Y}{A-g}\binom{n+Y}{M+g}, 0\leq Y\leq M$

[Proof]

$$=\frac{1}{A!M!}\nabla SUM\left(N,[X,X-1,\cdots,X-A+1,Y,Y-1,\cdots,Y-M+1],[1,2,\cdots]\right)$$

$$=\frac{1}{A!M!}\nabla SUM\left(N,[1,2,\cdots,Y,0,-1,\cdots,-(M-Y)+1,X,\cdots,X-A+1],[1,2,\cdots]\right)$$

$$=\frac{Y!}{A!M!}SUM\left(N,[0,-1,\cdots,-(M-Y)+1,X,\cdots,X-A+1],\right.$$
$$\left.[Y+1,Y+2,\cdots,M+A]\right)$$

$$H_1(q\geq M-Y,\sum K)\xrightarrow{Z=M+A-Y}\binom{Z-(M-Y)}{Z-q}[X+(M-Y)]_{Z-q}$$

$$\frac{Y!}{A!M!}H_1(q)$$

$$=\frac{Y!}{A!M!}\binom{A}{M+A-Y-q}[M+X-Y]_{M+A-Y-q}[Y+1]^q\xrightarrow{-g=M-Y-q}$$

$$=\frac{Y!}{A!M!}\binom{A}{A-g}\binom{M+X-Y}{A-g}(A-g)!\binom{M+g}{M-y+g}(M-y+g)!$$

$$=\binom{M+g}{g}\binom{M+X-Y}{A-g}$$

$$\binom{n+X}{A}\binom{n+Y}{M}=\frac{Y!}{A!M!}\nabla\sum_{q=M-Y}^{Z}H_1(q)\binom{N+(M+A)-Z}{(M+A)-Z+q+1}$$

$$=\frac{Y!}{A!M!}\sum_{g=0}^{A}H_1(g)\binom{n+Y}{M+g}$$

q.e.d.

$$M=Y,X=0\to\binom{n}{A}\binom{n+M}{M}=\binom{M+A}{A}\binom{n+M}{M+A}$$

$\sum_{n=0}^{N-1}\binom{n}{A}\binom{n+M}{M}=\binom{M+A}{A}\binom{N+M}{M+A+1}=\binom{N+M}{M}\binom{N}{A}\frac{N-A}{M+A+1}$, record at [3].

2.1.1) is for $\binom{n+A}{A}\binom{n+Y}{M}$, 2.1.2) is for $\binom{n+X}{A}\binom{n+Y}{M}$, $0\le Y\le M$.

There has no formula for $\binom{n+X}{A}\binom{n+Y}{M}$, $Y>M$, $X\ne A$.

2.1.3) $\binom{n}{B}\binom{A-n}{M-B}=\sum_{g=0}^{M-B}(-1)^{M-B-g}\binom{A-M+g}{g}\binom{M-g}{B}\binom{n}{M-g}, n\ge 0$

[Proof]

$PS=[0,-1,\cdots,-(B-1),A-(M-B-1):-1,\cdots,A:-1]$

$$\sum H_1(q,K)=\begin{cases}0, q<B\\ (-1)^q\binom{M-B}{M-q}[A-(M-B-1)]^{M-q}, q\ge B\end{cases}$$

q.e.d.

2.1.4) $[P+nD:D]_M=\sum_{q=0}^{M}D^q[M]_q[P:D]_{M-q}\binom{n}{q}$

2.1.5) $\binom{A-n}{M}=\frac{1}{M!}\sum_{q=0}^{M}(-1)^q q!\binom{M}{q}[A-M+1]^{M-q}\binom{n}{q}$

[Proof]

$PS=[A-M+1,\cdots,A-1,A]:-1$,

$PT=[1,2,\cdots,M]\to\binom{A-n}{M}=\frac{1}{M!}\nabla SUM(N)$,

$H_1(q,\sum K)=\binom{M}{q}[A-M+1]^{M-q}, H_1(q,T)=(-1)^q q!$

q.e.d.

$A=2M\to\binom{2M-n}{M}=\sum_{q=0}^{M}(-1)^q\binom{n}{q}\binom{2M-q}{M}$, record at [2]: (3.50).

2.1.6) $\binom{n}{A}\binom{n}{B}=\sum_{q=0}^{B}\binom{B}{q}\binom{A+B-q}{B-q}\binom{n}{A+B-q}$, record at [2]: (6.44).

[Proof]

$PS=[0,-1,\cdots,-A+1,0,-1,\cdots,-B+1], PT=[1,2,\cdots,A+B]$,

$\{X_1,X_2,\cdots,X_A\}\in T\to H_1(q<A)=0, H_1(q\ge A)=q!\binom{B}{q-A}[A]_{A+B-q}$

$$\binom{n}{A}\binom{n}{B}=\frac{1}{A!B!}\sum_{q=A}^{A+B}q!\binom{B}{q-A}[A]_{A+B-q}\binom{n}{q}$$

$$\xrightarrow{q:=A+q}\frac{1}{A!B!}\sum_{q=0}^{B}(A+q)!\binom{B}{q}[A]_{B-q}\binom{n}{A+q}\xrightarrow{q:=B-q}\text{conclusion}$$

q.e.d.

2.2. $P \geq 0$, $PT = [P + 1, P + 2, \ldots, P + M]$, $PS = [P + 2, P + 4, \ldots, P + 2M]$

$$H_2(q,\sum K)=SUM(q+1,[1,3,\cdots,2(M-q)-1],[1,3,\cdots,2(M-q)-1])$$

$$=(2(M-q)-1)!!\binom{2M-q}{q}$$

$$SUM(N)=\frac{1}{P!}SUM(N,[1,2,\cdots,P,PS],[1,2,\cdots,P,P+1,\cdots,P+M])$$

$$=\sum_{n=0}^{N-1}\binom{n+P}{P}(P+2+n)(P+4+n)\cdots(P+2M+n)$$

$$=\sum_{q=0}^{M}(2[M-q]-1)!!\binom{2M-q}{q}[P+1]^q\binom{N+P+q}{P+q+1}$$

2.2.1)

$$\binom{n+P}{P}\prod_{i=1}^{M}(P+2i+n)=\sum_{q=0}^{M}(2[M-q]-1)!!\binom{2M-q}{q}[P+1]^q\binom{n+P+q}{P+q}$$

2.2.2) $\prod_{i=1}^{M}(2i+n)=\sum_{q=0}^{M}(2[M-q]-1)!!\binom{2M-q}{q}q!\binom{n+q}{q}$

$$SUM(N,[1,3,\cdots,2M-1],[1,2,\cdots,M])=SUM(N,[3,\cdots,2M-1],[2,\cdots,M])$$

Change M to $M-1$ and $q!$ to $(q+1)!$ →

2.2.3) $\prod_{i=1}^{M}(2i+n-1)=\sum_{q=0}^{M-1}(2[M-q]-3)!!\binom{2M-q-2}{q}(q+1)!\binom{n+1+q}{q+1}$

2.3. $PT = [1, 3, \ldots, 2M - 1]$, $PS = [P + D, P + 3D, \ldots, P + T_M D]:D$

$$H_2(q,K)=[P:D]^{M-q}$$

$$H_2(q,\sum T)=D^q SUM(M-q+1,[1,3,\cdots,2q-1],[1,3,\cdots,2q-1])$$

2.3.1) $SUM(N)=\sum_{q=0}^{M}[P:D]^{M-q}D^q(2q-1)!!\binom{M+q}{2q}\binom{N+M-1+q}{M+q}$

$$P=1,D=1\to PS=[2,4,\cdots,2M]$$

2.3.2)

$$SUM(N)=\sum_{n_M=0}^{N-1}(2M+n_M)\cdots\sum_{n_2=0}^{n_3}(4+n_2)\sum_{n_1=0}^{n_2}(2+n_1)$$

$$=\sum_{q=0}^{M}\frac{(M+q)!}{(q)!2^q}\binom{N+M-1+q}{M+q}=\sum_{q=0}^{M}\binom{N-1+q}{q}\frac{[N+q]^M}{2^q}$$

(*) $N=1\to 2^M M!=\sum_{q=0}^{M}\frac{(M+q)!}{(q)!2^q}=\sum_{q=0}^{M}\frac{[1+q]^M}{2^q}$

This can also be obtained from 2.2.2).

2.4. $PT = [1, 3, \ldots, 2M - 1]$, $PS = [P, P + D, \ldots, P + (M - 1)D]:2D$

2.4.1) $SUM(N)=\binom{M+N-1}{M}[P+(M+N-2)D:D]_M$

[Proof]

$$H_1\left(q,\sum K,\left[P,P+D,\cdots,P+(M-1)D\right]:D,[1,2,\cdots,M]\right)$$
$$=SUM\left(q+1,\left[P,P+D,\cdots,P+(M-1-q)D\right]:2D,\left[1,3,\cdots,2(M-q)-1\right]\right)$$
$$=H_1\left(q,\sum K,\left[P+(M-1)D,\cdots,P+D,P\right]:D,[1,2,\cdots,M]\right)$$
$$=\binom{M}{q}H_1(q,K,\cdots)\xrightarrow{3.2)}\binom{M}{q}\left[P+(M-1)D:D\right]_{M-q}$$
$$SUM\left(q+1,\left[P,P+D,\cdots,P+(M-1-q)D\right]:2D,\left[1,3,\cdots,2(M-q)-1\right]\right)$$
$$=\binom{M}{q}\left[P+(M-1)D:D\right]_{M-q}\xrightarrow{M:=M-q}$$
$$SUM\left(q+1,\left[P,P+D,\cdots,P+(M-1)D\right]:2D,[1,3,\cdots,2M-1]\right)$$
$$=\binom{M+q}{q}\left[P+(M+q-1)D:D\right]_M$$
$$\xrightarrow{N:=q+1}\binom{M+N-1}{M}\left[P+(M+N-2)D:D\right]_M$$

q.e.d.

2.4.2)
$$SUM\left(N,[P+1,P+2,\cdots]:2,[1,3,\cdots,2M-1]\right)=M!\binom{N+M-1}{M}\binom{N+M+P-1}{M}$$

2.4.3) $SUM\left(N,[1,2,\cdots,M]:2,[1,3,\cdots,2M-1]\right)=M!\binom{N+M-1}{M}^2$

$$M=1\to 1+3+\cdots+(2N-1)=N^2$$
$$M=2\to SUM\left(N,[1,2]:2,[1,3]\right)=\sum_{n=0}^{N-1}(2+2n)SUM\left(n+1,[1]:2,[1]\right)$$
$$=\sum_{n=0}^{N-1}(2+2n)(1+n)^2=2\sum_{n=0}^{N-1}(1+n)^3\to\sum_{n=1}^{N}n^3=\binom{N+1}{2}^2$$

2.4.2) and 2.1.1)→

$$H\left(q,[P+1,P+2,\cdots]:2,[1,3,\cdots,2M-1]\right)$$
$$=M!H\left(q,[P+1,P+2,\cdots],[M+1,M+2,\cdots,2M]\right)$$

2.4.5) $\binom{n}{A}\binom{n}{B}=\sum_{q=0}^{B}\binom{A}{B-q}\binom{B}{q}\binom{n+B-q}{A+B}$, record at [2]: (6.45).

[Proof]

$PS=[0,-1,\cdots,-A+1,0,-1,\cdots,-B+1]$, $PT=[1,2,\cdots,A+B]$, use Form$_3$

$$\{X_1,X_2,\cdots,X_A\}\in T\to H_3(q<A)=0,q\ge A\to\prod_{i\le A}X_i\in T=A!$$
$$\sum\prod_{i>A}X_i\in T=SUM\left(A+B+1-q,[1,2,\cdots,q-A]:2,\left[1,3,\cdots,2(q-A)-1\right]\right)$$
$$=(q-A)!\binom{A+B-q+q-A}{q-A}^2$$
$$H_3(q\ge A)=A!(q-A)!\binom{B}{q-A}\binom{B}{q-A}[A]_{A+B-q}$$
$$\binom{n}{A}\binom{n}{B}=\frac{1}{A!B!}\sum_{q=A}^{A+B}H_3(q)\binom{n+A+B-q}{A+B}$$

$$\xrightarrow{q:=A+q} \frac{1}{A!B!}\sum_{q=0}^{B} A!q!\binom{B}{q}^2 [A]_{B-q}\binom{n+B-q}{A+B}$$

q.e.d.

2.5. $P \geq 0$, $PT = [1, 3, \ldots, 2M-1]$, $PS = [P+2, P+4, \ldots, P+2M]$:3

[1] has obtained: $H_1(q) = \sum_{x=q}^{M} H_2(x)\binom{x}{q} = \sum_{x=0}^{q} H_3(x)\binom{M-x}{M-q}$

2.5.1) $SUM(N) = \sum_{q=0}^{M}\binom{P+N-1+q}{q}\binom{N+M-1-q}{M-q}\frac{[M+N-q]^M}{2^{M-q}}$

[Proof]

For $SUM(N,[P+2,P+4,\cdots,P+2M],[P+1,P+2,\cdots,P+M])$

$H_1(q,T) = [P+1]^q$

$H_1(q,\sum K) = SUM(q+1,[P+2,P+4,\cdots]:3,[1,3,\cdots,2(M-q)-1])$

$H_1(q) = \sum_{x=q}^{M} H_2(x)\binom{x}{q}$

$$\xrightarrow{2.3.1),D=1} \sum_{x=q}^{M}(2[M-x]-1)!!\binom{2M-x}{x}[P+1]^x\binom{x}{q}$$

$SUM(q+1,[P+2,P+4,\cdots,P+2(M-q)]:3,[1,3,\cdots,2(M-q)-1])$

$$= \sum_{x=q}^{M}\frac{(2M-x)!}{(m-x)!2^{M-x}}\frac{[P+1]^x}{q!(x-q)![P+1]^q}$$

$$= \sum_{x=q}^{M}\frac{(2M-x)!}{(m-x)!2^{M-x}}\frac{(P+x)!}{q!(x-q)!(P+q)!}$$

$$\xrightarrow{x:=q+x} \sum_{x=0}^{M-q}\frac{(2M-q-x)!}{(M-q-x)!2^{M-q-x}}\frac{(P+x+q)!}{q!x!(P+q)!}$$

$SUM(q+1,[P+2,P+4,\cdots,P+2M]:3,[1,3,\cdots,2M-1])$

$$= \sum_{x=0}^{M}\frac{(2M+q-x)!}{(M-x)!2^{M-x}}\frac{(P+x+q)!}{q!x!(P+q)!} \to \text{conclusion}$$

q.e.d.

$SUM(N,[2,4,\cdots,2M]:3,[1,3,\cdots,2M-1])$

$$= \sum_{x=0}^{M}\frac{(2M+N-1-x)!}{(M-x)!2^{M-x}}\frac{(P+x+N-1)!}{x!(N-1)!(P+N-1)!} \to$$

2.5.2)

$SUM(N,[2,4,\cdots,2M]:3,[1,3,\cdots])$

$$= \sum_{q=0}^{M}\binom{N-1+q}{q}\binom{N+M-1-q}{M-q}\frac{[M+N-q]^M}{2^{M-q}}$$

2.5.3)

$SUM(N,[3,5,\cdots,2M-1]:3,[3,5,\cdots,2M-1])$

$$= \sum_{q=0}^{M-1}\binom{N+q}{q}\binom{N+M-2-q}{M-q-1}\frac{[M+N-q-1]^{M-1}}{2^{M-q-1}}$$

2.6. Summary

1.8)	SUM(N, [1, 2, 3], [1, 2, 3]) SUM(N, [1, 3, 5], [1, 3, 5])	
2.1	SUM(N, [1, 2, 3], [4, 5, 6]) SUM(N, [1, 3, 5]: 2, [1, 2, 3])	SUM(N, [2, 4, 6]: 2, [1, 2, 3])
2.2.2)		SUM(N, [2, 4, 6], [1, 2, 3])
2.2.3)	SUM(N, [1, 3, 5], [1, 2, 3])	
2.3.1)		SUM(N, [2, 4, 6], [1, 3, 5])
2.4.1)	SUM(N, [1, 2, 3]: 2, [1, 3, 5])	SUM(N, [2, 4, 6]: 4, [1, 3, 5])
2.5.2)		SUM(N, [2, 4, 6]: 3, [1, 3, 5])
2.5.3)	SUM(N, [3, 5, 7]: 3, [1, 3, 5])	

3. r-Flod Sum

Define $\sum_{(r)}^{N} f(k) = \sum_{k_r=1}^{N} \cdots \sum_{k_2=1}^{k_3} \sum_{k=1}^{k_2} f(k) = \sum_{k_r=0}^{N-1} \cdots \sum_{k_2=0}^{k_3} \sum_{k=0}^{k_2} f(k+1)$

$$\sum_{(r)}^{N} 1 = SUM\left(N,[1,1,\cdots,1]:0,[1,3,\cdots,2r-1]\right) \xrightarrow{H_1(q>0)=0} \binom{N+r-1}{r}$$

3.1) $\sum_{(r)}^{N} \nabla SUM(k+1,PS,PT) = \sum_{q=0}^{M} H_1(q,PS,PT)\binom{N+T_M-M+r-1}{T_M-M+r+q}$

[Proof]

$$PS1 = [PS,1:0,1:0,\cdots], PT1 = [PT,T_M+2,T_M+4,\cdots,T_M+2(r-1)]$$

$$\sum_{(r)}^{N} \nabla SUM(k+1,PS,PT) = SUM(N,PS1,PT1)$$

$$= \sum_{q=0}^{M+r-1} H_1(q,PS1,PT1)\binom{N+T_M-M+r-1}{T_M-M+r+q}$$

In $H_1(q,PS1,PT1)$, $i>M$, $\begin{cases}(T_i - X_{K-1})D_i = 0; X_i = T_i \\ K_i + X_{T-1}D_i = 1; X_i = K_i\end{cases}$

$$H_1(q>M,PS1,PT1) = 0, H_1(q\le M,PS1,PT1) = H_1(q,PS,PT)$$

q.e.d.

Another way: understand the definition of ∇^p and use three Forms.

3.2) $\sum_{(r)}^{N} \nabla SUM(k+1,PS,PT) = \nabla^{-(r-1)} SUM(N,PS,PT)$

$$\sum_{(r)}^{N} K^2 = \sum_{(r)}^{N} \nabla SUM(k+1,[1,1],[1,2])$$

$$= SUM(N,[1,1,1:0,1:0,\cdots],[1,2,4,6,\cdots,2r])$$

$$= 2\binom{N+r-1}{r+2} + 3\binom{N+r-1}{r+1} + \binom{N+r-1}{r} = 2\binom{N+r}{r+2} + \binom{N+r}{r+1}$$

$$= \frac{2(N+r)!(N-1)}{(r+2)!(N-1)!} + \frac{(N+r)!(r+2)}{(r+2)!(N-1)!} = \frac{(N+r)!(2N+r)}{(r+2)!(N-1)!} \rightarrow \text{record at [2].}$$

$$\sum_{(r)}^{N} K^3 = \nabla^{-(r-1)} SUM(N,[1,1],[2,3])$$

$$= \nabla^{-(r-1)}\left[6\binom{N+1}{4} + 6\binom{N+1}{3} + \binom{N+1}{2}\right]$$

$$=6\binom{N+r}{r+3}+6\binom{N+r}{r+2}+\binom{N+r}{r+1}$$

$$=\frac{\left(6N^2+r\left(6N+r-1\right)\right)\left(N+r\right)!}{\left(r+3\right)!\left(N-1\right)!}\rightarrow \text{ record at [2].}$$

3.3) $\sum_{k_r=0}^{N-1}\cdots\sum_{k_2=0}^{k_3}\sum_{k=0}^{k_2}\binom{k_i}{p}=\binom{p+i-1}{p}\binom{N+r-1}{r+p}\xrightarrow{p=1}i\binom{N+r-1}{r+1}$

[Proof]

$$\sum_{k_r=0}^{N-1}\cdots\sum_{k_2=0}^{k_3}\sum_{k=0}^{k_2}\binom{k_i}{p}=\sum_{k_r=0}^{N-1}\cdots\sum_{k_i=0}^{k_{i+1}}\binom{k_i}{p}\binom{k_i+i-1}{i-1}$$

$$PS1=\left[1,2,\cdots,i-1,0,-1,-2,\cdots,-p+1,1:0,1:0,\cdots,1:0\right]$$

$$PT1=\left[1,2,\cdots,i-1,i,i+1,\cdots,i+p-1,i+p+1,i+p+3,\cdots,i+p+2\left(r-i\right)-1\right]$$

$$=\frac{1}{p!\left(i-1\right)!}SUM\left(N,PS1,PT1\right)$$

$$PS=\left[0,-1,-2,\cdots,-p+1,1:0,1:0,\cdots,1:0\right]$$

$$PT=\left[i,i+1,i+2,\cdots,i+p-1,i+p+1,i+p+3,\cdots,i+p+2\left(r-i\right)-1\right]$$

$$=\frac{1}{p!}SUM\left(N,PS,PT\right)\xrightarrow{H_1(q\neq p)=0,\{X_1,X_2,\cdots,X_p\}\in T}\binom{p+i-1}{p}\binom{N+r-1}{r+p}$$

q.e.d.

This is the conclusion of [4] and the proof is simpler.

4. $\sum_{n=0}^{N-1}\nabla^{P_1}SUM\left(n+1,\left[K_1\right],\left[T_1\right]\right)\nabla^{P_2}SUM\left(n+1,\left[K_2\right],\left[T_2\right]\right)\cdots$

4.1) $P\le T,\nabla^P SUM\left(N,\left[K:D\right],\left[T\right]\right),D\neq 0$

$$=\frac{TD}{\left(T+1-P\right)!}\left(n+1\right)\left(n+2\right)\cdots\left(n+T-P\right)\left(n+\frac{T+1-P}{TD}K\right)$$

[Proof]

$$\nabla^P SUM\left(N,\left[K:D\right],\left[T\right]\right)=\nabla^P\left[K\binom{N+T-1}{T}+TD\binom{N+T-1}{T+1}\right]$$

$$=\left[K\binom{n+T-P}{T-P}+TD\binom{n+T-P}{T+1-P}\right]$$

$$=\frac{1}{\left(T-P\right)!}\left(n+1\right)\left(n+2\right)\cdots\left(n+T-P\right)\left(\frac{TD}{T+1-P}n+K\right)$$

$$=\frac{TD}{\left(T+1-P\right)!}\left(n+1\right)\left(n+2\right)\cdots\left(n+T-P\right)\left(n+\frac{T+1-P}{TD}K\right)$$

q.e.d.

This leads to:

4.2) $P_1\le T_1,P_2\le T_2,D_1\neq 0,D_2\neq 0$

$$PS=\left[1,2,\cdots,T_1-P_1,\frac{K_1\left(T_1+1-P_1\right)}{T_1D_1},1,2,\cdots,T_2-P_2,\frac{K_2\left(T_2+1-P_2\right)}{T_2D_2}\right]$$

$$PT=\left[1,2,3,\cdots,T_1+T_2+2-P_1-P_2\right]$$

$$\sum_{n=0}^{N-1}\nabla^{P_1}SUM\left(n+1,[K_1:D_1],[T_1]\right)\nabla^{P_2}SUM\left(n+1,[K_2:D_2],[T_2]\right)$$
$$=\frac{T_1T_2D_1D_2}{(T_1+1-P_1)!(T_2+1-P_2)!}SUM\left(N,PS,PT\right)$$

4.3)

$$\sum_{n=0}^{N-1}\nabla SUM\left(n+1,[K_1],[T_1]\right)\nabla SUM\left(n+1,[K_2],[T_2]\right)$$
$$=\sum_{n=0}^{N-1}\left[K_1\binom{n+T_1-1}{T_1-1}+T_1\binom{n+T_1-1}{T_1}\right]\left[K_2\binom{n+T_2-1}{T_2-1}+T_2\binom{n+T_2-1}{T_2}\right]$$
$$=\frac{1}{(T_1-1)!}\frac{1}{(T_2-1)!}SUM\left(N,[1,2,\cdots,T_1-1,K_1,1,2,\cdots,T_2-1,K_2],[1,2,\cdots,T_1+T_2]\right)$$
$$T_1=T_2=1\to\sum_{n=0}^{N-1}(n+K_1)(n+K_2)=SUM\left(N,[K_1,K_2],[1,2]\right)$$

4.4)

$$\sum_{n=0}^{N-1}SUM\left(n+1,[K_1],[T_1]\right)SUM\left(n+1,[K_2],[T_2]\right)$$
$$=\frac{T_1T_2}{(T_1+1)!(T_2+1)!}SUM\left(N,\left[1,\cdots,T_1,\frac{K_1(T_1+1)}{T_1},1,\cdots,T_2,\frac{K_2(T_2+1)}{T_2}\right],\right.$$
$$\left.[1,2,\cdots,T_1+T_2+2]\right)$$

The following calculation problems have been solved:

$$\sum_{n=0}^{N-1}\nabla^{P_1}SUM\left(n+1,[K_1:D_1],[T_1]\right)\nabla^{P_2}(\ldots)\nabla^{P_3}(\ldots)\nabla^{P_4}(\ldots)\cdots$$
$$\nabla\sum_{n=0}^{N-1}(K+n)=\nabla SUM\left(N,[K],[1]\right)=K+n$$
$$\nabla\sum_{n,n_1,n_2=0,n_1\le n_2=n}^{N-1}(K+n_1+n_2)\xrightarrow{n_1\in[0,n],\text{count of items}=n+1}$$
$$=(K+n_2)\binom{n+1}{1}+\sum_{n_1=0}^{n}n_1=(K+n)\binom{n+1}{1}+\binom{n+1}{2}$$
$$=K\binom{n+1}{1}+3\binom{n+1}{2}=\nabla^2SUM\left(N,[K],[3]\right)$$

$$\nabla\sum(K+n_1+\cdots+n_M)\xrightarrow{\text{Repeatable selection } M-1 \text{ in } [0,n],\text{count of items}=\binom{n+1+M-1-1}{M-1}}$$
$$=(K+n)\binom{n+M-1}{M-1}+\sum_{n_{M-1}=0}^{n}\cdots\sum_{n_1=0}^{n_2}(n_1+n_2+\cdots+n_{M-1})$$
$$\xrightarrow{3.3),P=1}(K+n)\binom{n+M-1}{M-1}+\sum_{i=1}^{M-1}i\binom{n+M-1}{M+1}$$
$$=(K+n)\binom{n+M-1}{M-1}+\binom{M}{2}\binom{n+M-1}{M}$$
$$=(K+n)\binom{n+M-1}{M-1}+\binom{M}{2}\binom{n+M-1}{M}$$
$$=K\binom{n+M-1}{M-1}+n\binom{n+M-1}{M-1}+\binom{M}{2}\binom{n+M-1}{M}$$
$$=K\binom{n+M-1}{M-1}+\left(M+\binom{M}{2}\right)\binom{n+M-1}{M}$$
$$=K\binom{n+M-1}{M-1}+\binom{M+1}{2}\binom{n+M-1}{M}$$

$$= K\binom{N+M-2}{M-1} + \binom{M+1}{2}\binom{N+M-2}{M}$$

$$= \nabla^{\binom{M}{2}+1} SUM\left(N,[K],\left[\binom{M+1}{2}\right]\right)$$

4.5) $\sum_{n_1,n_2,\cdots,n_M=0,n_1\le n_2\le\cdots\le n_M}^{N-1}(K+n_1+\cdots+n_M) = \nabla^{\binom{M}{2}} SUM\left(N,[K],\left[\binom{M+1}{2}\right]\right)$

4.6) $\sum_{n_1,n_2,\cdots,n_M=0,n_1\le n_2\le\cdots\le n_M}^{N-1}(K+n_1d_1+\cdots+n_Md_M)$
$= \nabla^{\binom{M}{2}} SUM\left(N,\left[K:(d_1+2d_2+\cdots+Md_M)\binom{M+1}{2}^{-1}\right],\left[\binom{M+1}{2}\right]\right)$

Use 4.1), the following calculation problems have been solved:

$$\sum_0^{N-1}\nabla^{P_1}\left(\sum\left(K+n_1d_{1,1}+\cdots+n_{M_1}d_{1,M}\right)\right)\nabla^{P_2}\left(\sum\left(K+n_1d_{2,1}+\cdots\right)\right)\nabla^{P_3}(\ldots)\cdots$$

Investigation

$$\sum_{n_1,n_2,\cdots,n_M=0,n_1\le n_2\le\cdots\le n_M}^{N-1}\left(K_1+n_1d_{1,1}+\cdots+n_Md_{M,1}\right)\left(K_2+n_1d_{1,2}+\cdots+n_Md_{M,2}\right)$$

$$\sum_{n=0}^{N-1}\left(K_1+nd_{1,1}\right)\left(K_2+nd_{1,2}\right) = SUM\left(N,\left[K_1:d_{1,1},K_2:d_{1,2}\right],[1,2]\right)$$

$$= 2d_{1,1}d_{12}\binom{N}{3} + \left(K_1d_{1,2}+K_2d_{1,1}+d_{1,1}d_{1,2}\right)\binom{N}{2} + K_1K_2\binom{N}{1}$$

Suppose

$$F(N,M) = \sum_{n_1,n_2,\cdots,n_M=0,n_1\le n_2\le\cdots\le n_M}^{N-1}\left(K_1+n_1d_{1,1}+\cdots\right)\left(K_2+n_1d_{1,2}+\cdots\right)$$

$$= A(M)\binom{N+M-1}{M+2} + B(M)\binom{N+M-1}{M+1} + K_1K_2\binom{N+M-1}{M}$$

Let $D_i = d_{1,i}+2d_{2,i}+\cdots+Md_{M,i}, D_i^M = d_{1,i}+2d_{2,i}+\cdots+(M-1)d_{M-1,i}$

$$F(N,M+1) = \sum_{n_{M+1}=0}^{N-1}F(N,M)$$

$$+\sum_{n_{M+1}=0}^{N-1}n_{M+1}d_{M+1,2}\sum_{n_1,n_2,\cdots,n_M=0,n_1\le n_2\le\cdots\le n_M}^{N-1}\left(K_1+n_1d_{1,1}+\cdots\right)$$

$$+\sum_{n_{M+1}=0}^{N-1}n_{M+1}d_{M+1,1}\sum_{n_1,n_2,\cdots,n_M=0,n_1\le n_2\le\cdots\le n_M}^{N-1}\left(K_2+n_1d_{2,1}+\cdots\right)$$

$$+\sum_{n_{M+1}=0}^{N-1}n_{M+1}d_{M+1,1}d_{M+1,2}\sum_{n_1,n_2,\cdots,n_M=0,n_1\le n_2\le\cdots\le n_M}^{N-1}1$$

$$= A(M)\binom{N+M}{(M+1)+2} + B(M)\binom{N+M}{(M+1)+1} + K_1K_2\binom{N+M}{M+1}$$

$$+SUM\left(N,\left[K_1:D_1\binom{M+1}{2}^{-1},0:d_{M+1,2}\right]\left[\binom{M+1}{2},\binom{M+1}{2}+2-\binom{M}{2}\right]\right)$$

$$+SUM\left(N,\left[K_2:D_2\binom{M+1}{2}^{-1},0:d_{M+1,1}\right]\left[\binom{M+1}{2},M+2\right]\right)$$

$$+SUM\left(N,\left[1:0,1:0,\cdots,1:0,0:d_{M+1,1},0:d_{M+1,2}\right][1,3,\cdots,2M-1,2M]\right)$$

$$= A(M)\binom{N+M}{(M+1)+2} + B(M)\binom{N+M}{(M+1)+1} + K_1K_2\binom{N+M}{M+1}$$

$$+D_1(M+2)d_{M+1,2}\binom{N+M}{(M+1)+2} + \left\{K_1(M+1)d_{M+1,2}+D_1d_{M+1,2}\right\}\binom{N+M}{(M+1)+1}$$

$$+D_2(M+2)d_{M+1,1}\binom{N+M}{(M+1)+2}+\left\{K_2(M+1)_{M+1,1}+D_2d_{M+1,1}\right\}\binom{N+M}{(M+1)+1}$$

$$+(M+1)(M+2)d_{M+1,1}d_{M+1,2}\binom{N+M}{(M+1)+2}+(M+1)d_{M+1,1}d_{M+1,2}\binom{N+M}{(M+1)+1}$$

$$A(0)=0,B(0)=0$$

$$A(M)=A(M-1)+M(M+1)d_{M,1}d_{M,2}+(M+1)\left(D_i^M d_{M,2}+D_i^M d_{M,1}\right)$$

$$=A(M-1)+(M+1)\left(D_1d_{M,2}+D_2d_{M,1}-Md_{M,1}d_{M,2}\right)$$

$$B(M)=B(M-1)+M(d_{M,1}d_{M,2}+K_1d_{M,2}+K_2d_{M,1})+D_i^M d_{M,2}+D_i^M d_{M,1}$$

$$=B(M-1)+M\left(K_1d_{M,2}+K_2d_{M,1}-d_{M,1}d_{M,2}\right)+D_1d_{M,2}+D_2d_{M,1}$$

4.7) $F(N,M)=A(M)\binom{N+M-1}{M+2}+B(M)\binom{N+M-1}{M+1}+K_1K_2\binom{N+M-1}{M}$

$$d_{i,1}=1,d_{i,2}=1\rightarrow$$

$$A(M)=\sum_{n=0}^{M}\left\{n(n+1)+2(n+1)\binom{n}{2}\right\}=2\binom{M+2}{3}+6\binom{M+2}{4}$$

$$B(M)=\sum_{n=0}^{M}\left\{n(K_1+K_2+1)+2\binom{n}{2}\right\}=\binom{M+1}{2}(K_1+K_2+1)+2\binom{M+1}{3}$$

5. Formal Calculation of Gaussian Coefficients

Define: $G_M^N=\begin{bmatrix}N\\M\end{bmatrix}_q=\dfrac{\left(q^N-1\right)\left(q^{N-1}-1\right)\cdots\left(q^{N-M+1}-1\right)}{\left(q^M-1\right)\left(q^{M-1}-1\right)\cdots(q-1)}$

$$G_0^N=1,G_{M<0\text{ or }M>N}^N=0,G_M^N=G_{N-M}^N,G_M^N=q^MG_M^{N-1}+G_{M-1}^{N-1}$$

5.1) $G_M^N=G_M^{N-1}+q^{N-M}G_{M-1}^{N-1}$

5.2) $\sum_{n=0}^{N-1}q^nG_M^{n+M}=G_{M+1}^{N+M}$

5.3) $\sum_{n=0}^{N-1}q^nG_M^{n+K}=q^{M-K}G_{M+1}^{N+K};\sum_{n=0}^{N-1}q^nG_M^n=q^MG_{M+1}^N$

5.4) $\sum_{n=0}^{N-1}q^nG_1^nG_M^{n+K},M>0,M\geq K$

① $=q^{2(M-K)+1}G_1^{M+1}G_{M+2}^{N+K}+q^{M-K}G_1^{M-K}G_{M+1}^{N+K}$,

② $=q^{M-2K-1}G_1^{M+1}G_{M+2}^{N+K+1}+q^{M-K}\left(G_1^{M-K}-q^{-K-1}G_1^{M+1}\right)G_{M+1}^{N+K}$,

③ $=\left(q^{2(M-K)+1}G_1^{M+1}-q^{2M-K+2}G_1^{M-K}\right)G_{M+2}^{N+K}+q^{M-K}G_1^{M-K}G_{M+2}^{N+K+1}$.

[1] has obtained the formal formula of $\sum_{n=0}^{N-1}q^n\prod_{i=1}^{M}\left(K_i+D_iq^n\right)$, but it cannot be generalized to $\sum_{n_M=0}^{N-1}\cdots\sum_{n_2=0}^{n_3}\cdots\sum_{n_1=0}^{n_2}\cdots$.

Notice $K_i+D_in=K_i+D_i\binom{n}{1}$, this inspired use G_1^n instead of q^n.

The difficulty lies in the definition of ∇_q^P.

Recursive define operator ∇_q^P:

$$\nabla_q^0f(N)=f(N),\sum_{n=0}^{N-1}q^n\nabla_q^1f(n+1)=f(N),\sum_{n=0}^{N-1}q^nf(n+1)=\nabla_q^{-1}f(N)$$

$$SUM_q\left(N,[K_1:D_1],[T_1=1]\right)=\sum_{n=0}^{N-1}q^n\left(D_1G_1^n+K_1\right)$$

$$SUM_q\left(N,[K_1:D_1,K_2:D_2],[T_1=1,T_2=T_1+2-P]\right)$$

$$=\sum_{n=0}^{N-1}q^n\nabla^PSUM_q\left(n+1,[K_1:D_1],[1]\right)\left(D_2G_2^n+K_2\right)$$

Recursive definie $SUM_q\left(N,\left[K_1:D_1,K_2:D_2,\cdots\right],PT\right)$.

5.5) $\nabla^1 SUM_q\left(N,\left[K_1:D_1,K_2:D_2,\cdots\right],\left[1,2,\cdots,M\right]\right)=\prod_{i=1}^{M}\left(D_iG_1^n+K_i\right)$

5.6) $$SUM_q\left(N,\left[K_1:D_1,K_2:D_2,\cdots\right],\left[1,3,\cdots,2M-1\right]\right)$$
$$=\sum_{n_M=0}^{N-1}\cdots\sum_{n_2=0}^{n_3}q^{n_2}\left(D_2G_1^{n_2}+K_2\right)\sum_{n_1=0}^{n_2}q^{n_1}\left(D_1G_1^{n_1}+K_1\right)$$

The Form: $\left(T_1+K_1\right)\left(T_2+K_2\right)\cdots\left(T_M+K_M\right)=\sum\prod_{i=1}^{M}X_i$

5.7) $H^q\left(g\right)=H^q\left(PS,PT,g\right)=\sum_{X(T)=g}\prod_{i=1}^{M}B_i,SUM_q\left(N\right)=$

$$\begin{cases}\xrightarrow{\text{Form}_1}\sum_{g=0}^{M}H_1^q\left(g\right)G_{N-1-\text{g}}^{N+T_M-M}=\sum_{g=0}^{M}H_1^q\left(g\right)G_{T_M-M+1+\text{g}}^{N+T_M-M}\\ \xrightarrow{\text{Form}_2}\sum_{g=0}^{M}H_2^q\left(g\right)G_{N-1}^{N+T_M-M+g}=\sum_{g=0}^{M}H_2^q\left(g\right)G_{T_M-M+1+\text{g}}^{N+T_M-M+g}\\ \xrightarrow{\text{Form}_3}\sum_{g=0}^{M}H_3^q\left(g\right)G_{N-1-g}^{N+T_M-g}=\sum_{g=0}^{M}H_3^q\left(g\right)G_{T_M+1}^{N+T_M+g}\end{cases}$$

$$\begin{cases}B_i\xrightarrow{\text{Form}_1}\begin{cases}q^{\{T_i-T_{i-1}\}X_{T-1}+1}G_1^{T_i-X_{K-1}}D_i;X_i=T_i\\ q^{(T_i-T_{i-1}-1)X_{T-1}}\left(K_i+G_1^{X_{T-1}}D_i\right);X_i=K_i\end{cases}\\ B_i\xrightarrow{\text{Form}_2}\begin{cases}q^{-(T_i-X_{K-1})}G_1^{T_i-X_{K-1}}D_i;X_i=T_i\\ K_i-q^{-(T_i-X_{K-1})}G_1^{T_i-X_{K-1}}D_i;X_i=K_i\end{cases}\\ B_i\xrightarrow{\text{Form}_3}\begin{cases}q^{(T_i-T_{i-1}-1)X_{T-1+1}}\left\{\left(q^{X_{T-1}}G_1^{T_i}-q^{T_i}G_1^{X_{T-1}}\right)D_i-K_iq^{T_i}\right\};X_i=T_i\\ q^{(T_i-T_{i-1}-1)X_{T-1}}\left(K_i+G_1^{X_{T-1}}D_i\right);X_i=K_i\end{cases}\end{cases}$$

[Proof]

$$SUM_q\left(N,\left[K_1:D_1\right],\left[1\right]\right)\xrightarrow{\text{Form}_1}q^1D_1G_2^N+K_1G_1^N$$
$$\xrightarrow{\text{Form}_2}q^{-1}D_1G_2^{N+1}+\left(K_1-q^{-1}D_1\right)G_1^N$$
$$\xrightarrow{\text{Form}_3}\left(q^1D_1-K_1q^2\right)G_2^N+K_1G_2^{N+1}$$

If $f\left(N\right)=\sum A_iG_{M_i+1}^{N+M_i}$, $\nabla_q^P f\left(N\right)=\sum A_iG_{M_i+1-P}^{N+M_i-P}$, Form₂ is simplest.

Assume $SUM_q\left(N,PS,PT\right)=\sum_{g=0}^{M}H_2^q\left(g\right)G_{T_M-M+1+g}^{N+T_M-M+g}, X=T_M-M+1-P$

$$\nabla_q^P SUM_q\left(n+1,PS,PT\right)=\sum_{g=0}^{M}H_2^q\left(g\right)G_{T_M-M+1+g-P}^{n+1+T_M-M+g-P}=\sum_{g=0}^{M}H_2^q\left(g\right)G_{X+g}^{n+X+g}$$

$$SUM_q\left(N,PSA,PTA\right)$$
$$=\sum_{n=0}^{N-1}q^n\sum_{g=0}^{M}H_2^q\left(g\right)G_{X+g}^{n+X+g}\left(D_{M+1}G_1^n+K_{M+1}\right)$$
$$=\sum_{g=0}^{M}D_{M+1}q^{-X-g-1}G_1^{X+g+1}H_2^q\left(g\right)G_{X+g+2}^{N+X+g+1}$$
$$+\sum_{g=0}^{M}\left(K_{M+1}-q^{-X-g-1}G_1^{X+g+1}D_{M+1}\right)H_2^q\left(g\right)G_{X+g+1}^{N+X+g}$$
$$=\sum_{g=0}^{M}D_{M+1}q^{-(T_{M+1}-[M-g])}G_1^{T_{M+1}-[M-g]}H_2^q\left(g\right)G_{T_{M+1}-(M+1)+g+2}^{N+T_{M+1}-(M+1)+g+1}$$
$$+\sum_{g=0}^{M}\left(K_{M+1}-q^{-(T_{M+1}-[M-g])}G_1^{T_{M+1}-[M-g]}D_{M+1}\right)H_2^q\left(g\right)G_{T_{M+1}-(M+1)+g+1}^{N+T_{M+1}-(M+1)+g}$$
$$=\sum_{g=0}^{M+1}H_2^q\left(PSA,PTA,g\right)G_{N-1}^{N+T_{M+1}-(M+1)+g}\to\text{Form}_2$$

If $f\left(N\right)=\sum A_iG_{M_i}^{N+K},\nabla_q^P f\left(N\right)=\sum A_iq^{-(M_i-K)P}G_{M_i-P}^{N+K-P}$

Assume $SUM_q\left(N,PS,PT\right)=\sum_{g=0}^{M}H_1^q\left(g\right)G_{T_M-M+1+g}^{N+T_M-M}, X=T_M-M+1-P$

$$\nabla_q^P SUM\left(n+1\right)=\sum_{g=0}^{M}q^{-Pg}H_1^q\left(g\right)G_{T_M-M+1+g-P}^{n+1+T_M-M-P}=\sum_{g=0}^{M}q^{-Pg}H_1^q\left(g\right)G_{X+g}^{n+X}$$

$$\begin{aligned}
&SUM_q(N,PSA,PTA)\\
&=\sum_{n=0}^{N-1}q^n\sum_{g=0}^{M}q^{-Pg}H_1^q(g)G_{X+g}^{n+X}\times\left(D_{M+1}G_1^n+K_{M+1}\right)\\
&=\sum_{g=0}^{M}q^{-Pg}D_{M+1}q^{2g+1}G_1^{X+g+1}H_1^q(g)G_{X+g+2}^{N+X}\\
&\quad+\sum_{g=0}^{M}q^{-Pg}\left(D_{M+1}q^gG_1^g+K_{M+1}q^g\right)H_1^q(g)G_{X+g+1}^{N+X}\\
&=\sum_{g=0}^{M}D_{M+1}q^{(2-P)g+1}G_1^{T_{M+1}-[M-g]}H_1^q(g)G_{N-1-(g+1)}^{N+T_{M+1}-(M+1)}\\
&\quad+\sum_{g=0}^{M}q^{g(1-P)}\left(G_1^gD_{M+1}+K_{M+1}\right)H_1^q(g)G_{N-1-g}^{N+T_{M+1}-(M+1)}\\
&=\sum_{g=0}^{M+1}H_1^q(PSA,PTA,g)G_{N-1-g}^{N+T_{M+1}-(M+1)}\to\text{Form}_1
\end{aligned}$$

If $f(N)=\sum A_iG_M^{N+K_i},\nabla_q^Pf(N)=\sum A_iq^{-(M-K_i)P}G_{M-P}^{N+K_i-P}\to\text{Form}_3$

q.e.d.

From the proof process→

5.8) $\sum_{g=0}^{M}H_1^q(g)G_{B-g}^{A}=\sum_{g=0}^{M}H_2^q(g)G_B^{A+g}=\sum_{g=0}^{M}H_3^q(g)G_{B-g}^{A+M-g}$

When regardless of the actual meaning, $\text{Form}_1=\text{Form}_2=\text{Form}_3$ is still established.

PT's domain can be extended to $\mathbb{C}$.

$$\begin{aligned}
&SUM_q(N,[K_1,K_2],[1,2])\\
&\xrightarrow{\text{Form}_2}q^{-1}q^{-2}G_1^1G_1^2G_3^{N+2}+q^{-1}\left[\left(K_1-q^{-1}G_1^1\right)+\left(K_2-q^{-2}G_1^2\right)\right]G_2^{N+1}\\
&\qquad+\left(K_1-q^{-1}G_1^1\right)\left(K_2-q^{-1}G_1^1\right)G_1^N
\end{aligned}$$

$$\begin{aligned}
&SUM_q(N,[K_1,K_2],[1,3])\\
&\xrightarrow{\text{Form}_2}q^{-1}q^{-3}G_1^1G_1^3G_4^{N+3}+\left[\left(K_1-q^{-1}G_1^1\right)q^{-2}G_1^2+q^{-1}\left(K_2-q^{-3}G_1^3\right)\right]G_3^{N+2}\\
&\qquad+\left(K_1-q^{-1}G_1^1\right)\left(K_2-q^{-2}G_1^2\right)G_2^{N+1}
\end{aligned}$$

$$\begin{aligned}
&SUM_q(N,[K_1,K_2,K_3],[1,2,3])\\
&\xrightarrow{\text{Form}_2}q^{-1}q^{-2}q^{-3}G_1^1G_1^2G_1^3G_4^{N+3}+q^{-1}q^{-2}G_1^1G_1^2\left[\left(K_1-q^{-1}G_1^1\right)+\left(K_2-q^{-2}G_1^2\right)\right.\\
&\qquad\left.+\left(K_3-q^{-3}G_1^3\right)\right]G_3^{N+2}+q^{-1}G_1^1G_2^{N+1}\left[\left(K_1-q^{-1}G_1^1\right)\left(K_2-q^{-1}G_1^1\right)\right.\\
&\qquad\left.+\left(K_1-q^{-1}G_1^1\right)\left(K_3-q^{-2}G_1^2\right)+\left(K_2-q^{-2}G_1^2\right)\left(K_3-q^{-2}G_1^2\right)\right]\\
&\qquad+\left(K_1-q^{-1}G_1^1\right)\left(K_2-q^{-1}G_1^1\right)\left(K_3-q^{-1}G_1^1\right)G_1^N
\end{aligned}$$

Form_2→

5.9) $SUM_q\left(N,\left[q^{-T_1}G_1^{T_1},q^{-T_2}G_1^{T_2},\cdots\right],PT\right)=\prod_{i=1}^{M}q^{-T_i}G_1^{T_i}G_{T_M+1}^{N+T_M}$

5.10) $SUM_q\left(N,\left[q^{-L_1}G_1^{L_1},\cdots,q^{-L_p}G_1^{L_p},PS\right],\left[L_1,\cdots,L_P,PT\right]\right)$
$=\prod_{i=1}^{P}q^{-L_i}G_1^{L_i}SUM_q(N,PS,PT)\to$ extends domain of T_1 from 1 to $\mathbb{N}$

5.11) $SUM_q(N,PS,[T+1,T+2,\cdots,T+M])$, K_i can exchange order

5.12) $q^{Mn}=\dfrac{q^{-M}}{q^{-1}-1}\sum_{g=0}^{M}q^g\left(q^{-\binom{M}{g}}-1\right)\prod_{i=1}^{g}\left(q^n-q^{-i}\right)$

[Proof]

$q^{Mn}=\nabla SUM_q(N,[1,1,\cdots]:q-1,[1,2,\cdots,M])$

$$H_2^q(g,T)=(q-1)^g\prod_{i=1}^g q^{-i}G_1^i$$

$$1-\frac{q^x-1}{q^x(q-1)}(q-1)=\frac{1}{q^x}\to H_2^q\left(g,\sum K\right)=q^{-M+g}\sum_{i=0}^{\binom{M}{g}-1}q^{-i}$$

$$q^{Mn}=\sum_{g=0}^M(q-1)^g\left(\prod_{i=1}^g q^{-i}G_1^i\right)q^{-M+g}\frac{q^{-\binom{M}{g}}-1}{q^{-1}-1}G_g^{n+g}$$

q.e.d.

$$SUM\left(N,\left[K_1+D_1:(q-1)D_1,K_2+D_2:(q-1)D_2,\cdots\right],[1,2,\cdots,M]\right)\xrightarrow{\text{Form}_1}$$

5.13) $\sum_{n=0}^{N-1}q^n\prod_{i=1}^M\left(K_i+D_iq^n\right)=\sum_{g=0}^M H(g)G_{g+1}^N\to$ Conclusion of [1]

$$H(g)=\sum_{X(T)=g}\prod_{i=1}^M\begin{cases}q^{X_T}\left(q^{X_T}-1\right)D_i;X\in T\\K_i+q^{X_{T-1}}D_i;X\in K\end{cases}$$

5.14) $q^{Mn}=\sum_{g=0}^M q^{-g}\prod_{i=1}^g q^i\left(q^{n-g+i}-1\right)G_g^M$

[Proof]

$$q^{Mn}=\nabla SUM_q\left(N,[1,1,\cdots]:q-1,[1,2,\cdots,M]\right)$$

$$H_1^q(g,T)=(q-1)^g\prod_{i=1}^g q^iG_1^i$$

$$1+\frac{q^x-1}{q-1}(q-1)=q^x\to H_1^q\left(g,\sum K\right)=\sum\prod q^{X_{T-1}}$$

In 1916 MacMahon [5] observed that $G_K^M=\sum_{w\in\Omega(M,K)}q^{inv(w)}$ $\Omega(M,K)$ denotes all permutations of the multiset $\{0^{M-K},1^K\}$ that is, all words $w=w_1,\cdots,w_n$ with $n-k$ zeroes and k ones, and $inv(\cdot)$ denotes the inversion statistic defined by $inv(w_1,\cdots,w_n)=\left|\left\{(i,j):1\le i<j\le n,w_i>w_j\right\}\right|$.

So $\sum q^{X_{T-1}}=G_{M-g}^M,H_1^q(g)=\sum_{g=0}^M(q-1)^g\left(\prod_{i=1}^g q^iG_1^i\right)G_{M-g}^M$

$$q^{Mn}=\nabla SUM_q(N)=\sum_{g=0}^M(q-1)^g\left(\prod_{i=1}^g q^iG_1^i\right)G_{M-g}^Mq^{-g}G_g^n$$

q.e.d.

$$\frac{q^{Mn}-1}{q^M-1}=\frac{\sum_{g=1}^M(q-1)^g\left(\prod_{i=1}^g q^iG_1^i\right)G_{M-g}^Mq^{-g}G_g^n}{q^M-1}$$

$$=\frac{\sum_{g=1}^M\left(\prod_{i=1}^g q^i\left(q^i-1\right)\right)G_g^Mq^{-g}G_g^n}{q^M-1}\to$$

5.15) $\frac{q^{Mn}-1}{q^M-1}=\sum_{g=1}^M\left(\prod_{i=1}^{g-1}q^i\left(q^i-1\right)\right)G_{M-g}^{M-1}G_g^n\to$ Conclusion of [1]

5.16) $P_1<T_1,P_2<T_2,T_1>0,T_2>0$

$$PS=\left[\frac{G_1^1}{q^1},\frac{G_1^2}{q^2},\cdots,\frac{G_1^{T_1-P_1}}{q^{T_1-P_1}},\frac{K_1G_1^{T_1+1-P_1}}{q^{1-P_1}G_1^{T_1}},\frac{G_1^1}{q^1},\frac{G_1^2}{q^2},\cdots,\frac{G_1^{T_2-P_2}}{q^{T_2-P_2}},\frac{K_2G_1^{T_2+1-P_2}}{q^{1-P_2}G_1^{T_2}}\right]$$

$$PT=\left[1,2,3,\cdots,T_1+T_2+2-P_1-P_2\right]$$

$$\sum_{n=0}^{N-1}q^n\nabla^{P_1}SUM_q\left(n+1,[K_1],[T_1]\right)\nabla^{P_2}SUM_q\left(n+1,[K_2],[T_2]\right)$$

$$=\frac{q^1}{G_1^1}\frac{q^2}{G_1^2}\cdots\frac{q^{T_1-P_1}}{G_1^{T_1-P_1}}\frac{q^{1-P_1}G_1^{T_1}}{G_1^{T_1-P_1+1}}\frac{q^1}{G_1^1}\frac{q^2}{G_1^2}\cdots\frac{q^{T_2-P_2}}{G_1^{T_2-P_2}}\frac{q^{1-P_2}G_1^{T_2}}{G_1^{T_2-P_2+1}}SUM(N,PS,PT)$$

[Proof]

$$\frac{q^{n+K}-1}{q^K-1}=\frac{q^{n+K}-1}{q^K-1}\frac{q^1-1}{q^1-1}=\frac{q^K\left(q^n-1\right)+q^K-1}{G_1^K q^1-1}$$
$$=\frac{q^K}{G_1^K}G_1^n+\frac{G_1^K}{G_1^K}=\frac{q^K}{G_1^K}\left(G_1^n+G_1^K q^{-K}\right)$$

$$\nabla^P SUM_q\left(N,[K],[T]\right)$$
$$\xrightarrow{\text{Form}_2} q^{-T}G_1^T\binom{n+T+1-P}{T+1-P}+\left(K-q^{-T}G_1^T\right)\binom{n+T-P}{T-P}$$
$$=q^{-T}G_1^T\frac{\left(q^{n+T+1-P}-1\right)\cdots\left(q^{n+1}-1\right)}{\left(q^{T+1-P}-1\right)\cdots\left(q-1\right)}+\left(K-q^{-T}G_1^T\right)\frac{\left(q^{n+T-P}-1\right)\cdots\left(q^{n+1}-1\right)}{\left(q^{T-P}-1\right)\cdots\left(q-1\right)}$$
$$=\frac{\left(q^{n+T-P}-1\right)\cdots\left(q^{n+1}-1\right)}{\left(q^{T-P}-1\right)\cdots\left(q-1\right)}\left(K-q^{-T}G_1^T+q^{-T}G_1^T\frac{q^{n+T+1-P}-1}{q^{T+1-P}-1}\right)$$
$$=\frac{\left(q^{n+T-P}-1\right)\cdots\left(q^{n+1}-1\right)}{\left(q^{T-P}-1\right)\cdots\left(q-1\right)}\left(K-q^{-T}G_1^T+q^{-T}G_1^T\frac{1}{G_1^{T+1-P}}\left(q^{T+1-P}G_1^n+G_1^{T+1-P}\right)\right)$$
$$=\frac{\left(q^{n+T-P}-1\right)\cdots\left(q^{n+1}-1\right)}{\left(q^{T-P}-1\right)\cdots\left(q-1\right)}\frac{q^{1-P}G_1^T}{G_1^{T+1-P}}\left(\frac{KG_1^{T+1-P}}{q^{1-P}G_1^T}+G_1^n\right)$$

q.e.d.

5.17) $\lim_{q\to 1} H^q\left(g\right)=H\left(g\right),\lim_{q\to 1} SUM_q\left(N\right)=SUM\left(N\right)$

[1] has a conclusion:

$$H_1\left(q\right)=\sum_{x=q}^{M}H_2\left(x\right)\binom{x}{q}=\sum_{x=0}^{q}H_3\left(x\right)\binom{M-x}{M-q}$$
$$H_2\left(q\right)=\sum_{x=q}^{M}\left(-1\right)^{x+q}H_1\left(x\right)\binom{x}{q}$$
$$H_3\left(q\right)=\sum_{x=0}^{q}\left(-1\right)^{x+q}H_1\left(x\right)\binom{M-x}{M-q}$$

Generally, $H^q(g)$ has no such attribute; things get complicated.
When $PT=\left[1,2,\cdots,M\right]$, the situation is relatively simple.

5.18) $PT=\left[1,2,\cdots,M\right],H_1^q\left(g\right)=\sum_{x=g}^{M}H_2^q\left(x\right)G_g^x q^{2\binom{g+1}{2}}$

[Proof]

This is to prove: $H_1^q\left(g\right)q^{-\binom{g+1}{2}}=\left(\sum_{x=g}^{M}H_2^q\left(x\right)G_g^x\right)q^{\binom{g+1}{2}}$

$$\to H_1^q\left(g\right)\left(\prod_{i=1}^{g}q^{\{T_i-T_{i-1}\}X_{T-1}+1}\right)^{-1}=\left(\sum_{x=g}^{M}H_2^q\left(x\right)G_g^x\right)\left(\prod_{i=1}^{g}q^{-(T_i-X_{K-1})}\right)^{-1}$$
$$PS1=\left[PS,K_{M+1}:D_{M+1}\right],PT1=\left[PT,M+1\right]$$

Suppose it is true at M, $H_1^q\left(g\right)=\left(\sum_{x=g}^{M}\cdots\right)=\left(\sum_{x=0}^{M}\cdots\right)$

$$H_1^q\left(PS1,PT1,g\right)$$
$$=q^{X_{T-1}+1}G_1^{T_{M+1}-X_{K-1},\text{choice }T}D_{M+1}H_1^q\left(g-1\right)+\left(K_{M+1}+G_1^{X_{T-1},\text{choice }K}D_{M+1}\right)H_1^q\left(g\right)$$
$$=q^g G_1^g D_{M+1}H_1^q\left(g-1\right)+\left(K_{M+1}+G_1^g D_{M+1}\right)H_1^q\left(g\right)$$

$$= q^{g} G_1^{g} D_{M+1} \sum_{x=0}^{M} H_2^{q}(x) \left\{ G_{g-1}^{x} = G_g^{x+1} - q^{g} G_g^{x} \right\} q^{2\binom{g}{2} = 2\binom{g+1}{2} - 2g}$$

$$+ \left(K_{M+1} + G_1^{g} D_{M+1} \right) \sum_{x=0}^{M} H_2^{q}(x) G_g^{x} q^{2\binom{g+1}{2}}$$

$$= q^{-g} G_1^{g} D_{M+1} \sum_{x=0}^{M} H_2^{q}(x) G_g^{x+1} q^{2\binom{g+1}{2}} + K_{M+1} \sum_{x=0}^{M} H_2^{q}(x) G_g^{x} q^{2\binom{g+1}{2}}$$

$$q^{2\binom{g+1}{2}} \sum_{x=0}^{M+1} H_2^{q}(PS1, PT1, x) G_g^{x}$$

$$= q^{2\binom{g+1}{2}} \left\{ \sum_{x=1}^{M+1} H_2^{q}(x-1) q^{-x} G_1^{x} D_{M+1} G_g^{x} \right.$$

$$\left. + \sum_{x=0}^{M} H_2^{q}(x) \left(K_{M+1} - q^{-x-1} G_1^{x+1} D_{M+1} \right) G_g^{x} \right\}$$

$$= q^{2\binom{g+1}{2}} \left\{ \sum_{x=0}^{M} H_2^{q}(x) q^{-x-1} G_1^{x+1} D_{M+1} G_g^{x+1} \right.$$

$$\left. + \sum_{x=0}^{M} H_2^{q}(x) \left(K_{M+1} - q^{-x-1} G_1^{x+1} D_{M+1} \right) G_g^{x} \right\}$$

$$H_1^{q}(PS1, PT1, g) - q^{2\binom{g+1}{2}} \sum_{x=0}^{M+1} H_2^{q}(PS1, PT1, x) G_g^{x}$$

$$= q^{2\binom{g+1}{2}} D_{M+1} \sum_{x=0}^{M} H_2(x) \left\{ \left(q^{-g} G_1^{g} - q^{-x-1} G_1^{x+1} \right) G_g^{x+1} + q^{-x-1} G_1^{x+1} G_g^{x} \right\}$$

$$\xrightarrow{\{..\}=0} = 0$$

q.e.d.

6. Multiparameter Forms

(1.1) and (5.7) Use the Form: $(T_1 + K_1)(T_2 + K_2)\cdots(T_M + K_M)$

The form has T and K parameters, and more parameters will be used in this section. This section moves the D_i to PT.

$$PS = [K_1, K_2, \cdots, K_M], PSA = [PS, K_{M+1}]$$

$$PT_1 = [T_{1,1} : D_{1,1}, T_{2,1} : D_{2,1}, \cdots, T_{M,1} : D_{M,1}] = [T_1 : D_{1,1}, T_2 : D_{2,1}, \cdots, T_M : D_{M,1}]$$

$$PT_2 = [T_{1,2} : D_{1,2}, T_{2,2} : D_{2,2}, \cdots, T_{M,2} : D_{M,2}] = [T_1 : D_{1,2}, T_2 : D_{1,2}, \cdots, T_M : D_{M,2}]$$

$$PTA_1 = [PT_1, T_{M+1} = T_M + 2 - p : D_{M+1,1}], PTA_2 = [PT_2, T_{M+1} : D_{M+1,2}]$$

Define

$$SUM\left(N, [K_1], [T_{1,1} = 1 : D_{1,1}], [T_{1,2} = 1 : D_{1,2}]\right) = \sum_{n=0}^{N-1} \left(K_1 + nD_{1,1} + \binom{n}{2} D_{1,2} \right)$$

$$SUM(N) = SUM(N, PS, PTA_1, PTA_2)$$

$$= \sum_{n=0}^{N-1} \left(K_{M+1} + n \times D_{M+1,1} + \binom{n}{2} \times D_{M+1,1} \right) \times \nabla^{p} SUM(n+1)$$

$$(K_1 + T_{1,1} + T_{1,2})(K_2 + T_{2,1} + T_{2,2}) \cdots (K_M + T_{M,1} + T_{M,2}) = \sum \prod_{i=1}^{M} X_i$$

$$X(PT_1), X(PT_2) = \text{count of } X \in PT_1, PT_2; X(PT) = X(PT_1) + 2X(PT_2)$$

$$X_{PT_1}, X_{PT_2} = \text{count of } \{X_1, X_2, \cdots, X_i\} \in PT_1, PT_2; X_{PT} = X_{PT_1} + 2X_{PT_2}$$

$$H(q) = \sum_{\prod X_i \text{ with } X(PT)=q} \prod_{i=1}^{M} B_i,$$

6.1)

$$SUM(N) \xrightarrow{\text{Form}_1} \sum_{q=0}^{2M} H_1(q) \binom{N + T_M - M}{T_M - M + 1 + q}$$

$$\xrightarrow{\text{Form}_1} B_i = \begin{cases} \binom{X_{PT}}{0} K_i + \binom{X_{PT}}{1} D_{i,1} + \binom{X_{PT}}{2} D_{i,2}; X_i = K_i \\ \binom{T_i + X_{PT} - i}{1} D_{i,1} + \binom{T_i + X_{PT} - i}{1} \binom{X_{PT} - 1}{1} D_{i,2}; X_i = PT_{i,1} \\ \binom{T_i + X_{PT} - i}{2} D_{i,2}; X_i = PT_{i,2} \end{cases}$$

[Proof]

(*) $\sum_{n=0}^{N-1} n \binom{n+K}{M} = (M+1)\binom{N+K}{M+2} + (M-K)\binom{N+K}{M+1}$

(**) $$\sum_{n=0}^{N-1} \binom{n}{2}\binom{n+K}{M} = \binom{M+2}{2}\binom{N+K}{M+3} + (M-k)(M+1)\binom{N+K}{M+2} + \binom{M-K}{2}\binom{N+K}{M+1}$$

Still use induction, Let $Y = 1 + T_M - M - P = T_{M+1} - (M+1)$

$$SUM(N) \xrightarrow{\text{Form}_1} \sum_{q=0}^{2M} H_1(q) \binom{N + T_M - M}{T_M - M + 1 + q}$$

$$SUM(N, PSA, PTA_1, PTA_2)$$

$$= \sum_{n=0}^{N-1} \sum_{q=0}^{2M} \left(K_{M+1} + nD_{M+1,1} + \binom{n}{2} D_{M+1,2} \right) H_1(q) \binom{n+Y}{Y+q}$$

$$= \sum_{q=0}^{2M} \left(K_{M+1} + \binom{q}{1} D_{M+1,1} + \binom{q}{2} D_{M+1,2} \right) H_1(q) \binom{N+Y}{Y+q+1}$$

$$+ \sum_{q=0}^{2M} \left((Y+q+1) D_{M+1,1} + q(Y+q+1) D_{M+1,2} \right) H_1(q) \binom{N+Y}{Y+q+2}$$

$$+ \sum_{q=0}^{2M} \binom{Y+q+2}{2} H_1(q) \binom{N+Y}{Y+q+3} D_{M+1,2}$$

$$\xrightarrow{Y+q+1 = T_{M+1} - (M+1) + (q+1),\, Y+q+2 = T_{M+1} - (M+1) + (q+2)}$$

$$= \sum_{q=0}^{2M} \left(K_{M+1} + \binom{q}{1} D_{M+1,1} + \binom{q}{2} D_{M+1,2} \right) H_1(q) \binom{N + T_{M+1} - (M+1)}{T_{M+1} - (M+1) + 1 + q}$$

$$+ \sum_{q=0}^{2M} \left(T_{M+1} + (q+1) - (M+1) \right) D_{M+1,1} H_1(q) \binom{N + T_{M+1} - (M+1)}{T_{M+1} - (M+1) + 1 + (q+1)}$$

$$+ \sum_{q=0}^{2M} \left(T_{M+1} + (q+1) - (M+1) \right)(q+1-1) D_{M+1,2} H_1(q) \binom{N + T_{M+1} - (M+1)}{T_{M+1} - (M+1) + 1 + (q+1)}$$

$$+ \sum_{q=0}^{2M} \binom{T_{M+1} - (M+1) + (q+2)}{2} H_1(q) \binom{N + T_{M+1} - (M+1)}{T_{M+1} - (M+1) + 1 + (q+2)} D_{M+1,2}$$

q.e.d.

T_1 can be greater than 1,

$$SUM(N, [K_1], [T_1], [T_1]) = SUM(N, [1, K_1], [1:0, T_1], [1:0, T_1])$$

In the same way, we can get the other two forms and extend them to more parameters.

We can also get the three forms of the Gaussian coefficient with a multiparameter. (1.1) and (5.7) can be rewritten by X_T instead of X_{T-1} and X_{K-1}.

Conflicts of Interest

The author declares no conflicts of interest.

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