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Ground State Solutions for the Fractional Klein-Gordon-Maxwell System with Steep Potential Well

Qingying Shi

School of Mathematics, Liaoning Normal University, Dalian, China
Email: Shiqy5108@163.com

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Abstract

In this paper, we study the fractional Klein-Gordon-Maxwell system with steep potential well. On the basis of overcoming the lack of compactness, the ground state solution is obtained by proving that the solution satisfies the mountain pass level.

Subject Areas

Partial Differential Equation

Keywords

Fractional Laplacian, Klein-Gordon-Maxwell System, Ground state Solution, Steep Potential Well, Variational Methods

1. Introduction and Main Results

In the present paper, we are concerned the following fractional Klein-Gordon-Maxwell system

$$\begin{cases} (-\Delta)^s u + (\lambda a(x)+1)u - (2\omega+\phi)\phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ (-\Delta)^s \phi = -(\omega+\phi)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $s \in \left(\frac{3}{4}, 1\right)$, $(-\Delta)^s$ denotes the fractional Laplacian, $\omega > 0$ is a parameter, $\phi, u : \mathbb{R}^3 \to \mathbb{R}$ are functions. Recently, a great attention has been focused on the study of nonlinear problems involving the fractional Laplacian, in view of concrete real-world applications. For instance, this type of operators arises in the thin obstacle problem, optimization, finance, phase transitions, stratified materials, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, materials science and water waves, see [1]. The study of existence of positive solutions for problems related to the fractional Laplacian operator has been vigorous in the past three decades, see [1] [2] [3] [4] and references therein. In [5], Miyagaki, de Moura and Ruviaor studied the fractional Klein-Gordon-Maxwell system

$$\begin{cases} (-\Delta)^s u + V(x)u - (2\omega+\phi)\phi u = K(x) f(u), & \text{in } \mathbb{R}^3, \\ \Delta^s \phi = (\omega+\phi)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

and the existence of positive ground state solutions was obtained.

The Klein-Gordon-Maxwell system has been introduced in [6] as a model describing solitary waves for the non-linear stationary Klein-Gordon equation coupled with Maxwell equation in the three-dimensional space interacting with the electrostatic field. Some existence results for Klein-Gordon-Maxwell system have been investigated extensively. In [7] [8] [9], the authors studied the existence of ground stated solutions for Klein-Gordon-Maxwell system with periodic potential. Moura [10] obtained the same result involving zero mass potential.

In [11], Liu, Chen and Tang studied the following Klein-Gordon-Maxwell system

$$\begin{cases} -\Delta u + (\lambda a(x)+1)u - (2\omega+\phi)\phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = -(\omega+\phi)u^2 & \text{in } \mathbb{R}^3 \end{cases} \tag{1.3}$$

and the existence of a ground state solution was proved by using variational methods. Later, In [12], Zhang, Gan, Xiao and Jia expand the range of ω, and the existence of a ground state solution for the above system is established under suitable conditions on $a(x)$ and *f*.

We are going to explore problem (1) showing the existence of the ground state solution with steep potential well. Moreover, we will treat the problem Klein-Gordon-Maxwell using the fractional Laplacian operator instead of classical Laplacian operator. The interest in this kind of problem is: the vast range of applications; the mathematical challenge the nonlocal problem; the challenge when working in the domain like $\mathbb{R}^3$ and also fractional Laplacian.

In case $\phi \neq 0$, our problem becomes doubly nonlocal because of the term ϕ and the fractional operator. Classical compactness arguments are not available and the equation cannot be treated point wisely. We overcome these difficulties using the reduction method introduced by Caffarelli and Silvestre [3], for the fractional Laplacian. When $\phi \neq 0$ a process of plugging the ϕ into the main equation is used, allowing look at the system as a single equation. This technique was also employed in [6] [13] [14], and so on.

Inspired by the works in the above references, our main purpose in this paper is to study the existence of ground state solution for problem (1.1). In order to state our main results, we assume that

(a_1) $a \in C^1(\mathbb{R}^3, \mathbb{R})$, $a \geq 0$ for all $x \in \mathbb{R}^3$.

(a_2) There is $M > 0$ such that $\text{meas}(\{x \in \mathbb{R}^3 : a(x) \leq M\}) < \infty$.

(a_3) The set $\Omega_0 = \{x \in \mathbb{R}^3 : a(x) = 0\}$ is nonempty and has smooth boundary with $\bar{\Omega}_0 = a^{-1}(0)$.

(f_1) $f \in C^1(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and $|f(x,t)| \le C(1+|t|^{q-1})$ for some $C > 0$ and $2 < q < 2_s^*$, where $2_s^* = \frac{6}{3-2s}$.

(f_2) $\lim_{t \to 0} \frac{f(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^3$.

(f_3) There exists $\mu > 2$ such that $0 < \mu F(x,t) := \mu \int_0^t f(x,s)\,\mathrm{d}s \le f(x,t)t$, $\forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}$.

(f_4) $\inf_{x \in \mathbb{R}^3, |t|=1} F(x,t) > 0$.

Remark 1.1. *The conditions* (a_1)-(a_3) *were first introduced in* [15] *and* $\lambda a(x)$ *was called a steep potential well when* λ *was large.*

Now we state our main results as following.

Theorem 1.1 *Suppose* $s \in \left(\frac{3}{4}, 1\right)$ *and assume that* (a_1)-(a_3) *and* (f_1)-(f_4) *hold. There exists* $\Lambda > 0$ *such that problem* (1.1) *has a ground state solution for* $\lambda \ge \Lambda$ *if one of the following conditions is satisfied*:

1) $4 \le \mu < \infty$;

2) $2 < \mu < 4$ *and* $0 < \omega < \frac{\sqrt{8(\mu-2)}}{4-\mu}$.

The plan of the paper is as follows. In Section 2, we give the variational framework for problem (1.1) and some preliminary results. In Section 3, we prove some basic lemmas. In Section 4, we complete the proof of Theorem 1.1.

Throughout the paper, we give the following notations:

- C and $C_k\ (k = 1, 2, \cdots)$ for psositive constants.
- $\to$ ($\rightharpoonup$) denote the strong (weak) convergence.
- $B_r := \{x \in \mathbb{R}^3 : |x| < r\}$.
- The integral $\int_{\mathbb{R}^3} u\mathrm{d}x$ is represented by $\int_{\mathbb{R}^3} u$.
- $|u|_p = \left(\int_{\mathbb{R}^3} |u|^p\right)^{\frac{1}{p}}$ for $1 \le p < \infty$.

2. Variational Setting and Preliminaries

We reformulate the nonlocal Klein-Gordon-Maxwell system (1.1) into a local system using the local reduction due to Caffarelli and Silvestre [3], that is,

$$\begin{cases} -\mathrm{div}\left(y^{1-2s}\nabla v_1\right) = 0, & \text{in } \mathbb{R}_+^4, \\ v_1 = u, & \text{on } \mathbb{R}^3 \times \{0\}, \\ k_s y^{1-2s} \frac{\partial v_1}{\partial \eta} = -(\lambda a(x)+1)u + (2\omega+\phi)\phi u + f(x,u), & \text{on } \mathbb{R}^3 \times \{0\}, \\ -\mathrm{div}\left(y^{1-2s}\nabla v_2\right) = 0, & \text{in } \mathbb{R}_+^4, \\ v_2 = \phi, & \text{on } \mathbb{R}^3 \times \{0\}, \\ k_s y^{1-2s} \frac{\partial v_2}{\partial \eta} = (\omega+\phi)u^2, & \text{on } \mathbb{R}^3 \times \{0\}. \end{cases} \tag{2.1}$$

Here $k_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$, such that

$$-k_s \lim_{y\to 0^+} y^{1-2s}\frac{\partial v_1(x,y)}{\partial y} = (-\Delta)^s u(x),$$

where $u(x) = v_1(x,0) := \tilde{v}_1$, $\phi(x) = v_2(x,0) := \tilde{v}_2$ and the outward normal derivative should be understood as

$$y^{1-2s}\frac{\partial v_1}{\partial \eta} = -\lim_{y\to 0^+} y^{1-2s}\frac{\partial v_1}{\partial y}.$$

Similar definition is given for v_2.

The fractional Laplacian $(-\Delta)^s$ with $s \in \left(\frac{3}{4},1\right)$ of a function $\Psi:\mathbb{R}^3 \to \mathbb{R}$ is defined by

$$\mathcal{F}\left((-\Delta)^s \Psi\right)(\xi) = |\xi|^{2s}\mathcal{F}(\xi), \quad \xi \in \mathbb{R}^3,$$

where $\mathcal{F}$ is the Fourier transform, that is

$$\mathcal{F}(\Psi)(\xi) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\exp(-2\pi i\xi\cdot x)\Psi(x)\mathrm{d}x,$$

i is the imaginary unit. If Ψ is smooth enough, $(-\Delta)^s$ can be computed by the following singular integral

$$(-\Delta)^s \Psi(x) = c_\alpha P.V.\int_{\mathbb{R}^3}\frac{\Psi(x)-\Psi(y)}{|x-y|^{3+2s}}\mathrm{d}y, \quad x \in \mathbb{R}^3,$$

where c_α is a normalization constant and $P.V.$ stands the principal value. For any $s \in \left(\frac{3}{4},1\right)$. About fractional Sobolev space a very complete introduction can be found in [1].

The spaces $X^{2s}\left(\mathbb{R}^4_+\right)$ is defined as the completion of $C_0^\infty\left(\overline{\mathbb{R}^4_+}\right)$, under the norms (which actually coincide, see ([16], Lemma A.2))

$$\|z_1\|_{X^{2s}} := \left(\int_{\mathbb{R}^4_+} k_s y^{1-2s}|\nabla z_1|^2\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}},$$

The Sobolev space $D^s\left(\mathbb{R}^4_+\right)$ is defined by

$$D^s\left(\mathbb{R}^4_+\right) = \left\{u \in L^{2_s^*}\left(\mathbb{R}^4_+\right) : |\xi|^s \hat{u} \in L^2\left(\mathbb{R}^4_+\right)\right\},$$

where $\hat{u}$ is the usual Fourier transforms of u, which is the completion of $C_0^\infty\left(\mathbb{R}^4_+\right)$ under the norm

$$\|u\|^2_{D^s\left(\mathbb{R}^4_+\right)} = \left|(-\Delta)^{\frac{s}{2}} u\right|_2^2 = \int_{\mathbb{R}^4_+} y^{1-2s}|\nabla u|^2\mathrm{d}x\mathrm{d}y, \quad u \in D^s\left(\mathbb{R}^4_+\right).$$

We are looking for a solution in the Hilbert space E defined by

$$E = \left\{z \in X^{2s}\left(\mathbb{R}^4_+\right) : \int_{\mathbb{R}^3}\lambda a(x) z^2(x,0)\mathrm{d}x < \infty\right\}$$

endowed with norm

$$\|z\| := \left(\int_{\mathbb{R}^4_+} k_s y^{1-2s} |\nabla z|^2 \,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} (\lambda a(x)+1) z(x,0)^2 \,\mathrm{d}x\right)^{\frac{1}{2}},$$

It follows from Poincaré inequality and (a_1)-(a_3) that the embedding $E \hookrightarrow X^{2s}(\mathbb{R}^4_+)$ is continuous (Its proof is similar to [17]). Thus there exists $C_l > 0$ for any $l \in [2, 2_s^*]$ such that

$$\left(\int_{\mathbb{R}^3} |z|^l\right)^{\frac{1}{l}} \le C_l \|z\|, \quad \forall\, z \in E. \tag{2.2}$$

E is a Hilbert space. In the following, for convenience, for any u, let $\tilde{u} := u(x,0)$, furthermore $\tilde{\phi} := \phi(x,0)$ for any $\phi \in D^s(\mathbb{R}^4_+)$. By a standard argument, solution $(u,\phi) \in E \times D^s(\mathbb{R}^3)$ of problem (1.1) is a critical point of the energy functional $I_\lambda : E \times D^s(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$\begin{aligned} I_\lambda(u,\phi) = {} & \frac{k_s}{2}\int_{\mathbb{R}^4_+} y^{1-2s} |\nabla u|^2 \,\mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}^3} (\lambda a(x)+1)\tilde{u}^2 \mathrm{d}x \\ & - \frac{k_s}{2}\int_{\mathbb{R}^4_+} y^{1-2s} |\nabla \phi|^2 \,\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int_{\mathbb{R}^3} (2\omega + \tilde{\phi})\tilde{\phi}\tilde{u}^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(x,\tilde{u})\mathrm{d}x. \end{aligned}$$

We need the following lemma to reduce the functional I_λ in the only variable u.

Lemma 2.1. *For every* $u(x,y) \in X^{2s}(\mathbb{R}^4_+)$, *there exists a unique* $\phi = \phi_u(x,y) \in D^s(\mathbb{R}^4_+)$ *which solves*

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla v) = 0, & \text{in } \mathbb{R}^4_+, \\ k_s y^{1-2s} \dfrac{\partial v}{\partial \eta} = (\omega+\phi)u^2, & \text{on } \mathbb{R}^3 \times \{0\}. \end{cases} \tag{2.3}$$

Furthermore, in the set $\{(x,0) : u(x,0) \not\equiv 0\}$ we have $-\omega \le \phi_u \le 0$ if $\omega > 0$.

Proof. Its proof is the same as ([5], Lemma 2.1), and we omit it.

We rewrite I_λ as a C^1 functional $I : E \to \mathbb{R}$ defined as

$$\begin{aligned} I(u) = I_\lambda(u,\phi_u) = {} & \frac{k_s}{2}\int_{\mathbb{R}^4_+} y^{1-2s} |\nabla u|^2 \,\mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}^3} (\lambda a(x)+1)\tilde{u}^2 \mathrm{d}x \\ & - \frac{1}{2}\int_{\mathbb{R}^3} \omega\tilde{\phi}_u \tilde{u}^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(x,\tilde{u})\mathrm{d}x, \end{aligned}$$

where $F(x,u) = \int_0^u f(x,s)\mathrm{d}s$.

From (f_1), I is well defined C^1-functional with derivative given by

$$\begin{aligned} \langle I'(u),\varphi\rangle = {} & k_s \int_{\mathbb{R}^4_+} y^{1-2s} \langle \nabla\tilde{u}, \nabla\varphi\rangle \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} (\lambda a(x)+1)\tilde{u}\tilde{\varphi}\mathrm{d}x \\ & - \int_{\mathbb{R}^3} (2\omega + \tilde{\phi}_u)\tilde{\phi}_u \tilde{u}\tilde{\varphi}\mathrm{d}x - \int_{\mathbb{R}^3} f(x,\tilde{u})\tilde{\varphi}\mathrm{d}x, \ \forall\, u,\varphi \in E \end{aligned}$$

Lemma 2.2. *If* $u_n(x,y) \rightharpoonup u_0(x,y)$ *in E, as* $n \to \infty$, *then passing to a subsequence if necessary,* $\phi_{u_n}(x,y) \rightharpoonup \phi_{u_0}(x,y)$ *weakly in* $D^s(\mathbb{R}^4_+)$, *as* $n \to \infty$. *As a consequence* $I'(u_n) \to I'(u_0)$ *in the sense of distributions.*

Proof. The proof of this lemma is similar to ([5], Lemma 2.3], but we exhibit it here for completeness. Consider $u_n(\cdot,\cdot), u_0(\cdot,\cdot) \in X^{2s}(\mathbb{R}^4_+)$ such that $u_n(x,y) \rightharpoonup u_0(x,y)$ in $X^{2s}(\mathbb{R}^4_+)$, as $n \to \infty$. It follows that

$$\tilde{u}_n \rightharpoonup \tilde{u}_0 \quad \text{weakly in } L^r\left(\mathbb{R}^3 \times \{0\}\right), \text{ as } n \to \infty, \quad 2 \le r \le \frac{6}{3-2s}.$$

Since that $H^s\left(\mathbb{R}_+^4\right) \hookrightarrow L^r\left(\mathbb{R}^3\right)$ it is compact for bounded domain, we have

$$\tilde{u}_n \to \tilde{u}_0 \quad \text{in } L_{loc}^r\left(\mathbb{R}^3 \times \{0\}\right), \text{ as } n \to \infty, \quad 2 \le r < \frac{6}{3-2s}.$$

We denote by ϕ_n the function ϕ_{u_n}. From Lemma 2.1, note that for any $n \ge 1$ we get

$$\left\|\phi_n\right\|_{D^s\left(\mathbb{R}_+^4\right)}^2 \le C\left\|\phi_n\right\|_{D^s\left(\mathbb{R}_+^4\right)}\left|u_n\right|_{\frac{12}{3+2s}}^2.$$

It means that $\{\phi_n\}$ is bounded in $D^s\left(\mathbb{R}_+^4\right)$. Since that $D^s\left(\mathbb{R}_+^4\right)$ is a Hilbert space, there is a $\xi \in D^s\left(\mathbb{R}_+^4\right)$ such that

$$\tilde{\phi}_n \rightharpoonup \xi \quad \text{weakly in } L^r\left(\mathbb{R}^3 \times \{0\}\right), \text{ as } n \to \infty, \quad 2 \le r \le \frac{6}{3-2s}$$

and

$$\phi_n \to \xi \quad \text{in } L_{loc}^r\left(\mathbb{R}^3 \times \{0\}\right), \text{ as } n \to \infty, \quad 2 \le r < \frac{6}{3-2s}.$$

We want to prove the following equality $\phi_0 = \xi$. To this end, it is necessary to show, in the sense of distributions,

$$\begin{cases} -\operatorname{div}\left(y^{1-2s}\nabla v\right) = 0, & \text{in } \mathbb{R}_+^4, \\ k_s y^{1-2s}\dfrac{\partial v}{\partial \eta} = (\omega + \xi)u_0^2, & \text{on } \mathbb{R}^3 \times \{0\}, \end{cases} \tag{2.4}$$

and use the uniqueness of the solution given from Lemma 2.1.

Consider a test function $\psi \in C_0^\infty\left(\mathbb{R}_+^4\right)$ and $\tilde{\psi} \in C_0^\infty\left(\mathbb{R}^3 \times \{0\}\right)$. We know by Lemma 2.1 the following equality

$$\begin{cases} -\operatorname{div}\left(y^{1-2s}\nabla v\right) = 0, & \text{in } \mathbb{R}_+^4, \\ k_s y^{1-2s}\dfrac{\partial v}{\partial \eta} = (\omega + \phi_n)u_n^2, & \text{on } \mathbb{R}^3 \times \{0\}. \end{cases} \tag{2.5}$$

Then, we just need to see how each term of the equality above converges. To verify that

$$\int_{\mathbb{R}_+^4} y^{1-2s}\nabla\phi_n \nabla\psi \,\mathrm{d}x\mathrm{d}y \to \int_{\mathbb{R}_+^4} y^{1-2s}\nabla\xi\nabla\psi \,\mathrm{d}x\mathrm{d}y, \quad \text{as } n \to \infty$$

it follows from the weak convergence, also

$$\int_{\mathbb{R}^3}\left(\omega + \tilde{\phi}_n\right)\tilde{u}_n^2\tilde{\psi}\,\mathrm{d}x \to \int_{\mathbb{R}^3}\left(\omega + \tilde{\xi}\right)\tilde{u}_0^2\tilde{\psi}\,\mathrm{d}x.$$

By the strong convergence in $L_{loc}^r\left(\mathbb{R}^3 \times \{0\}\right)$, as $n \to \infty$, $2 \le r < \frac{6}{3-2s}$, we have

$$\int_{\mathbb{R}^3}\tilde{u}_n^2\tilde{\psi}\,\mathrm{d}x \to \int_{\mathbb{R}^3}\tilde{u}_0^2\tilde{\psi}\,\mathrm{d}x, \text{ as } n \to \infty.$$

Whereas

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(\tilde{\phi}_n\tilde{u}_n^2-\tilde{\xi}\tilde{u}_0^2\right)\tilde{\psi}\,\mathrm{d}x\\&=\int_{\mathbb{R}^3}\tilde{\phi}_n\left(\tilde{u}_n^2-\tilde{u}_0^2\right)\tilde{\psi}\,\mathrm{d}x+\int_{\mathbb{R}^3}\tilde{u}_0^2\left(\tilde{\phi}_n-\tilde{\xi}\right)\tilde{\psi}\,\mathrm{d}x\\&\le C\|\phi_n\|_{D^s\left(\mathbb{R}_+^4\right)}\left(\int_{\mathbb{R}^3}\left|\tilde{u}_n^2-\tilde{u}_0^2\right|^{\frac{6}{3+2s}}|\tilde{\psi}|^{\frac{6}{3+2s}}\,\mathrm{d}x\right)^{\frac{3+2s}{6}}\\&\quad+\int_{\mathbb{R}^3}\tilde{u}_0^2\left(\tilde{\phi}_n-\tilde{\xi}\right)\tilde{\psi}\,\mathrm{d}x\to 0,\ \text{ as } n\to\infty.\end{aligned}$$

Now we pass to prove the second part of the Lemma. Consider a test function $\tilde{\psi}\in C_0^\infty\left(\mathbb{R}^3\times\{0\}\right)$. Using boundedness of $\{\phi_n\}$, the strong convergences in $L_{loc}^r\left(\mathbb{R}^3\times\{0\}\right)$, $2\le r<\dfrac{6}{3-2s}$ and the Sobolev embeddings follow that as $n\to\infty$, it has

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(\tilde{\phi}_n\tilde{u}_n-\tilde{\xi}\tilde{u}_0\right)\tilde{\psi}\,\mathrm{d}x\\&=\int_{\mathbb{R}^3}\tilde{\phi}_n\left(\tilde{u}_n-\tilde{u}_0\right)\tilde{\psi}\,\mathrm{d}x+\int_{\mathbb{R}^3}\tilde{u}_0\left(\tilde{\phi}_n-\tilde{\xi}\right)\tilde{\psi}\,\mathrm{d}x\\&\le C\left\|\phi_{u_n}\right\|_{D^s\left(\mathbb{R}_+^4\right)}\left(\int_{\mathbb{R}^3}\left|\tilde{u}_n-\tilde{u}_0\right|^{\frac{6}{3+2s}}|\tilde{\psi}|^{\frac{6}{3+2s}}\,\mathrm{d}x\right)^{\frac{3+2s}{6}}\\&\quad+\int_{\mathbb{R}^3}\tilde{u}_0\left(\tilde{\phi}_n-\tilde{\xi}\right)\tilde{\psi}\,\mathrm{d}x\to 0.\end{aligned}$$

Analogously, we prove that

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(\tilde{\phi}_n^2\tilde{u}_n-\tilde{\xi}^2\tilde{u}_0\right)\tilde{\psi}\,\mathrm{d}x\\&=\int_{\mathbb{R}^3}\tilde{\phi}_n^2\left(\tilde{u}_n-\tilde{u}_0\right)\tilde{\psi}\,\mathrm{d}x+\int_{\mathbb{R}^3}\tilde{u}_0\left(\tilde{\phi}_n^2-\tilde{\xi}^2\right)\tilde{\psi}\,\mathrm{d}x\\&\le C\|\phi_n\|_{D^s\left(\mathbb{R}_+^4\right)}\left(\int_{\mathbb{R}^3}\left|\tilde{u}_n-\tilde{u}_0\right|^{\frac{3}{2s}}|\tilde{\psi}|^{\frac{3}{2s}}\,\mathrm{d}x\right)^{\frac{2s}{3}}\\&\quad+\int_{\mathbb{R}^3}\tilde{u}_0\left(\tilde{\phi}_n^2-\tilde{\xi}^2\right)\tilde{\psi}\,\mathrm{d}x\to 0,\end{aligned}$$

as $n\to\infty$. For density, $\forall\,\psi\in X^{2s}\left(\mathbb{R}_+^4\right)$ we infer that,

$$\int_{\mathbb{R}_+^4}y^{1-2s}\nabla\phi_n\nabla\psi\,\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^3}\left(\lambda a(x)+1\right)\tilde{u}_n\tilde{\psi}\,\mathrm{d}x$$

converges to

$$\int_{\mathbb{R}_+^4}y^{1-2s}\nabla\phi_0\nabla\psi\,\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^3}\left(\lambda a(x)+1\right)\tilde{u}_0\tilde{\psi}\,\mathrm{d}x$$

and

$$\int_{\mathbb{R}^3}\left(2\omega+\tilde{\phi}_n\right)\tilde{\phi}_n\tilde{u}_n\tilde{\psi}\,\mathrm{d}x$$

converges to

$$\int_{\mathbb{R}^3}\left(2\omega+\tilde{\xi}\right)\tilde{\xi}\tilde{u}_0\tilde{\psi}\,\mathrm{d}x,$$

as $n\to\infty$, thus $\left\langle I'\left(u_n\right),\psi\right\rangle\to\left\langle I'\left(u_0\right),\psi\right\rangle$ in the sense of distributions.

3. Basic Lemmas

In this section, we first begin proving that I satisfies the assumptions of the mountain pass theorem.

Lemma 3.1. *Suppose that* (a_1)-(a_3) *and* (f_1)-(f_4) *are satisfied. Then the functional I satisfies the mountain pass geometry, that is,*

1) There exist $r, \alpha > 0$ such that $I(u) \geq \alpha$ for any $u \in E$ such that $\|u\| = r$;

2) There exists $e \in E \setminus \{0\}$ with $\|u\| > r$ such that $I(e) \leq 0$.

Proof. From (f_1) and (f_2), given $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$F(s) \leq \varepsilon s^2 + C_\varepsilon |s|^p, \quad \forall s \in \mathbb{R}.$$

By sobolev embedding, we have

$$\begin{aligned} I(u) &= \frac{k_s}{2}\int_{\mathbb{R}_+^4} y^{1-2s}|\nabla u|^2 \mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}^3}(\lambda a(x)+1)\tilde{u}^2 \mathrm{d}x \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^3} \omega\tilde{\phi}_u \tilde{u}^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(x,\tilde{u})\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - \varepsilon\int_{\mathbb{R}^3}|\tilde{u}|^2 - C_\varepsilon\int_{\mathbb{R}^3}|\tilde{u}|^p \\ &\geq \left[\left(\frac{1}{2} - \varepsilon C_1\right) - C_2 C_\varepsilon \|u\|^{p-2}\right]\|u\|^2, \end{aligned} \tag{3.1}$$

then we can choose $r, \alpha > 0$ such that $I(u) \geq \alpha$ for $\|u\| = r$. On the other hand, from (f_3), for any $u \in E$, there exists $C > 0$ such that $F(s) \geq C|s|^\mu$, for all $s \in \mathbb{R}$. Hence $u \in E \setminus \{0\}$, by Lemma 2.1, one has

$$\begin{aligned} I(tu) &= \frac{t^2}{2}\int_{\mathbb{R}_+^4} k_s y^{1-2s}|\nabla u|^2 \mathrm{d}x\mathrm{d}y + \frac{t^2}{2}\int_{\mathbb{R}^3}(\lambda a(x)+1)\tilde{u}^2 \mathrm{d}x \\ &\quad - \frac{t^2}{2}\int_{\mathbb{R}^3} \omega\tilde{\phi}_{tu}\tilde{u}^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(x,t\tilde{u})\mathrm{d}x \\ &\leq \frac{t^2}{2}\|u\|^2 + \frac{t^2\omega^2}{2}\|u\|^2 - Ct^\mu\|u\|^\mu \\ &\to -\infty, \quad \text{as } t \to +\infty. \end{aligned} \tag{3.2}$$

It is obvious that $I(tu) \to -\infty$ as $t \to +\infty$. Thus, there exists $e \in E \setminus \{0\}$ such that $I(e) \leq 0$. This completes the proof of Lemma 3.1.

So, there is a Cerami sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \text{ and } (1 + \|u_n\|)\|I'(u_n)\| \to 0, \quad n \to \infty, \tag{3.3}$$

where

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$$

is the Mountain-Pass level, with $\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = 0, I(\gamma(1)) \leq 0\}$.

Lemma 3.2. *Under the assumptions of Theorem* 1.1, *the Cerami sequence* $\{u_n\} \subset E$ *given in* (3.3) *is bounded.*

Proof. With the fact that (a_1) and (a_3) hold, there exists $v \in E \setminus \{0\}$ such that v has support in $\overline{\Omega}_0$. Then,

$$\begin{aligned} c &\le \max_{t\ge 0} I(tv) \\ &\le \max_{t\ge 0}\left[\frac{t^2}{2}\left(\int_{\mathbb{R}_+^4} k_s y^{1-2s}|\nabla v|^2 \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3}\tilde{v}^2\mathrm{d}x - \int_{\mathbb{R}^3}\omega\phi_{tv}\tilde{v}^2\mathrm{d}x\right) - \int_{\mathbb{R}^3}F(x,t\tilde{v})\mathrm{d}x\right] \\ &=: c_0. \end{aligned}$$

If $\mu \ge 4$, by (3.3), (f_3) and Lemma 2.1, we have

$$\begin{aligned} c+o_n(1) &= I(u_n) - \frac{1}{\mu}\langle I'(u_n),u_n\rangle \\ &= \left(\frac{1}{2}-\frac{1}{\mu}\right)\left[\int_{\mathbb{R}_+^4} k_s y^{1-2s}|\nabla u_n|^2\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3}(\lambda a(x)+1)\tilde{u}_n^2\mathrm{d}x\right] \\ &\quad -\left(\frac{1}{2}-\frac{2}{\mu}\right)\int_{\mathbb{R}^3}\omega\tilde{\phi}_n\tilde{u}_n^2\mathrm{d}x + \frac{1}{\mu}\int_{\mathbb{R}^3}\tilde{\phi}_n^2\tilde{u}_n^2\mathrm{d}x \\ &\quad +\frac{1}{\mu}\int_{\mathbb{R}^3}\left(f(x,\tilde{u}_n)\tilde{u}_n - \mu F(x,\tilde{u}_n)\right)\mathrm{d}x \\ &\ge \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|^2. \end{aligned} \tag{3.4}$$

Thus,

$$\|u_n\|^2 \le \frac{2\mu}{\mu-2}c \le \frac{2\mu}{\mu-2}c_0. \tag{3.5}$$

If $2<\mu<4$ and $0<\omega<\frac{\sqrt{8(\mu-2)}}{4-\mu}$. Using Lemma 2.1 and ($f_3$), it obtains

$$\begin{aligned} c+o_n(1) &= I(u_n) - \frac{1}{\mu}\langle I'(u_n),u_n\rangle \\ &= \left(\frac{1}{2}-\frac{1}{\mu}\right)\left[\int_{\mathbb{R}_+^4} k_s y^{1-2s}|\nabla u_n|^2\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3}(\lambda a(x)+1)\tilde{u}_n^2\mathrm{d}x\right] \\ &\quad -\int_{\mathbb{R}^3}\frac{(4-\mu)^2}{16\mu}\omega^2\tilde{u}_n^2\mathrm{d}x + \int_{\mathbb{R}^3}\frac{(4-\mu)^2}{16\mu}\omega^2\tilde{u}_n^2\mathrm{d}x \\ &\quad -\left(\frac{1}{2}-\frac{2}{\mu}\right)\int_{\mathbb{R}^3}\omega\tilde{\phi}_n\tilde{u}_n^2\mathrm{d}x + \frac{1}{\mu}\int_{\mathbb{R}^3}\tilde{\phi}_n^2\tilde{u}_n^2\mathrm{d}x \\ &\quad +\frac{1}{\mu}\int_{\mathbb{R}^3}\left(f(x,\tilde{u}_n)\tilde{u}_n - \mu F(x,\tilde{u}_n)\right)\mathrm{d}x \\ &\ge \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|^2 - \int_{\mathbb{R}^3}\frac{(4-\mu)^2}{16\mu}\omega^2\tilde{u}_n^2\mathrm{d}x \\ &\quad +\int_{\mathbb{R}^3}\left[\frac{(4-\mu)^2}{16\mu}\omega^2\tilde{u}_n^2 - \left(\frac{1}{2}-\frac{2}{\mu}\right)\omega\tilde{\phi}_n\tilde{u}_n^2 + \frac{1}{\mu}\tilde{\phi}_n^2\tilde{u}_n^2\right]\mathrm{d}x \\ &\ge \left[\left(\frac{1}{2}-\frac{1}{\mu}\right) - \frac{(4-\mu)^2}{16\mu}\omega^2\right]\|u_n\|^2. \end{aligned} \tag{3.6}$$

Similar to (3.5), one sees

$$\|u_n\|^2 \le \frac{16\mu c}{8(\mu-2)-(4-\mu)^2\omega^2} \le \frac{16\mu c_0}{8(\mu-2)-(4-\mu)^2\omega^2}. \tag{3.7}$$

In light of (3.5) and (3.7) we conclude that $\{u_n\}$ is bounded. □

Lemma 3.3. *For any boundedness of* $(C)_c$ *sequence* $\{u_n\}$, *there exists* $u_1 \in E\setminus\{0\}$ *such that* $I'(u_1)=0$.

Proof. Inspired by [15]. Let $\{u_n\}$ be a bounded $(C)_c$ sequence, there exists $u_1 \in E$ such that $u_n \rightharpoonup u_1$. Lemma 2.1 implies that $I'(u_1)=0$. By (f_1) and (f_2), there exists $C_\varepsilon > 0$ such that

$$\frac{1}{2} f(x,t)t - F(x,t) \le \varepsilon t^2 + C_\varepsilon |t|^p, \quad \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{3.8}$$

If $2<\mu<4$ and $0<\omega<\frac{\sqrt{8(\mu-2)}}{4-\mu}$. Let $\varepsilon = \frac{8(\mu-2)-(4-\mu)^2\omega^2}{32\mu}$, it follows from (3.9) and (3.10) that

$$\begin{aligned}
c &= \lim_{n\to\infty}\left(I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle \right) \\
&= \lim_{n\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}^3} (\omega+\tilde{\phi}_n)\tilde{\phi}_n \tilde{u}_n^2 \mathrm{d}x + \int_{\mathbb{R}^3} \frac{1}{2} f(x,\tilde{u}_n)\tilde{u}_n - F(x,\tilde{u}_n)\mathrm{d}x \right) \\
&\le \lim_{n\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}^3} (\omega+\tilde{\phi}_n)\tilde{\phi}_n \tilde{u}_n^2 \mathrm{d}x + \frac{8(\mu-2)-(4-\mu)^2\omega^2}{32\mu}\int_{\mathbb{R}^3} |\tilde{u}_n|^2 \mathrm{d}x + C\int_{\mathbb{R}^3}|\tilde{u}_n|^p \mathrm{d}x \right) \\
&\le \liminf_{n\to\infty}\left[\frac{8(\mu-2)-(4-\mu)^2\omega^2}{32\mu}\left(\int_{\mathbb{R}^4_+} k_s y^{1-2s} |\nabla u_n|^2 \mathrm{d}x\mathrm{d}y \right.\right. \\
&\qquad \left.\left. + \int_{\mathbb{R}^3} (\lambda a(x)+1)\tilde{u}_n^2 \mathrm{d}x \right) + C\int_{\mathbb{R}^3} |\tilde{u}_n|^p \mathrm{d}x \right] \\
&\le \frac{1}{2}c + \liminf_{n\to\infty} C\int_{\mathbb{R}^3} |\tilde{u}_n|^p \mathrm{d}x.
\end{aligned} \tag{3.9}$$

Thus,

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} |\tilde{u}_n|^p \mathrm{d}x \ge \frac{c}{2C}. \tag{3.10}$$

If $\mu \ge 4$. Let $\varepsilon = \frac{\mu-2}{4\mu}$, as in proof of (3.9), we can also get (3.10).

For $r>0$, let

$$A(r) := \{x \in \mathbb{R}^3 : |x| > r, a(x) \ge M\},$$
$$B(r) := \{x \in \mathbb{R}^3 : |x| > r, a(x) < M\}.$$

Since $\frac{2\mu}{\mu-2} < \frac{16\mu}{8(\mu-2)-(4-\mu)^2}$, from ($a_2$), (2.2) and (3.7), for $\lambda \ge 1$, one sees

$$\begin{aligned}
\int_{A(r)} |\tilde{u}_n|^2 \mathrm{d}x &\le \frac{1}{\lambda M+1}\int_{A(r)} (\lambda a(x)+1)|\tilde{u}_n|^2 \mathrm{d}x \\
&\le \frac{1}{\lambda M+1}\int_{\mathbb{R}^3} (\lambda a(x)+1)|\tilde{u}_n|^2 \mathrm{d}x \\
&\le \frac{1}{\lambda M+1}\|u_n\|^2
\end{aligned}$$

$$\le \frac{1}{\lambda M+1}\left(\frac{16\mu c}{8(\mu-2)-(4-\mu)^2\omega^2}+o_n(1)\right) \le \frac{1}{\lambda M+1}\left(\frac{16\mu c_0}{8(\mu-2)-(4-\mu)^2\omega^2}+o_n(1)\right) \quad (3.11)$$

and using the Hölder inequality and (3.7), we obtain

$$\begin{aligned}\int_{B(r)}|\tilde{u}_n|^2\,\mathrm{d}x &\le \left(\int_{\mathbb{R}^3}|\tilde{u}_n|^{\frac{6}{3-2s}}\,\mathrm{d}x\right)^{\frac{3-2s}{3}}\left(\int_{B(r)}1\mathrm{d}x\right)^{\frac{2s}{3}} \\ &\le C\|u_n\|^2\cdot\mu(B(r))^{\frac{2s}{3}} \\ &\le C\frac{16\mu c}{8(\mu-2)-(4-\mu)^2\omega^2}\cdot\mu(B(r))^{\frac{2s}{3}}+o_n(1) \\ &\le C\frac{16\mu c_0}{8(\mu-2)-(4-\mu)^2\omega^2}\cdot\mu(B(r))^{\frac{2s}{3}}+o_n(1),\end{aligned}$$

where $\mu(B(r))$ is the measure of $B(r)$. Hence

$$\begin{aligned}\int_{\mathbb{R}^3\setminus B_r}|\tilde{u}_n|^2\,\mathrm{d}x &= \int_{A(r)}|\tilde{u}_n|^2\,\mathrm{d}x+\int_{B(r)}|\tilde{u}_n|^2\,\mathrm{d}x \\ &\le \frac{16\mu c}{8(\mu-2)-(4-\mu)^2\omega^2}\left(\frac{1}{\lambda M+1}+C\mu(B(r))^{\frac{2s}{3}}\right)+o_n(1).\end{aligned} \quad (3.12)$$

When λ and r are large enough, we can obtain that $\int_{\mathbb{R}^3\setminus B_r}|u_n|^2\,\mathrm{d}x$ is small enough. Hölder inequality and (2.2) imply that

$$\begin{aligned}\int_{\mathbb{R}^3\setminus B_r}|\tilde{u}_n|^p\,\mathrm{d}x &\le \left(\int_{\mathbb{R}^3\setminus B_r}|\tilde{u}_n|^{\frac{6}{3-2s}}\,\mathrm{d}x\right)^{\frac{3p-2sp-6+4s}{4s}}\left(\int_{\mathbb{R}^3\setminus B_r}|\tilde{u}_n|^2\,\mathrm{d}x\right)^{\frac{2sp-3p+6}{4s}} \\ &\le C\|u_n\|^{\frac{3p-6}{2s}}\left(\int_{\mathbb{R}^3\setminus B_r}|\tilde{u}_n|^2\,\mathrm{d}x\right)^{\frac{2sp-3p+6}{4s}}.\end{aligned}$$

Since $\|u_n\|$ is bounded which is independent of n and λ, there are $\Lambda>0$ and $r_1>0$ such that $\int_{\mathbb{R}^3\setminus B_r}|\tilde{u}_n|^p\,\mathrm{d}x<\frac{c}{4C}$, $\lambda\ge\Lambda$, $r\ge r_1$. From (3.10), we obtain $\int_{B_r}|\tilde{u}_n|^p\,\mathrm{d}x\ge\frac{c}{4C}$, $\lambda\ge\Lambda$, $r\ge r_1$, which implies $u_1\in E\setminus\{0\}$. □

4. Proof of Theorem 1.1

In this section, we prove that problem (1.1) has a ground state solution.

Proof of Theorem 1.1. By the previous discussion of Lemmas 3.1-3.3, it follows that there exists a sequence $\{v_n\}$ and $v\in E\setminus\{0\}$ such that $v_n\rightharpoonup v$ and $\langle I'(v),v\rangle=0$. Define

$$m:=\inf_{u\in\mathcal{N}}I(u), \quad (4.1)$$

where $\mathcal{N}=\{u\in E\setminus\{0\}:\langle I'(u),u\rangle=0\}$.

Clearly, $\mathcal{N}$ is not an empty set. Let $\{u_n\}\subset\mathcal{N}$ be a minimizing sequence for I, that is, $I(u_n)\to m$ and $\langle I'(u_n),u_n\rangle=0$. It follows from Lemmas 3.2 and

3.3, $\{u_n\}$ is bounded in E and there exists $u_0 \neq 0$ such that $u_n \rightharpoonup u_0$ in E, $u_n \to u_0$ in $L^s_{loc}(\mathbb{R}^3)$ for $1 \le s < \frac{6}{3-2s}$, $u_n \to u_0$ a.e. in $\mathbb{R}^3$. Using the definition of m together with the fact that $\langle I'(u_n), u_n \rangle = 0$ and $I(u_n) \to m$, it has $\langle I'(u_0), u_0 \rangle = 0$ and $I(u_0) \ge m$. Next, we will claim $I(u_0) = m$.

If $\mu \ge 4$. Combining (f_3) and Fatou's Lemma, it gets

$$\begin{aligned}
m &= \lim_{n\to\infty}\left(I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \right) \\
&= \lim_{n\to\infty}\Bigg\{ \left(\frac{1}{2}-\frac{1}{\mu}\right)\left[\int_{\mathbb{R}^4_+} k_s y^{1-2s}|\nabla u_n|^2 \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3}(\lambda a(x)+1)\tilde{u}_n^2 \mathrm{d}x\right] \\
&\quad - \left(\frac{1}{2}-\frac{2}{\mu}\right)\int_{\mathbb{R}^3}\omega\tilde{\phi}_n\tilde{u}_n^2\mathrm{d}x + \frac{1}{\mu}\int_{\mathbb{R}^3}\tilde{\phi}_n^2\tilde{u}_n^2\mathrm{d}x \\
&\quad + \frac{1}{\mu}\int_{\mathbb{R}^3}\left(f(x,\tilde{u}_n)\tilde{u}_n - \mu F(x,\tilde{u}_n)\right)\mathrm{d}x \Bigg\} \\
&\ge \liminf_{n\to\infty}\Bigg\{ \left(\frac{1}{2}-\frac{1}{\mu}\right)\left[\int_{\mathbb{R}^4_+} k_s y^{1-2s}|\nabla u_n|^2 \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3}(\lambda a(x)+1)\tilde{u}_n^2 \mathrm{d}x\right] \\
&\quad - \left(\frac{1}{2}-\frac{2}{\mu}\right)\int_{\mathbb{R}^3}\omega\tilde{\phi}_n\tilde{u}_n^2\mathrm{d}x + \frac{1}{\mu}\int_{\mathbb{R}^3}\tilde{\phi}_n^2\tilde{u}_n^2\mathrm{d}x \\
&\quad + \frac{1}{\mu}\int_{\mathbb{R}^3}\left(f(x,\tilde{u}_n)\tilde{u}_n - \mu F(x,\tilde{u}_n)\right)\mathrm{d}x \Bigg\} \\
&\ge \left(\frac{1}{2}-\frac{1}{\mu}\right)\left[\int_{\mathbb{R}^4_+} k_s y^{1-2s}|\nabla u_0|^2 \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3}(\lambda a(x)+1)\tilde{u}_0^2 \mathrm{d}x\right] \\
&\quad - \left(\frac{1}{2}-\frac{2}{\mu}\right)\int_{\mathbb{R}^3}\omega\tilde{\phi}_0\tilde{u}_0^2\mathrm{d}x + \frac{1}{\mu}\int_{\mathbb{R}^3}\tilde{\phi}_0^2\tilde{u}_0^2\mathrm{d}x \\
&\quad + \frac{1}{\mu}\int_{\mathbb{R}^3}\left(f(x,\tilde{u}_0)\tilde{u}_0 - \mu F(x,\tilde{u}_0)\right)\mathrm{d}x \\
&= I(u_0) - \frac{1}{\mu}\langle I'(u_0), u_0\rangle = I(u_0) \ge m.
\end{aligned} \tag{4.2}$$

If $2 < \mu < 4$ and $0 < \omega < \frac{\sqrt{8(\mu-2)}}{4-\mu}$. It follows from ($a_2$) that $\Omega := \{x \in \mathbb{R}^3 : a(x) < M\}$ is a bounded domain. Therefore,

$$\frac{(4-\mu)^2\omega^2}{16\mu}\int_\Omega |\tilde{u}_n|^2 \mathrm{d}x \to \frac{(4-\mu)^2\omega^2}{16\mu}\int_\Omega |\tilde{u}_0|^2 \mathrm{d}x. \tag{4.3}$$

From (f_3), (4.3) and Fatou's Lemma, we have

$$\begin{aligned}
m &= \lim_{n\to\infty}\left(I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \right) \\
&= \lim_{n\to\infty}\Bigg\{ \left(\frac{1}{2}-\frac{1}{\mu}\right)\left(\int_{\mathbb{R}^4_+} k_s y^{1-2s}|\nabla u_n|^2 \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3}(\lambda a(x)+1)\tilde{u}_n^2 \mathrm{d}x\right) \\
&\quad - \int_{\mathbb{R}^3}\frac{(4-\mu)^2\omega^2}{16\mu}|\tilde{u}_n|^2 \mathrm{d}x + \int_{\mathbb{R}^3}\frac{(4-\mu)^2\omega^2}{16\mu}|\tilde{u}_n|^2 \mathrm{d}x
\end{aligned}$$

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$$
\begin{aligned}
&-\left(\frac{1}{2}-\frac{2}{\mu}\right)\int_{\mathbb{R}^3}\omega\tilde{\phi}_n\tilde{u}_n^2\mathrm{d}x+\frac{1}{\mu}\int_{\mathbb{R}^3}\tilde{\phi}_n^2\tilde{u}_n^2\mathrm{d}x \\
&+\int_{\mathbb{R}^3}\left(\frac{1}{\mu}f(x,\tilde{u}_n)\tilde{u}_n-F(x,\tilde{u}_n)\right)\mathrm{d}x\Bigg\} \\
&=\lim_{n\to\infty}\Bigg\{\left(\frac{1}{2}-\frac{1}{\mu}\right)\left(\int_{\mathbb{R}_+^4}k_s y^{1-2s}|\nabla u_n|^2\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^3\setminus\Omega}(\lambda a(x)+1)\tilde{u}_n^2\mathrm{d}x\right) \\
&+\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\Omega}(\lambda a(x)+1)\tilde{u}_n^2\mathrm{d}x-\int_{\mathbb{R}^3\setminus\Omega}\frac{(4-\mu)^2\omega^2}{16\mu}|\tilde{u}_n|^2\mathrm{d}x \\
&-\int_{\Omega}\frac{(4-\mu)^2\omega^2}{16\mu}|\tilde{u}_n|^2\mathrm{d}x+\int_{\mathbb{R}^3}\left(\frac{1}{\mu}f(x,\tilde{u}_n)\tilde{u}_n-F(x,\tilde{u}_n)\right)\mathrm{d}x \\
&+\int_{\mathbb{R}^3}\frac{(4-\mu)^2\omega^2}{16\mu}|\tilde{u}_n|^2\mathrm{d}x-\left(\frac{1}{2}-\frac{2}{\mu}\right)\int_{\mathbb{R}^3}\omega\tilde{\phi}_n\tilde{u}_n^2\mathrm{d}x+\frac{1}{\mu}\int_{\mathbb{R}^3}\tilde{\phi}_n^2\tilde{u}_n^2\mathrm{d}x\Bigg\} \\
&\geq\liminf_{n\to\infty}\Bigg\{\left(\frac{1}{2}-\frac{1}{\mu}\right)\left(\int_{\mathbb{R}_+^4}k_s y^{1-2s}|\nabla u_n|^2\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^3\setminus\Omega}(\lambda a(x)+1)\tilde{u}_n^2\mathrm{d}x\right) \\
&+\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\Omega}(\lambda a(x)+1)\tilde{u}_n^2\mathrm{d}x-\int_{\mathbb{R}^3\setminus\Omega}\frac{(4-\mu)^2\omega^2}{16\mu}|\tilde{u}_n|^2\mathrm{d}x \\
&-\int_{\Omega}\frac{(4-\mu)^2\omega^2}{16\mu}|\tilde{u}_n|^2\mathrm{d}x+\int_{\mathbb{R}^3}\left(\frac{1}{\mu}f(x,\tilde{u}_n)\tilde{u}_n-F(x,\tilde{u}_n)\right)\mathrm{d}x \\
&+\int_{\mathbb{R}^3}\frac{(4-\mu)^2\omega^2}{16\mu}|\tilde{u}_n|^2\mathrm{d}x-\left(\frac{1}{2}-\frac{2}{\mu}\right)\int_{\mathbb{R}^3}\omega\tilde{\phi}_n\tilde{u}_n^2\mathrm{d}x+\frac{1}{\mu}\int_{\mathbb{R}^3}\tilde{\phi}_n^2\tilde{u}_n^2\mathrm{d}x\Bigg\} \\
&\geq\left(\frac{1}{2}-\frac{1}{\mu}\right)\left(\int_{\mathbb{R}_+^4}k_s y^{1-2s}|\nabla u_0|^2\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^3\setminus\Omega}(\lambda a(x)+1)\tilde{u}_0^2\mathrm{d}x\right) \\
&+\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\Omega}(\lambda a(x)+1)\tilde{u}_0^2\mathrm{d}x-\int_{\mathbb{R}^3\setminus\Omega}\frac{(4-\mu)^2\omega^2}{16\mu}|\tilde{u}_0|^2\mathrm{d}x \\
&-\int_{\Omega}\frac{(4-\mu)^2\omega^2}{16\mu}|\tilde{u}_0|^2\mathrm{d}x+\int_{\mathbb{R}^3}\left(\frac{1}{\mu}f(x,\tilde{u}_0)\tilde{u}_0-F(x,\tilde{u}_0)\right)\mathrm{d}x \\
&+\int_{\mathbb{R}^3}\left(\frac{(4-\mu)^2}{16\mu}\omega^2|\tilde{u}_0|^2-\left(\frac{1}{2}-\frac{2}{\mu}\right)\omega\tilde{\phi}_0\tilde{u}_0^2+\frac{1}{\mu}\tilde{\phi}_0^2\tilde{u}_0^2\right)\mathrm{d}x \\
&=\left(\frac{1}{2}-\frac{1}{\mu}\right)\left(\int_{\mathbb{R}_+^4}k_s y^{1-2s}|\nabla u_0|^2\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^3}(\lambda a(x)+1)\tilde{u}_0^2\mathrm{d}x\right) \\
&-\left(\frac{1}{2}-\frac{2}{\mu}\right)\int_{\mathbb{R}^3}\omega\tilde{\phi}_0\tilde{u}_0^2\mathrm{d}x+\frac{1}{\mu}\int_{\mathbb{R}^3}\tilde{\phi}_0^2\tilde{u}_0^2\mathrm{d}x \\
&+\int_{\mathbb{R}^3}\left(\frac{1}{\mu}f(x,\tilde{u}_0)\tilde{u}_0-F(x,\tilde{u}_0)\right)\mathrm{d}x \\
&=I(u_0)-\frac{1}{\mu}\langle I'(u_0),u_0\rangle=I(u_0)\geq m.
\end{aligned}
\tag{4.4}
$$

Obviously, this proves that u_0 is a ground state solution of problem (1.1). Hence, Theorem 1.1 is proved. □

Conflicts of Interest

The author declares no conflicts of interest.

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