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Generalized Hyers-Ulam-Rassisa Stability of an Additive (β_1,β_2)-Functional Inequalities with *n*-Variables in Complex Banach Space

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Abstract

In this paper, we study to solve the additive (β_1,β_2)-functional inequality with *n*-variables and their Hyers-Ulam stability. First are investigated in complex Banach spaces with a fixed point method and last are investigated in complex Banach spaces with a direct method. These are the main results of this paper.

Subject Areas

Mathematics

Keywords

Additive (β_1,β_2)-Functional Inequality, Fixed Point Method, Direct Method, Banach Space, Hyers-Ulam Stability,

Mathemtics Subject Classification

39B52, 39B62, 47H10, 46S10

1. Introduction

Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces on the same field $\mathbb{K}$, and $f:\mathbb{X}\to\mathbb{Y}$. We use the notation $\|\cdot\|$ for all the norms on both $\mathbb{X}$ and $\mathbb{Y}$. In this paper, we investigate additive (β_1,β_2)-functional inequality when $\mathbb{X}$ is a real or complex normed space and $\mathbb{Y}$ a complex Banach space. We solve and prove the Hyers-Ulam stability of following additive (β_1,β_2)-functional inequality.

$$\begin{aligned}&\left\|f\left(x_1+x_2+\cdots+x_n\right)-f\left(x_1\right)-f\left(x_2+\cdots+x_n\right)\right\|_{\mathbb{Y}}\\&\le\left\|\beta_1\left(f\left(x_1+x_2+\cdots+x_n\right)-f\left(x_1-x_2-\cdots-x_n\right)-2f\left(x_1\right)\right)\right\|_{\mathbb{Y}}\\&\quad+\left\|\beta_2\left(2f\left(\frac{x_1+x_2+\cdots+x_n}{2}\right)-f\left(x_1\right)-f\left(x_2+\cdots+x_n\right)\right)\right\|_{\mathbb{Y}}\end{aligned}\tag{1}$$

In which β_1,β_2 are fixed nonzero complex numbers with $g\left(\beta_1,\beta_2\right)$-functional inequality. Note that in the preliminaries we just recap some of the most essential properties for the above problem and for the specific problem, please see the document. The Hyers-Ulam stability was first investigated for functional equation of Ulam in [1] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping.

The Hyers [2] gave first affirmative partial answer to the equation of Ulam in Banach spaces. After that, Hyers' Theorem was generalized by Aoki [3] additive mappings and by Rassias [4] for linear mappings considering an unbounded Cauchy diffrence. Ageneralization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference with a general control function in the spirit of Rassias' approach.

The stability of quadratic functional equation was proved by Skof [6] for mappings $f:\mathbb{X}\to\mathbb{Y}$, where $\mathbb{X}$ is a normed space and $\mathbb{Y}$ is a Banach space. Park [7] [8] defined additive γ-functional inequalities and proved the Hyers-Ulam stability of the additive γ-functional inequalities in Banach spaces and nonArchimedean Banach spaces. The stability problems of various functional equations have been extensively investigated by a number of authors on the world even term [4] [5] [6] [8]-[20]. We recall a fundamental result in fixed point theory. The authors studied the Hyers-Ulam stability for the following functional inequalities

$$\left\|f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x+y}{2}\right)-f(z)\right\|\le\left\|f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)-\frac{1}{2}f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(z)\right\|\tag{2}$$

$$\left\|f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)-\frac{1}{2}f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(z)\right\|\le\left\|f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x+y}{2}\right)-f(z)\right\|\tag{3}$$

$$\left\|f(x+y)-f(x)-f(y)\right\|\le\left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\|\tag{4}$$

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\|\le\left\|\rho\left(f(x+y)-f(x)-f(y)\right)\right\|\tag{5}$$

and

$$\begin{aligned}&\left\|f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f(z)\right\|\\&\le\left\|\beta\left(2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f(z)\right)\right\|\end{aligned}\tag{6}$$

$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f(z)\right\|\\&\le\left\|\beta\left(f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f(z)\right)\right\|\end{aligned}\tag{7}$$

finaly

$$\begin{aligned}&\left\|f(x+y)-f(x)-f(y)\right\|\\&\le\left\|\beta_1\left(f(x+y)+f(x-y)-2f(x)\right)\right\|+\left\|\beta_2\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\|\end{aligned}\tag{8}$$

in complex Banach spaces

In this paper, we solve and proved the Hyers-Ulam stability for (β_1,β_2)-functional inequalities (1), ie the (β_1,β_2)-functional inequalities with n-variables. Under suitable assumptions on spaces $\mathbb{X}$ and $\mathbb{Y}$, we will prove that the mappings satisfy the (β_1,β_2)-functional inequalities (1). Thus, the results in this paper are generalization of those in [21] [22] [23] [24] for (β_1,β_2)-functional inequalities with n-variables.

The goal of the paper is to develop functional inequalities with higher number of variables to solve problems of general nonlinear functional equations in order to develop the field of nonlinear analysis.

The paper is organized as follows: In section preliminaries we remind some basic notations in [21] [22] [25] such as complete generalized metric space and Solutions of the inequalities.

Section 3: In this section, I use the method of the fixed to prove the Hyers-Ulam stability of the addive (β_1,β_2)-functional inequalities (1) when $\mathbb{X}$ is a real or complete normed space and $\mathbb{Y}$ complex Banach space.

Section 4: In this section, I use the method of directly determining the solution for (1) when $\mathbb{X}$ is a real or complete normed space and $\mathbb{Y}$ complex Banach space.

2. Preliminaries

2.1. Complete Generalized Metric Space and Solutions of the Inequalities

Theorem 1. Let $(\mathbb{X},d)$ be a complete generalized metric space and let $J:\mathbb{X}\to\mathbb{X}$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in\mathbb{X}$, either

$$d\left(J^n,J^{n+1}\right)=\infty$$

for all nonegative integers n or there exists a positive integer n_0 such that

1) $d\left(J^n,J^{n+1}\right)<\infty$, $\forall n\ge n_0$;

2) The sequence $\left\{J^n x\right\}$ converges to a fixed point y^* of J;

3) y^* is the unique fixed point of J in the set $Y=\left\{y\in\mathbb{X}\mid d\left(J^n,J^{n+1}\right)<\infty\right\}$;

4) $d\left(y,y^*\right)\le\frac{1}{1-l}d\left(y,Jy\right)\quad\forall y\in\mathbb{Y}$.

2.2. Solutions of the Inequalities

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the cauchuy equation. In particular, every solution of the cauchuy equation is said to be an additive mapping.

3. Establish the Solution of the Additive (β_1,β_2)-Function Inequalities Using a Fixed Point Method

Now, we first study the solutions of (1). Note that for these inequalities, when $\mathbb{X}$ is a real or complex normed space and $\mathbb{Y}$ complex Banach space.

Lemma 2. *A mapping* $f:\mathbb{X}\to\mathbb{Y}$ *satisfies* $f(0)=0$ *and*

$$\begin{aligned}&\left\|f(x_1+x_2+\cdots+x_n)-f(x_1)-f(x_2+\cdots+x_n)\right\|_{\mathbb{Y}}\\&\le\left\|\beta_1\left(f(x_1+x_2+\cdots+x_n)-f(x_1-x_2-\cdots-x_n)-2f(x_1)\right)\right\|_{\mathbb{Y}}\\&+\left\|\beta_2\left(2f\left(\frac{x_1+x_2+\cdots+x_n}{2}\right)-f(x_1)-f(x_2+\cdots+x_n)\right)\right\|_{\mathbb{Y}}\end{aligned}\tag{9}$$

for all $x_j\in\mathbb{X}, j=1\to n$, then $f:\mathbb{X}\to\mathbb{Y}$ is additive.

Proof. Assume that $f:\mathbb{X}\to\mathbb{Y}$ satisfies (9)

Replacing $(x_1,\cdots,x_n)$ by $(x,x,0,\cdots,0)$ in (9), we get

$$\left\|f(2x)-2f(x)\right\|_{\mathbb{Y}}\le|\beta_1|\left\|f(2x)-2f(x)\right\|_{\mathbb{Y}}$$

and so f(2x) = 2f(x) for all $x\in\mathbb{X}$.

Thus

$$f\left(\frac{x}{2}\right)=\frac{1}{2}f(x)\tag{10}$$

for all $x\in X$. It follows from (9) and (10) that

$$\begin{aligned}&\left\|f(x_1+x_2+\cdots+x_n)-f(x_1)-f(x_2+\cdots+x_n)\right\|_{\mathbb{Y}}\\&\le\left\|\beta_1\left(f(x_1+x_2+\cdots+x_n)-f(x_1-x_2-\cdots-x_n)-2f(x_1)\right)\right\|_{\mathbb{Y}}\\&+\left\|\beta_2\left(2f\left(\frac{x_1+x_2+\cdots+x_n}{2}\right)-f(x_1)-f(x_2+\cdots+x_n)\right)\right\|_{\mathbb{Y}}\\&=\left\|\beta_1\left(f(x_1+x_2+\cdots+x_n)-f(x_1-x_2-\cdots-x_n)-2f(x_1)\right)\right\|_{\mathbb{Y}}\\&+\left\|\beta_2\left(f(x_1+x_2+\cdots+x_n)-f(x_1)-f(x_2+\cdots+x_n)\right)\right\|_{\mathbb{Y}}\end{aligned}\tag{11}$$

for all $x_j\in\mathbb{X}, j=1\to n$ and so

$$\begin{aligned}&(1-|\beta_2|)\left\|f(x_1+x_2+\cdots+x_n)-f(x_1)-f(x_2+\cdots+x_n)\right\|_{\mathbb{Y}}\\&\le\left\|\beta_1\left(f(x_1+x_2+\cdots+x_n)-f(x_1-x_2-\cdots-x_n)-2f(x_1)\right)\right\|_{\mathbb{Y}}\end{aligned}\tag{12}$$

Next we letting $u=x_1+x_2+\cdots+x_n, v=x_1-x_2-\cdots-x_n$ in (12), we get

$$\begin{aligned}&\left(1-|\beta_2|\right)\left\|f(u)-f\left(\frac{u+v}{2}\right)-f\left(\frac{u-v}{2}\right)\right\|_{\mathbb{Y}}\\&\le|\beta_1|\left\|f(u)+f(v)-2f\left(\frac{u+v}{2}\right)\right\|_{\mathbb{Y}}\end{aligned}\tag{13}$$

for all $u,v\in X$
and so

$$\begin{aligned}&\frac{1}{2}\left(1-|\beta_2|\right)\left\|f(u+v)+f(u-v)-2f(u)\right\|_{\mathbb{Y}}\\&\le|\beta_1|\left\|f(u+v)-f(u)-f(v)\right\|_{\mathbb{Y}}\end{aligned}\tag{14}$$

for all $u,v\in X$. It follows from (12) and (14) that

$$\begin{aligned}&\frac{1}{2}\left(1-|\beta_2|\right)^2\left\|f(x_1+x_2+\cdots+x_n)-f(x_1)-f(x_2+\cdots+x_n)\right\|_{\mathbb{Y}}\\&\le|\beta_1|^2\left\|f(x_1+x_2+\cdots+x_n)-f(x_1)-f(x_2+\cdots+x_n)\right\|_{\mathbb{Y}}\end{aligned}\tag{15}$$

Since $\sqrt{2}|\beta_1|+|\beta_2|<1$
and so

$$f(x_1+x_2+\cdots+x_n)=f(x_1)+f(x_2+\cdots+x_n).$$

Thus *f* is additive.

Theorem 3. Let $\varphi:\mathbb{X}^n\to[0,\infty)$ be a function such that there exists an $L<1$ with

$$\varphi\left(\frac{x_1}{2},\frac{x_2}{2},\cdots,\frac{x_n}{2}\right)\le\frac{L}{2}\varphi(x_1,x_2,\cdots,x_n)\tag{16}$$

for all $x,y,z\in\mathbb{X}$. Let $f:\mathbb{X}\to\mathbb{Y}$ be a mapping satisfy $f(0)=0$ and

$$\begin{aligned}&\left\|f(x_1+x_2+\cdots+x_n)-f(x_1)-f(x_2+\cdots+x_n)\right\|_{\mathbb{Y}}\\&\le\left\|\beta_1\left(f(x_1+x_2+\cdots+x_n)-f(x_1-x_2-\cdots-x_n)-2f(x_1)\right)\right\|_{\mathbb{Y}}\\&+\left\|\beta_2\left(2f\left(\frac{x_1+x_2+\cdots+x_n}{2}\right)-f(x_1)-f(x_2+\cdots+x_n)\right)\right\|_{\mathbb{Y}}\\&+\varphi(x_1,x_2,\cdots,x_n)\end{aligned}\tag{17}$$

for all $x_j\in\mathbb{X},j=1\to n$.

Then there exists a unique mapping $\psi:\mathbb{X}\to\mathbb{Y}$ such that

$$\left\|f(x)-\psi(x)\right\|_{\mathbb{Y}}\le\frac{L}{2(1-L)\left(1-|\beta_1|\right)}\varphi(x,x,\cdots,x)\tag{18}$$

for all $x\in\mathbb{X}$

Proof. Replacing $(x_1,x_2,\cdots,x_n)$ by $(x,x,0,\cdots,0)$ in (37), we get

$$\left(1-|\beta_1|\right)\left\|f(2x)-2f(x)\right\|_{\mathbb{Y}}\le\varphi(x,x,0,\cdots,0)\tag{19}$$

for all $x\in\mathbb{X}$.

Consider the set

$$\mathbb{S}:=\left\{h:\mathbb{X}\to\mathbb{Y},h(0)=0\right\}$$

and introduce the generalized metric on $\mathbb{S}$:

$$d(g,h) := \inf\left\{\lambda \in \mathbb{R} : \|g(x)-h(x)\| \le \lambda\varphi(x,x,0,\cdots,0), \forall x \in X\right\},$$

where, as usual, $\inf\phi = +\infty$. It is easy to show that $(\mathbb{S},d)$ is complete ([17]) Now we consider the linear mapping $J:\mathbb{S}\to\mathbb{S}$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in \mathbb{X}$. Let $g,h \in \mathbb{S}$ be given such that $d(g,h) = \varepsilon$ then

$$\|g(x)-h(x)\| \le \varepsilon\varphi(x,x,0,\cdots,0)$$

for all $x \in \mathbb{X}$.

Hence

$$\|Jg(x)-Jh(x)\| = \left\|2g\left(\frac{x}{2}\right)-2hf\left(\frac{x}{2}\right)\right\| \le 2\varepsilon\varphi\left(\frac{x}{2},\frac{x}{2},0,\cdots,0\right)$$
$$\le 2\varepsilon\frac{L}{2}\varphi(x,x,0,\cdots,0) \le L\varepsilon\varphi(x,x,0,\cdots,0)$$

for all $x \in \mathbb{X}$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \le L\cdot\varepsilon$. This means that

$$d(Jg,Jh) \le Ld(g,h)$$

for all $g,h \in X$ It follows from (19) that

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| \le \frac{1}{1-|\beta_1|}\varphi\left(\frac{x}{2},\frac{x}{2},0,\cdots,0\right) \le \frac{L}{2(1-|\beta_1|)}\varphi(x,x,0,\cdots,0)$$

for all $x \in \mathbb{X}$. So $d(f,Jf) \le \frac{L}{2(1-|\beta_1|)}$ for all $x \in \mathbb{X}$ By Theorem 1, there exists a mapping $\psi:\mathbb{X}\to\mathbb{Y}$ satisfying the following:

1) ψ is a fixed point of *J*, *i.e.*,

$$\psi(x) = 2\psi\left(\frac{x}{2}\right) \tag{20}$$

for all $x \in X$. The mapping ψ is a unique fixed point *J* in the set

$$\mathbb{M} = \{g \in \mathbb{S} : d(f,g) < \infty\}$$

This implies that ψ is a unique mapping satisfying (20) such that there exists a $\lambda \in (0,\infty)$ satisfying

$$\|f(x)-\psi(x)\| \le \lambda\varphi(x,x,0,\cdots,0)$$

for all $x \in \mathbb{X}$.

2) $d(J^l f,\psi) \to 0$ as $l \to \infty$. This implies equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = \psi(x)$$

for all $x \in \mathbb{X}$.

3) $d(f,\psi)\le \frac{1}{1-L}d(f,Jf)$. which implies

$$\|f(x)-\psi(x)\|\le \frac{L}{2(1-L)(1-|\beta_1|)}\varphi(x,x,0,\cdots,0)$$

for all $x\in X$. It follows (16) and (37) that

$$\begin{aligned}
&\|\psi(x_1+x_2+\cdots+x_n)-\psi(x_1)-\psi(x_2+\cdots+x_n)\|_{\mathbb{Y}}\\
&=\lim_{n\to\infty}2^n\left\|f\left(\frac{x_1+x_2+\cdots+x_n}{2^n}\right)-f\left(\frac{x_1}{2^n}\right)-f\left(\frac{x_2+\cdots+x_n}{2^n}\right)\right\|_{\mathbb{Y}}\\
&\le\lim_{n\to\infty}2^n|\beta_1|\left\|f\left(\frac{x_1+x_2+\cdots+x_n}{2^n}\right)-f\left(\frac{x_1-x_2-\cdots-x_n}{2^n}\right)-2f\left(\frac{x_1}{2^n}\right)\right\|_{\mathbb{Y}}\\
&\quad+\lim_{n\to\infty}2^n|\beta_2|\left\|2f\left(\frac{x_1+x_2+\cdots+x_n}{2^{n+1}}\right)-f\left(\frac{x_1}{2^n}\right)-f\left(\frac{x_2+\cdots+x_n}{2^n}\right)\right\|_{\mathbb{Y}}\\
&\quad+\lim_{n\to\infty}2^n\varphi\left(\frac{x_1}{2^n},\frac{x_2}{2^n},\cdots,\frac{x_n}{2^n}\right)\\
&=\|\beta_1(\psi(x_1+x_2+\cdots+x_n)-\psi(x_1-x_2-\cdots-x_n)-2\psi(x_1))\|_{\mathbb{Y}}\\
&\quad+\left\|\beta_2\left(2\psi\left(\frac{x_1+x_2+\cdots+x_n}{2}\right)-\psi(x_1)-\psi(x_2+\cdots+x_n)\right)\right\|_{\mathbb{Y}}
\end{aligned}\tag{21}$$

for all $x_j\in\mathbb{X}, j=1\to n$. So

$$\begin{aligned}
&\|\psi(x_1+x_2+\cdots+x_n)-\psi(x_1)-\psi(x_2+\cdots+x_n)\|_{\mathbb{Y}}\\
&\le\|\beta_1(\psi(x_1+x_2+\cdots+x_n)-\psi(x_1-x_2-\cdots-x_n)-2\psi(x_1))\|_{\mathbb{Y}}\\
&\quad+\left\|\beta_2\left(2\psi\left(\frac{x_1+x_2+\cdots+x_n}{2}\right)-\psi(x_1)-\psi(x_2+\cdots+x_n)\right)\right\|_{\mathbb{Y}}
\end{aligned}$$

for all $x_j\in\mathbb{X}, j=1\to n$. By Lemma 2, the mapping $\psi: X\to Y$ is additive. Ei

$$\psi(x_1+x_2+\cdots+x_n)=\psi(x_1)+\psi(x_2+\cdots+x_n).\ \square$$

Theorem 4. Let $\varphi:\mathbb{X}^n\to[0,\infty)$ be a function such that there exists an $L<1$ with

$$\varphi(x_1,x_2,\cdots,x_n)\le 2L\varphi\left(\frac{x_1}{2},\frac{x_2}{2},\cdots,\frac{x_n}{2}\right)\tag{22}$$

for all $x,y,z\in\mathbb{X}$. Let $f:\mathbb{X}\to\mathbb{Y}$ be a mapping satisfy $f(0)=0$ and

$$\begin{aligned}
&\|f(x_1+x_2+\cdots+x_n)-f(x_1)-f(x_2+\cdots+x_n)\|_{\mathbb{Y}}\\
&\le\|\beta_1(f(x_1+x_2+\cdots+x_n)-f(x_1-x_2-\cdots-x_n)-2f(x_1))\|_{\mathbb{Y}}\\
&\quad+\left\|\beta_2\left(2f\left(\frac{x_1+x_2+\cdots+x_n}{2}\right)-f(x_1)-f(x_2+\cdots+x_n)\right)\right\|_{\mathbb{Y}}\\
&\quad+\varphi(x_1,x_2,\cdots,x_n)
\end{aligned}\tag{23}$$

for all $x_j\in\mathbb{X}, j=1\to n$.

Then there exists a unique mapping $\psi:\mathbb{X}\to\mathbb{Y}$ such that

$$\|f(x)-\psi(x)\|_{\mathbb{Y}} \le \frac{L}{2(1-L)(1-|\beta_1|)}\varphi(x,x,\cdots,x) \tag{24}$$

for all $x \in \mathbb{X}$

Proof. Replacing $(x_1,x_2,\cdots,x_n)$ by $(x,x,0,\cdots,0)$ in (23), we get

$$(1-|\beta_1|)\|f(2x)-2f(x)\|_{\mathbb{Y}} \le \varphi(x,x,0,\cdots,0) \tag{25}$$

for all $x \in \mathbb{X}$.

Suppose (S,d) be the generalized metric space defined in the proof of Theeorem 1 Now we cosider the linear mapping $J:\mathbb{S} \to \mathbb{S}$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in \mathbb{X}$. It follows from (25)

$$\left\|f(x)-\frac{1}{2}f(2x)\right\| = \frac{1}{2(1-|\beta_1|)}\varphi(x,x,0,\cdots,0)$$

The rest of the proof is similar to proof of Theorem 3. □

From proving the theorems we have consequences:

Corollary 1. Let $r > 1$ and θ be nonnegative real numbers and let $f:\mathbb{X} \to \mathbb{Y}$ be a mapping satisfy $f(0)=0$ and

$$\begin{aligned}&\|f(x_1+x_2+\cdots+x_n)-f(x_1)-f(x_2+\cdots+x_n)\|_{\mathbb{Y}}\\ &\le \|\beta_1(f(x_1+x_2+\cdots+x_n)-f(x_1-x_2-\cdots-x_n)-2f(x_1))\|_{\mathbb{Y}}\\ &\quad+\left\|\beta_2\left(2f\left(\frac{x_1+x_2+\cdots+x_n}{2}\right)-f(x_1)-f(x_2+\cdots+x_n)\right)\right\|_{\mathbb{Y}}\\ &\quad+\theta\left(\|x_1\|^r+\|x_2\|^r+\cdots+\|x_n\|^r\right)\end{aligned} \tag{26}$$

for all $x_j \in \mathbb{X}$.

Then there exists a unique mapping $\psi:\mathbb{X} \to \mathbb{Y}$ such that

$$\|f(x)-\psi(x)\|_{\mathbb{Y}} \le \frac{2\theta}{(2^r-2)(1-|\beta_1|)}\|x\|_{\mathbb{X}}^r \tag{27}$$

for all $x \in \mathbb{X}$

Corollary 2. Let $r < 1$ and θ be nonnegative real numbers and let $f:\mathbb{X} \to \mathbb{Y}$ be a mapping satisfy $f(0)=0$ and

$$\begin{aligned}&\|f(x_1+x_2+\cdots+x_n)-f(x_1)-f(x_2+\cdots+x_n)\|_{\mathbb{Y}}\\ &\le \|\beta_1(f(x_1+x_2+\cdots+x_n)-f(x_1-x_2-\cdots-x_n)-2f(x_1))\|_{\mathbb{Y}}\\ &\quad+\left\|\beta_2\left(2f\left(\frac{x_1+x_2+\cdots+x_n}{2}\right)-f(x_1)-f(x_2+\cdots+x_n)\right)\right\|_{\mathbb{Y}}\\ &\quad+\theta\left(\|x_1\|^r+\|x_2\|^r+\cdots+\|x_n\|^r\right)\end{aligned} \tag{28}$$

for all $x_j \in \mathbb{X}$.

Then there exists a unique mapping $\psi:\mathbb{X} \to \mathbb{Y}$ such that

$$\|f(x)-\psi(x)\|_{\mathbb{Y}} \le \frac{2\theta}{(2-2^r)(1-|\beta_1|)}\|x\|_{\mathbb{X}}^r \tag{29}$$

for all $x \in \mathbb{X}$.

4. Establish the Solution of the Additive (β_1, β_2)-Function Inequalities Using a Direct Method

Next, we study the solutions of (1). Note that for these inequalities, when $\mathbb{X}$ is a real or complete normed space and $\mathbb{Y}$ complex Banach space.

Theorem 5. Let $\varphi : \mathbb{X}^n \to [0, \infty)$ be a function and let $f : \mathbb{X} \to \mathbb{Y}$ be a mapping such that

$$\phi(x_1, x_2, \cdots, x_n) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x_1}{2^j}, \frac{x_2}{2^j}, \cdots, \frac{x_n}{2^j}\right) < \infty \tag{30}$$

and let $f : \mathbb{X} \to \mathbb{Y}$ be a mapping $f(0) = 0$ and

$$\begin{aligned} &\left\| f(x_1 + x_2 + \cdots + x_n) - f(x_1) - f(x_2 + \cdots + x_n) \right\|_{\mathbb{Y}} \\ &\leq \left\| \beta_1 \left(f(x_1 + x_2 + \cdots + x_n) - f(x_1 - x_2 - \cdots - x_n) - 2f(x_1) \right) \right\|_{\mathbb{Y}} \\ &\quad + \left\| \beta_2 \left(2f\left(\frac{x_1 + x_2 + \cdots + x_n}{2}\right) - f(x_1) - f(x_2 + \cdots + x_n) \right) \right\|_{\mathbb{Y}} \\ &\quad + \varphi(x_1, x_2, \cdots, x_n) \end{aligned} \tag{31}$$

for all $x_j \in \mathbb{X}, j = 1 \to n$.

Then there exists a unique mapping $\psi : \mathbb{X} \to \mathbb{Y}$ such that

$$\left\| f(x) - \psi(x) \right\|_{\mathbb{Y}} \leq \frac{1}{2(1 - |\beta_1|)} \phi(x, x, \cdots, x) \tag{32}$$

for all $x \in \mathbb{X}$

Proof. Replacing $(x_1, x_2, \cdots, x_n)$ by $(x, x, 0, \cdots, 0)$ in (31), we get

$$(1 - |\beta_1|) \left\| f(2x) - 2f(x) \right\|_{\mathbb{Y}} \leq \varphi(x, x, 0, \cdots, 0) \tag{33}$$

for all $x \in \mathbb{X}$. So

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\|_{\mathbb{Y}} \leq \frac{1}{1 - |\beta_1|} \varphi\left(\frac{x_1}{2}, \frac{x_2}{2}, 0, \cdots, 0\right) \tag{34}$$

for all $x \in \mathbb{X}$. Hence

$$\begin{aligned} &\left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\|_{\mathbb{Y}} \\ &\leq \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\|_{\mathbb{Y}} \\ &\leq \sum_{j=l}^{m-1} \frac{2^j}{2(1 - |\beta_1|)} \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0, \cdots, 0\right) \end{aligned} \tag{35}$$

for all nonnegative integers *m* and *l* with $m > l$ and all $x \in \mathbb{X}$. It follows from (35) that the sequence $\left\{2^k f\left(\frac{x}{2^k}\right)\right\}$ is a Cauchy sequence for all $x \in \mathbb{X}$. Since $\mathbb{Y}$ is complete, the sequence $\left\{2^k f\left(\frac{x}{2^k}\right)\right\}$ coverages. So one can define the mapping $\psi:\mathbb{X}\to\mathbb{Y}$ by

$$\psi(x):=\lim_{k\to\infty}\frac{1}{2^k}f\left(2^k x\right) \tag{36}$$

for all $x\in\mathbb{X}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (35), we get (32) It follows from (30) and (31) that

$$\begin{aligned}
&\left\|\psi(x_1+x_2+\cdots+x_n)-\psi(x_1)-\psi(x_2+\cdots+x_n)\right\|_{\mathbb{Y}}\\
&=\lim_{n\to\infty}2^n\left\|f\left(\frac{x_1+x_2+\cdots+x_n}{2n}\right)-f\left(\frac{x_1}{2n}\right)-f\left(\frac{x_2+\cdots+x_n}{2n}\right)\right\|_{\mathbb{Y}}\\
&\le\lim_{n\to\infty}2^n|\beta_1|\left\|f\left(\frac{x_1+x_2+\cdots+x_n}{2n}\right)-f\left(\frac{x_1-x_2-\cdots-x_n}{2n}\right)-2f\left(\frac{x_1}{2n}\right)\right\|_{\mathbb{Y}}\\
&\quad+\lim_{n\to\infty}2^n|\beta_2|\left\|2f\left(\frac{x_1+x_2+\cdots+x_n}{2^{n+1}}\right)-f\left(\frac{x_1}{2n}\right)-f\left(\frac{x_2+\cdots+x_n}{2^n}\right)\right\|_{\mathbb{Y}}\\
&\quad+\lim_{n\to\infty}2^n\varphi\left(\frac{x_1}{2^n},\frac{x_2}{2^n},\cdots,\frac{x_n}{2^n}\right)\\
&=\left\|\beta_1\left(\psi(x_1+x_2+\cdots+x_n)-\psi(x_1-x_2-\cdots-x_n)-2\psi(x_1)\right)\right\|_{\mathbb{Y}}\\
&\quad+\left\|\beta_2\left(2\psi\left(\frac{x_1+x_2+\cdots+x_n}{2}\right)-\psi(x_1)-\psi(x_2+\cdots+x_n)\right)\right\|_{\mathbb{Y}}
\end{aligned} \tag{37}$$

for all $x_j\in\mathbb{X}, j=1\to n$. So

$$\begin{aligned}
&\left\|\psi(x_1+x_2+\cdots+x_n)-\psi(x_1)-\psi(x_2+\cdots+x_n)\right\|_{\mathbb{Y}}\\
&=\left\|\beta_1\left(\psi(x_1+x_2+\cdots+x_n)-\psi(x_1-x_2-\cdots-x_n)-2\psi(x_1)\right)\right\|_{\mathbb{Y}}\\
&\quad+\left\|\beta_2\left(2\psi\left(\frac{x_1+x_2+\cdots+x_n}{2}\right)-\psi(x_1)-\psi(x_2+\cdots+x_n)\right)\right\|_{\mathbb{Y}}
\end{aligned}$$

for all $x_j\in\mathbb{X}, j=1\to n$. By Lemma 2, the mapping $\psi:\mathbb{X}\to\mathbb{Y}$ is additive. Implied

$$\begin{aligned}
&\psi(x_1+x_2+\cdots+x_n)\\
&=\psi(x_1)+\psi(x_2+\cdots+x_n)
\end{aligned}$$

Now, let $\psi':\mathbb{X}\to\mathbb{Y}$ be another additive mapping satisfying (33). Then we have

$$\begin{aligned}
\left\|\psi(x)-\psi'(x)\right\|&=\left\|2^q\psi\left(\frac{x}{2^q}\right)-2^q\psi'\left(\frac{x}{2^q}\right)\right\|_{\mathbb{Y}}\\
&\le\left\|2^q\psi\left(\frac{x}{2^q}\right)-2^q f\left(\frac{x}{2^q}\right)\right\|+\left\|2^q\psi'\left(\frac{x}{2^q}\right)-2^q f\left(\frac{x}{2^q}\right)\right\|_{\mathbb{Y}}\\
&\le\frac{2^q}{1-|\beta_1|}\phi\left(\frac{x}{2^q},\frac{x}{2^q},0,\cdots,0\right)
\end{aligned}$$

which tends to zero as $q\to\infty$ for all $x\in\mathbb{X}$. So we can conclude that $\psi(x)=\psi'(x)$ for all $x\in\mathbb{X}$. This proves the uniqueness of ψ. □

Theorem 6. *Let* $\varphi:\mathbb{X}^n\to[0,\infty)$ *be a function and let* $f:\mathbb{X}\to\mathbb{Y}$ *be a* *mapping such that* $f(0)=0$,

$$\psi(x_1,x_2,\cdots,x_n):=\sum_{j=0}^{\infty}\frac{1}{2^j}\varphi\left(2^j x_1,2^j x_2,\cdots,2^j x_n\right) \\ <\infty \tag{38}$$

and

$$\begin{aligned}&\left\|f(x_1+x_2+\cdots+x_n)-f(x_1)-f(x_2+\cdots+x_n)\right\|_{\mathbb{Y}}\\ &\le\left\|\beta_1\left(f(x_1+x_2+\cdots+x_n)-f(x_1-x_2-\cdots-x_n)-2f(x_1)\right)\right\|_{\mathbb{Y}}\\ &+\left\|\beta_2\left(2f\left(\frac{x_1+x_2+\cdots+x_n}{2}\right)-f(x_1)-f(x_2+\cdots+x_n)\right)\right\|_{\mathbb{Y}}\\ &+\varphi(x_1,x_2,\cdots,x_n)\end{aligned} \tag{39}$$

for all $x_j\in\mathbb{X}, j=1\to n$.

Then there exists a unique mapping $\psi:\mathbb{X}\to\mathbb{Y}$ such that

$$\begin{aligned}&\left\|f(x)-\psi(x)\right\|_{\mathbb{Y}}\\ &\le\frac{1}{2\left(1-|\beta_1|\right)}\phi(x,x,\cdots,x)\end{aligned} \tag{40}$$

for all $x\in\mathbb{X}$

Proof. Replacing $(x_1,x_2,\cdots,x_n)$ by $(x,x,0,\cdots,0)$ in (39), we get

$$\begin{aligned}&\left(1-|\beta_1|\right)\left\|f(2x)-2f(x)\right\|_{\mathbb{Y}}\\ &\le\varphi(x,x,0,\cdots,0)\end{aligned} \tag{41}$$

for all $x\in\mathbb{X}$. So

$$\begin{aligned}&\left\|f(x)-\frac{1}{2}f(2x)\right\|_{\mathbb{Y}}\\ &\le\frac{1}{2\left(1-|\beta_1|\right)}\varphi(x,x,0,\cdots,0)\end{aligned} \tag{42}$$

for all $x\in\mathbb{X}$. Hence

$$\begin{aligned}&\left\|\frac{1}{2^l}f\left(2^l x\right)-\frac{1}{2^m}f\left(2^m x\right)\right\|_{\mathbb{Y}}\\ &\le\sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right)-\frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\|_{\mathbb{Y}}\\ &\le\sum_{j=l}^{m-1}\frac{1}{2^{j+1}\left(1-|\beta_1|\right)}\varphi\left(2^j x,2^j x,0,\cdots,0\right)\end{aligned} \tag{43}$$

for all nonnegative integers *m* and *l* with $m>l$ and all $x\in\mathbb{X}$. It follows from (43) that the sequence $\left\{\frac{1}{2^k}f\left(2^k x\right)\right\}$ is a Cauchy sequence for all $x\in\mathbb{X}$. Since $\mathbb{Y}$ is complete, the sequence $\left\{\frac{1}{2^k}f\left(2^k x\right)\right\}$ coverges. So one can define the mapping $\psi:\mathbb{X}\to\mathbb{Y}$ by

$$\psi(x):=\lim_{k\to\infty}\frac{1}{2^k}f\left(2^k x\right) \tag{44}$$

Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (43), we get (40).

The rest of the proof is similar to the proof of theorem 5. ☐

From proving the theorems we have consequences:

Corollary 3. Let $r>1$ and θ be nonnegative real numbers and let $f:\mathbb{X}\to\mathbb{Y}$ be a mapping satisfy $f(0)=0$ and

$$\begin{aligned}&\left\|f(x_1+x_2+\cdots+x_n)-f(x_1)-f(x_2+\cdots+x_n)\right\|_{\mathbb{Y}}\\&\le\left\|\beta_1\left(f(x_1+x_2+\cdots+x_n)-f(x_1-x_2-\cdots-x_n)-2f(x_1)\right)\right\|_{\mathbb{Y}}\\&\quad+\left\|\beta_2\left(2f\left(\frac{x_1+x_2+\cdots+x_n}{2}\right)-f(x_1)-f(x_2+\cdots+x_n)\right)\right\|_{\mathbb{Y}}\\&\quad+\theta\left(\|x_1\|^r+\|x_2\|^r+\cdots+\|x_n\|^r\right)\end{aligned}\tag{45}$$

for all $x_j\in\mathbb{X}$.

Then there exists a unique mapping $\psi:\mathbb{X}\to\mathbb{Y}$ such that

$$\left\|f(x)-\psi(x)\right\|_{\mathbb{Y}}\le\frac{2\theta}{(2^r-2)(1-|\beta_1|)}\|x\|_{\mathbb{X}}^r\tag{46}$$

for all $x\in\mathbb{X}$

Corollary 4. Let $r<1$ and θ be nonnegative real numbers and let $f:\mathbb{X}\to\mathbb{Y}$ be a mapping satisfy $f(0)=0$ and

$$\begin{aligned}&\left\|f(x_1+x_2+\cdots+x_n)-f(x_1)-f(x_2+\cdots+x_n)\right\|_{\mathbb{Y}}\\&\le\left\|\beta_1\left(f(x_1+x_2+\cdots+x_n)-f(x_1-x_2-\cdots-x_n)-2f(x_1)\right)\right\|_{\mathbb{Y}}\\&\quad+\left\|\beta_2\left(2f\left(\frac{x_1+x_2+\cdots+x_n}{2}\right)-f(x_1)-f(x_2+\cdots+x_n)\right)\right\|_{\mathbb{Y}}\\&\quad+\theta\left(\|x_1\|^r+\|x_2\|^r+\cdots+\|x_n\|^r\right)\end{aligned}\tag{47}$$

for all $x_j\in\mathbb{X}$.

Then there exists a unique mapping $\psi:\mathbb{X}\to\mathbb{Y}$ such that

$$\left\|f(x)-\psi(x)\right\|_{\mathbb{Y}}\le\frac{2\theta}{(2-2^r)(1-|\beta_1|)}\|x\|_{\mathbb{X}}^r\tag{48}$$

for all $x\in\mathbb{X}$.

Conflicts of Interest

The author declares no conflicts of interest.

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