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Geometric Characteristics and Constructions of Cubic Indirect-PH Curves

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Abstract

The geometric characteristics and the construction for cubic indirect Pythagorean-hodograph (indirect-PH) curves are presented in this study. By introducing an auxiliary control point and a parameter respectively, two geometric characteristics in terms of quantities related to Bézier control polygon of the curve are given. Furthermore, based on the derived conditions we provide a new geometric modeling approach for the construction of cubic indirect-PH curves in detail. And at the end of this paper several numerical examples are presented to show the feasibility and validity of our algorithm.

Subject Areas

Mathematics

Keywords

Indirect Pythagorean-Hodograph, Bézier Curves, Geometric Characteristic, Control Polygon

1. Introduction

Offsetting has many applications such as Computer Numerical Control (CNC) machining of digital motion along curved paths, robotics and the design of highways and railways. The Bézier model has been exhaustively studied in computer aided geometric design (CAGD) [1], but the Bézier curve sometimes may not have a rational offset [2]. This inspired a lot of research on offset approximation [3]-[8].

In this paper, we discuss a class of planar polynomial curves with rational offsets. A cubic Bézier curve $\mathbf{P}(t)=(x(t),y(t))$ have rational offsets if and only if the squared norm of its hodograph $(x'(t),y'(t))$ has at most two complex roots of odd multiplicity, which can be classified into two classes: Pythagorean Hodographs (PH) curves and indirect-PH curves. The former is familiar to us and the latter can have rational PHs after reparameterization by a fractional quadratic transformation.

So far, PH curves and their applications have been studied intensively. Farouki and Sakkalis proposed an intuitive geometric condition on the Bézier control polygon for a cubic curve to be a PH curve, which is conditions in terms of lengths and inner angles of its control polygon legs [9], Fang and Wang presented a necessary and sufficient geometric characterization of PH quartics in terms of Bernstein-Bézier forms [10]. This method was also proposed for designing quintic PH curves in [11]. Qin *et al.* had already established the necessary and sufficient conditions for cubic H-Bézier curves to be PH curves. Based on the H-Bézier curve, a new method for their construction is provided in [12]. However, to date, Much less is known for indirect-PH curves. An in-depth analysis of geometric conditions for properly parameterized cubic indirect-PH curves was given in [13]. This provided a geometric method for the construction of G1 Hermite interpolation. Hormann took a closer look at quartic indirect-PH curves and derived algebraic as well as geometric characterizations for an important subset of these curves [14]. With the given C1 Hermite data, a new method for the construction of the quintic indirect-PH curves was derived by specifying a real parameter [15]. By employing Gaussian elimination and geometric approaches, Li got geometric characteristics of quintic indirect-PH curves [16]. Except for some brief description of cubic indirect-PH curves in [13], no work has been published on the full geometric characterization of them and no attention has been paid to the construction of cubic indirect-PH curves based on the simple geometric constraints of Bézier control polygons.

In this paper we focus on cubic indirect-PH curves. The auxiliary control points are introduced to deal with this problem. It leads to a more succinct proof of the necessary and sufficient conditions for a planar cubic Bézier curve to be an indirect-PH curve. And based on these conditions a new geometric modeling approach for the construction of the cubic indirect-PH curves is presented. This paper is organized as follows. In Section 2, the polynomial curves with rational offsets are briefly reviewed, which are the foundation of our following work. Section 3 presents the geometric relationship among the Bézier control points of indirect-PH cubics. Based on these, the geometric construction of indirect-PH curves is discussed in Section 4. Section 5 provides some examples and Section 6 concludes the paper.

2. Preliminaries

Let $\mathbf{R}$ and $\mathbf{C}$ denote the sets of all real numbers and complex numbers, respectively. In this paper, we will be denoting a complex number by a single bold character.

In the complex representation of $\mathbf{R}^2$, a planar parametric curve

$\mathbf{P}(t)=(x(t),y(t))$ can be identified with a complex-valued function $\mathbf{P}(t)=x(t)+iy(t)$. The necessary and sufficient condition for a planar cubic curve to have rational offsets can be described by the following theorem [16].

Theorem 2.1. In the complex representation, a planar curve $\mathbf{P}(t)=x(t)+iy(t)$ has rational parameterized offsets, if and only its hodograph $P'(t)$ can be written in the complex form as

$$P'(t)=\rho(t)R(t)W^2(t) \tag{2.1}$$

where

$$\mathbf{R}(t)=\lambda t+1+i\mu t,\ \mathbf{W}(t)=u(t)+iv(t), \tag{2.2}$$

and $\rho(t)$ is the polynomial of t with real coefficient, $\lambda,\mu\in\mathbf{R}$ and $u(t)$ and $v(t)$ are relatively prime.

Specially, where $\mathbf{R}(t)=1$, such $\mathbf{P}(t)$ is called a PH curve. Otherwise, $\mathbf{P}(t)$ is a complex polynomial, the curve $\mathbf{P}(t)$ is not a PH curve. Therefore, let

$$b=\sqrt{(\lambda+1)^2+\mu^2},\ c=\sqrt{(\lambda+2)^2+\mu^2} \tag{2.3}$$

and $B_i^2(t)=C_2^i t^i(1-t)^{n-i}$ are quadratic Bernstein polynomials, we introduce a quadratic transformation:

$$t(s)=\frac{B_1^2(s)+\mu B_2^2(s)}{\lambda B_0^2(s)+(1+\|\alpha\|)B_1^2(s)+\mu B_2^2(s)} \tag{2.4}$$

the plannar curve $\mathbf{P}(t)$ has a rational Pythagorean hodograph. In fact

$$\begin{aligned}&(\lambda t+1)^2+(\mu t)^2\\&=\left(\frac{(c+1-b)B_0^2(s)+\frac{1}{2}\left(c^2-(1-b)^2\right)B_1^2(s)+b(c-1+b)B_2^2(s)}{(c+1-b)B_0^2(s)+(1+b)B_1^2(s)+(c+1-b)B_2^2(s)}\right)^2\end{aligned} \tag{2.5}$$

Therefore we called such curves indirect-PH curves.

In this paper, we focus on cubic indirect-PH curves.

3. Geometric Characteristics of Cubic Indirect-PH Curves

In this section, geometric constraints on the control polygon is formulated that will guarantee the indirect-PH property. The control points of indirect-PH curves are $\mathbf{P}_0,\cdots,\mathbf{P}_3$.

The necessary and sufficient condition for a cubic properly-parameterized curve to be an indirect-PH curve is given in [13] as follows:

Theorem 3.1. A cubic properly-parameterized curve $\mathbf{P}(t)=(x(t),y(t))$ is an indirect-PH curve if and only if

$$(x'(t),y'(t))=[\alpha(1-t)+\beta][\mathbf{T}_0(1-t)+\mathbf{T}_1 t] \tag{3.1}$$

where $\mathbf{T}_0,\mathbf{T}_1$ are two non-parallel vectors and $\mathbf{T}_0\neq 0,\mathbf{T}_1\neq 0$. α,β are two numbers satisfying $\alpha^2+\beta^2\neq 0$.

For a given cubic Bézier curve $\mathbf{P}(t)$, with control points $\mathbf{P}_i, i=0,1,2,3$,

$$\mathbf{P}(t) = (x(t), y(t)) = \sum_{i=0}^{3} P_i B_i^3(t), \quad t \in [0,1] \tag{3.2}$$

where $B_i^3(t) = \binom{3}{i}(1-t)^{3-i} t^i$ are Bernstein polynomials.

Note that Equation (3.1) can be re-written as

$$(x'(t), y'(t)) = \alpha \mathbf{T}_0 (1-t)^2 + (\alpha \mathbf{T}_1 + \beta \mathbf{T}_0) t (1-t) + \beta \mathbf{T}_1 t^2$$

By matching the coefficients of its Bernstein polynomials with (3.2), we have

$$\begin{cases} 3(\mathbf{P}_1 - \mathbf{P}_0) = \alpha \mathbf{T}_0 \\ 3(\mathbf{P}_3 - \mathbf{P}_2) = \beta \mathbf{T}_1 \\ 6(\mathbf{P}_2 - \mathbf{P}_1) = \alpha \mathbf{T}_1 + \beta \mathbf{T}_0 \end{cases} \tag{3.3}$$

Hence, the curve is a cubic indirect-PH curve if and only if system of Equation (3.3) has a solution. Notably, system of Equation (3.3) is a constraint system of three nonlinear equations, which is difficult to be directly solved in general. However, the above system can be solved by using geometric methods owing to the introduction of the auxiliary point $\mathbf{Q}$.

We give an auxiliary point $\mathbf{Q}$, and $\mathbf{Q}$ is on lines $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_2\mathbf{P}_3$, respectively, such that:

$$\mathbf{Q} = \mathbf{P}_1 + \frac{1}{6}\beta \mathbf{T}_0 = \mathbf{P}_2 - \frac{1}{6}\alpha \mathbf{T}_0 \tag{3.4}$$

In Figure 1 it is an example of the auxiliary point for a cubic indirect-PH curve.

Now we give a geometric characteristic condition for a cubic Bézier curve to be an indirect-PH curve.

Theorem 3.2. A cubic Bézier curve $\mathbf{P}(t)$ in (3.2) is an indirect-PH curve if and only if

$$\|\Delta \mathbf{P}_0\| \cdot \|\Delta \mathbf{P}_2\| = 4\|\mathbf{Q} - \mathbf{P}_1\| \|\mathbf{P}_2 - \mathbf{Q}\| \tag{3.5}$$

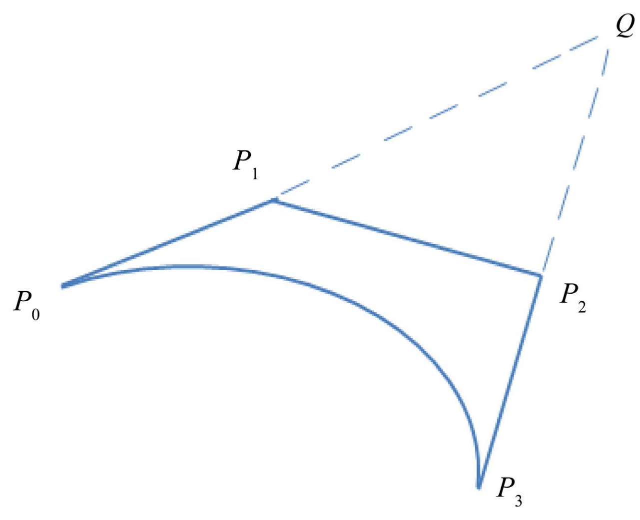

Figure 1. The Bézier control polygon with auxiliary point Q.

where $\Delta\mathbf{P}_0 = \mathbf{P}_1 - \mathbf{P}_0, \Delta\mathbf{P}_2 = \mathbf{P}_3 - \mathbf{P}_2$.

Proof. From Equation (3.3) and Equation (3.4), we have

$$\begin{cases} \alpha\mathbf{T}_0 = 3\Delta\mathbf{P}_0 \\ \beta\mathbf{T}_1 = 3\Delta\mathbf{P}_2 \\ \beta\mathbf{T}_0 = 6(\mathbf{Q} - \mathbf{P}_1) \\ \alpha\mathbf{T}_1 = 6(\mathbf{P}_2 - \mathbf{Q}) \end{cases} \tag{3.6}$$

The system has solutions if and only if all the equations are compatible. Because the constraints on angles have been satisfied following the scheme of selection of auxiliary point $\mathbf{Q}$, the conditions on lengths of legs (3.5) can be immediately derived.

According to $\frac{T_0}{T_1} = \frac{\alpha T_0}{\alpha T_1} = \frac{\beta T_0}{\beta T_1}$, we derive:

$$\frac{\mathbf{Q} - \mathbf{P}_1}{\Delta\mathbf{P}_0} \cdot \frac{\mathbf{P}_2 - \mathbf{Q}}{\Delta\mathbf{P}_2} = \frac{1}{4} \tag{3.7}$$

In other words, system (3.7) is equivalent to the system (3.5).

Every step above is reversible, so the proof of sufficiency is omitted.

Next we get the equivalent geometric characteristic condition by introducing parameter $h = \frac{\beta}{\alpha}$. In order to satisfy the criterion that the integral of the square norm of the second order derivative of the curve is minimized, the reference [13] has proved that h is within $(0.49, 2.04)$. The parameter h can be used for shape control.

Theorem 3.3. A properly parameterized cubic curve $\mathbf{P}(t)$ is an indirect-PH curve, if and only if its Bézier control edges satisfy

$$2\Delta\mathbf{P}_1 = \frac{1}{h}\Delta\mathbf{P}_2 + h\Delta\mathbf{P}_0$$

where $h = \frac{\beta}{\alpha}$ and $h \in \mathbf{R}$.

Proof. From the first two equations of (3.3) we can get

$$\begin{cases} 3\frac{\beta}{\alpha}\Delta\mathbf{P}_0 = \beta\mathbf{T}_0 \\ 3\frac{\alpha}{\beta}\Delta\mathbf{P}_2 = \alpha\mathbf{T}_1 \end{cases} \tag{3.8}$$

Substituting them into the third equation of (3.3) gives

$$6\Delta\mathbf{P}_0 = 3\frac{\alpha}{\beta}\Delta\mathbf{P}_2 + 3\frac{\beta}{\alpha}\Delta\mathbf{P}_0$$

Let $h = \frac{\beta}{\alpha}$. We have

$$2\Delta\mathbf{P}_1 = \frac{1}{h}\Delta\mathbf{P}_2 + h\Delta\mathbf{P}_0$$

Similarly, every step above is reversible, so the proof of sufficiency is omitted.

4. Geometric Construction of Cubic Indirect-PH Curves

The edges of polygons are widely used in CAGD due to their intrinsic invariance. The cubic indirect-PH curves can be obtained by constructing the control polygon. In this section, according to geometric modeling and the establishment of Cartesian coordinates, a new algorithm for construction is proposed by Theorem 3.2.

Given triangle ΔP_0QP_3, let the ratio of the lengths of its two sides be $\frac{\|QP_3\|}{\|P_0Q\|}=\rho\geq 1$ and $0<\angle P_0QP_3=\theta<\pi$. $\mathbf{P}_1$ and $\mathbf{P}_2$ along sides $\mathbf{P}_0\mathbf{Q}$ and $\mathbf{QP}_3$ should be properly chosen in order to assure $\mathbf{P}(t)$ to be an indirect-PH curve.

Give two vectors $\mathbf{e}_1=\mathbf{Q}-\mathbf{P}_0$ and $\mathbf{e}_2=\mathbf{P}_3-\mathbf{Q}$. The coordinates of $\mathbf{P}_1$ and $\mathbf{P}_2$ are obtained:

$$\begin{cases}\mathbf{P}_1=\lambda\mathbf{e}_1+\mathbf{P}_0\\ \mathbf{P}_2=\mu\mathbf{e}_2+\mathbf{P}_3\end{cases}$$

where $\lambda\geq 0,\mu\leq 1$.

Cartesian coordinates can be established with the origin at vertex $\mathbf{Q}$ of the triangle ΔP_0QP_3 and $\mathbf{QP}_0$ as the X axis. Let $\mathbf{P}_1=(0,1)$, $\mathbf{e}_1=(-1,0)$ and $\mathbf{e}_2=\rho(\cos\theta,\sin\theta)$. In this case, the coordinates of four control points are as follows:

$$\begin{cases}\mathbf{P}_0=(1,0)\\ \mathbf{P}_1=(1-\lambda,0)\\ \mathbf{P}_2=(1-\mu)\rho(\cos\theta,\sin\theta)\\ \mathbf{P}_3=\rho(\cos\theta,\sin\theta)\end{cases}\tag{4.1}$$

The drawing of the triangle and control polygon are shown in Figure 2.

In order to make sure that constraint (3.5) holds, let

$$\frac{\|\mathbf{Q}-\mathbf{P}_1\|}{\|\Delta\mathbf{P}_0\|}=\frac{h}{2},\quad \frac{\|\mathbf{P}_2-\mathbf{Q}\|}{\|\Delta\mathbf{P}_2\|}=\frac{1}{2h}$$

Solving these two equations gives:

$$\lambda=\frac{2}{h+2},\quad \mu=\frac{2h}{1+2h}\tag{4.2}$$

According to Theorem 3.3, h is a parameter that can be used for shape control. By substituting (4.2) into (4.1), the coordinates of $\mathbf{P}_1$ and $\mathbf{P}_2$ can be obtained.

5. Numerical Examples

We shall illustrate with some graphical examples in this section. The condition for a cubic indirect-PH curve to have no cusp is h is within (0.49, 2.04) in [13]. When the angle between two edges of the control polygon is acute, for example, $\theta=\frac{\pi}{3}$. Set $\rho=1$ and the parameter $h=\frac{3}{2}$, $h=2$ respectively. We can obtain the coordinates of $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, as shown in Figure 3.

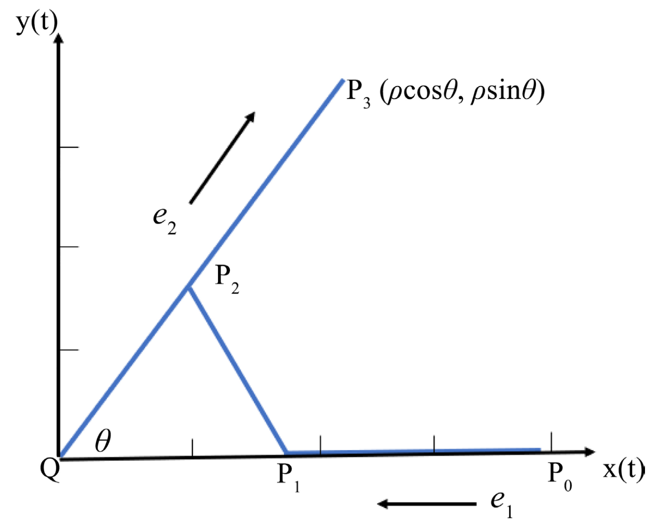

Figure 2. Triangle and control polygon.

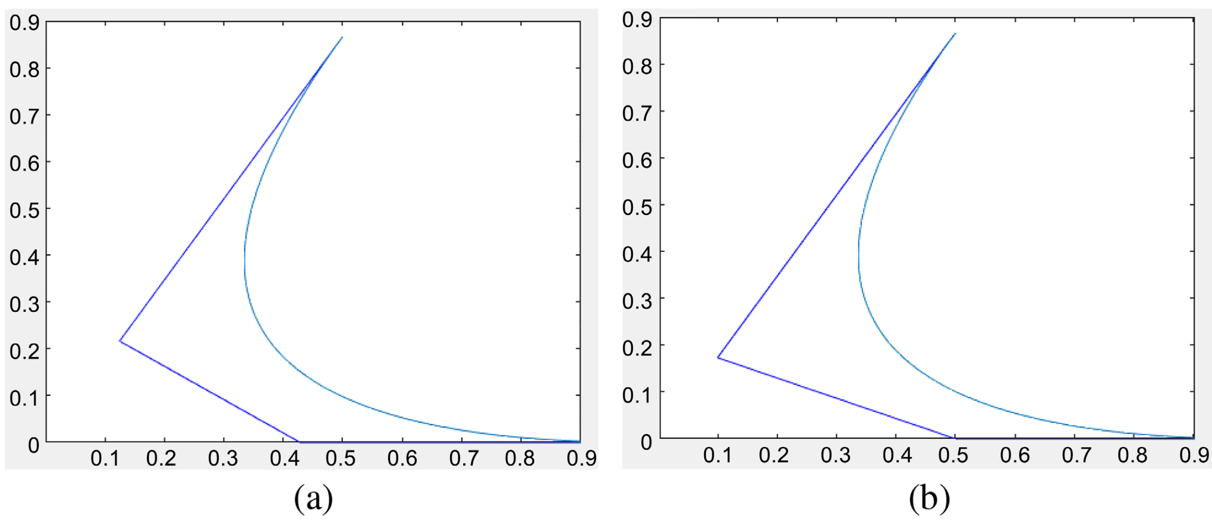

Figure 3. The angle between two edges of the control polygon is acute. (a) $h=\frac{3}{2}$; (b) $h=2$.

When the angle between two edges of the control polygon is a right angle. Set $\rho=1$ and the parameter $h=1$, $h=\frac{3}{2}$ respectively. By the method presented in Section 4 we can obtain the coordinates of $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, as shown in Figure 4.

When the angle between two edges of the control polygon is obtuse, for example, $\theta=\frac{2\pi}{3}$. Set $\rho=1$ and the parameter $h=1$, $h=\frac{3}{2}$ respectively. We can obtain the coordinates of $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, as shown in Figure 5.

Therefore, the cubic Bézier curves can be converted to a cubic indirect-PH curves by modifying the value of the parameter of h. illustrative examples demonstrate that it is feasible to construct a good approximate cubic indirect-PH curves using this algorithmm.

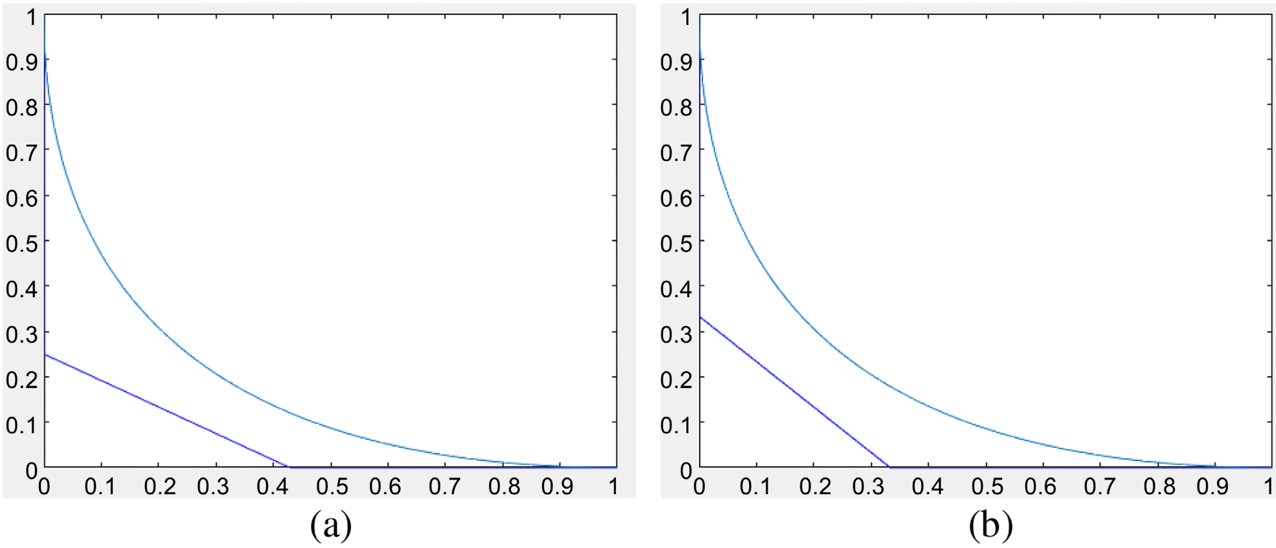

Figure 4. The angle between two edges of the control polygon is a right angle. (a) $h=\frac{3}{2}$; (b) $h=1$.

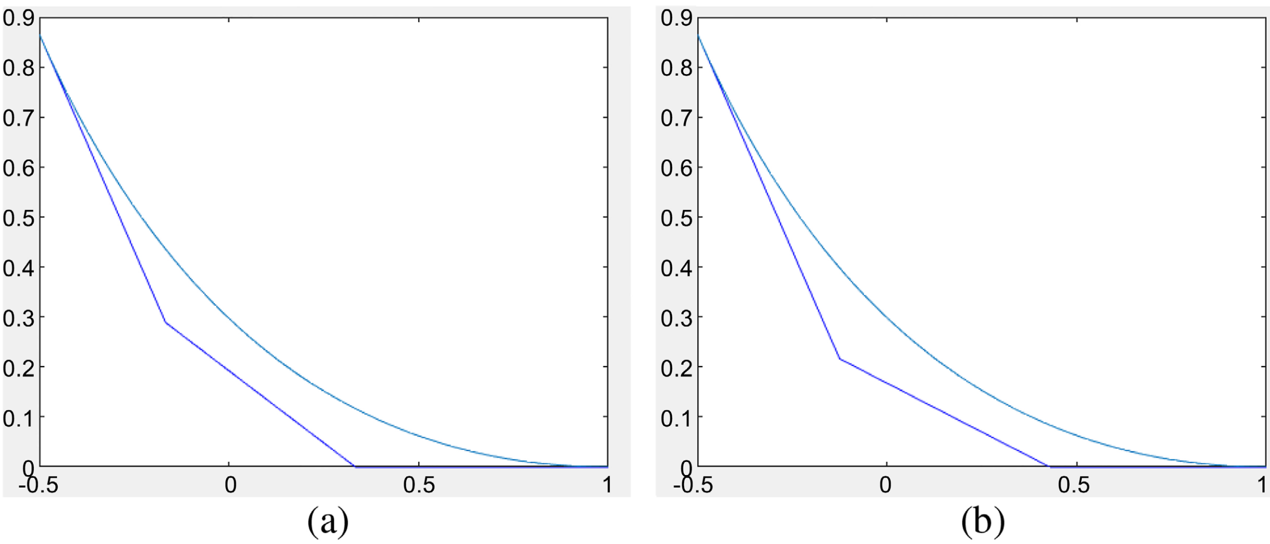

Figure 5. The angle between two edges of the control polygon is obtuse. (a) $h=1$ (a) $h=\frac{3}{2}$.

6. Conclusion

We have made some efforts to give geometric characteristics for a planar cubic curve to be a cubic indirect-PH curve. They are expressed by the control polygon and auxiliary control point and a parameter respectively. Furthermore, we provide the method for the construction of cubic indirect-PH curves, which is shown effective. Based on this, a future study of freedom curve design by cubic indirect-PH curves could be considered.

Conflicts of Interest

All authors declare no conflicts of interest in this paper.

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