Open Access Library Journal
2022, Volume 9, e9120
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Existence of Nontrivial Solution for Klein-Gordon-Maxwell System with Logarithmic Nonlinearity

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How to cite this paper: Shi, Q.Y. (2022) Existence of Nontrivial Solution for Klein-Gordon-Maxwell System with Logarithmic Nonlinearity. *Open Access Library Journal*, **9**: e9120.
https://doi.org/10.4236/oalib.1109120

Received: July 19, 2022
Accepted: August 14, 2022
Published: August 17, 2022

Abstract

In this paper, we study the nonautonomous Klein-Gordon-Maxwell system with logarithmic nonlinearity. We obtain the existence of nontrivial solution for this system by logarithmic Sobolev inequality and variational method.

Subject Areas

Partial Differential Equation

Keywords

Klein-Gordon-Maxwell System, Logarithmic Nonlinearity, Variational Methods, Nontrivial Solution, Mountain Pass Theorem

1. Introduction and Main Results

Our main purpose in this paper is to study the existence of nontrivial solution for the following system:

$$\begin{cases} -\Delta u + V(x)u - (2\omega+\phi)\phi u = u\ln|u| + |u|^{p-2}u, & \text{in } \Omega, \\ -\Delta\phi = -(\omega+\phi)u^2, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $\omega > 0$ is parameters. We assume:

(V) $V \in L^{\frac{3}{2}}(\Omega)$ and $V_0 = \inf_{x\in\Omega} V(x) > -\infty$.

In recent years, the following Klein-Gordon-Maxwell system:

$$\begin{cases} -\Delta u + V(x)u - (2\omega+\phi)\phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = -(\omega+\phi)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

has been object of interest for many researchers, $\omega > 0$ is a parameter, $\phi, u : \mathbb{R}^3 \to \mathbb{R}$, $V : \mathbb{R}^3 \to \mathbb{R}$ and $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. Such a system was first introduced by Benci and Fortunato [1] as a model describing solitary waves for the nonlinear Klein-Gordon equation interacting with an electromagnetic field. The unknowns of the system are the field u associated to the particle and the electric potential ϕ. The presence of the nonlinear term f simulates the interaction between many particles or external nonlinear perturbations. By applying a well known equivariant version of mountain pass theorem, Benci and Fortunato [1] [2] first studied the following special Klein-Gordon-Maxwell system with constant potential $m_0^2 - \omega^2$,

$$\begin{cases} -\Delta u + \left[m_0^2 - (\omega + \phi)^2 \right] u = |u|^{p-2} u, & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi) u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.3}$$

They considered $|\omega| < |m_0|$ and $f(u) = |u|^{p-2} u$, $4 < p < 2^* = 6$, and proved that system (1.3) has infinitely many radially symmetric solutions. In [3], D'Aprile and Mugnai extended the interval of definition of the power in the nonlinearity for the case $2 < p \le 4$. A nonexistence result has been established by the same authors in [4]. In [5] [6] the existence of ground state solutions of (1.3) was established.

Furthermore, if system (1.3) is added by a lower order perturbation, in the year 2004, Cassani [7] studied this kind of Klein-Gordon-Maxwell system:

$$\begin{cases} -\Delta u + \left[m_0^2 - (\omega + \phi)^2 \right] u = \mu |u|^{p-2} u + |u|^4 u, & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi) u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

he obtained the existence of a radially symmetric solution. Later, Wang [8] improved the result of [7]. Without need of symmetry, other related results about the autonomous Klein-Gordon-Maxwell system with a more general function $f(x, u)$ can be found in [9] [10] [11] and references therein. Very recently, the existence result for the (1.2) with a nonconstant potential $V(x)$ can be found in [12] [13] [14] [15] [16] [17] and so on. Furthermore, on bounded domains about Klein-Gordon-Maxwell system we refer to research in [18] [19].

Moreover, logarithmic nonlinearity is widely used in partial differential equations which describe the mathematical and physical phenomena. For elliptic equations with logarithmic nonlinearity, we can refer to [20] [21] [22] [23] [24] and the references therein. Compared with polynomial nonlinearity, logarithmic nonlinearity has both advantages and disadvantages. However, because logarithmic nonlinearity didn't satisfy the monotonicity condition and Ambrosetti-Rabinowitz condition which is quite different from these in the polynomial case.

Remark 1.1. *In this paper, we have:*

$$\lim_{t \to 0} \frac{t \ln|t| + |t|^{p-2} t}{t} = -\infty, \tag{1.5}$$

it is obviously different from the usual conditions,

$$\lim_{t\to 0}\frac{f(x,t)}{t}=0. \tag{1.6}$$

In other words, $t\ln|t|+|t|^{p-2}t$ *cannot be a special case of* $f(x,t)$ *in general.*

Now we state our main results as following.

Theorem 1.1. *Assume* (*V*) *hold. Then when* $4<p<6$ *problem* (1.1) *has a nontrivial solution* $(u,\phi)\in E\times\mathcal{D}^{1,2}(\Omega)$.

The plan of the paper is as follows. In Section 2, we give the variational framework for problem (1.1) and some preliminary results. In Section 3, we prove the some basic lemmas. In Section 4, we complete the proof of Theorem 1.1.

Throughout the paper, we give the following natations:

- C and $C_k\ (k=1,2,\cdots)$ for psositive constants;
- $\|u\|=\|u\|_{\mathcal{D}^{1,2}(\Omega)}=\left(\int_\Omega|\nabla u|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$ denote the norm of $\mathcal{D}^{1,2}(\Omega)$;
- $\|u\|_{H^1}=\|u\|_{H^1(\Omega)}=\left(\int_\Omega\left(|\nabla u|^2+u^2\right)\mathrm{d}x\right)^{\frac{1}{2}}$ denote the norm of $H^1(\Omega)$;
- $|u|_s=\left(\int_\Omega u^s\mathrm{d}x\right)^{\frac{1}{s}}$ denotes the norm of $L^s(\Omega),(1\le s\le\infty)$;
- S denotes the Sobolev constant.

2. Variational Setting and Preliminaries

Let

$$E:=\left\{u\in H^1(\Omega):\int_\Omega\left(|\nabla u|^2+V(x)u^2\right)\mathrm{d}x<\infty\right\},$$

E is a Hilbert space with the inner product,

$$(u,v)_E=\int_\Omega\left(\nabla u\cdot\nabla v+V(x)uv\right)\mathrm{d}x$$

and the norm,

$$\|u\|_E=\left(\int_\Omega|\nabla u|^2+V(x)u^2\mathrm{d}x\right)^{\frac{1}{2}}=\left(\|u\|^2+\int_\Omega V(x)u^2\mathrm{d}x\right)^{\frac{1}{2}},$$

which is equivalent to the standard norm in $H^1(\Omega)$. Obviously, the embedding $E\hookrightarrow L^s(\Omega)$ is continuous, for any $s\in[2,2^*]$, where $2^*=6$. Consequently, for each $s\in[2,6]$, there exists a constant $S_s>0$ such that,

$$|u|_s=S_s\|u\|_E,\quad\forall\,u\in E. \tag{2.1}$$

The solutions $(u,\phi)\in E\times\mathcal{D}^{1,2}(\Omega)$ of the (1.1) system are critical points of the functional $J:E\times\mathcal{D}^{1,2}(\Omega)\to\mathbb{R}$ defined as:

$$\begin{aligned}J(u,\phi)=&\frac{1}{2}\int_\Omega\left(|\nabla u|^2+V(x)u^2-|\nabla\phi|^2-(2\omega+\phi)\phi u^2\right)\mathrm{d}x\\&-\frac{1}{2}\int_\Omega u^2\ln|u|\,\mathrm{d}x+\frac{1}{4}\int_\Omega u^2\mathrm{d}x-\frac{1}{p}|u|^p\,\mathrm{d}x.\end{aligned} \tag{2.2}$$

By standard argument we can see that $J\in C^1\left(E\times\mathcal{D}^{1,2}(\Omega),\mathbb{R}\right)$ and that weak solutions of (1.1) turn out to be critical points for the energy functional *J*.

To obtain our main result, we have to overcome several difficulties. The first difficulty is that *J* is strongly indefinite, that is, is unbounded both from below and from above on infinite-dimensional subspaces. In order to avoid this indefiniteness, which rules out many of the usual tools of critical point theory, a reduction method is performed in [2] which we now recall.

Similar to [[4], Lemma 2.1], which deal with in the case of entire domain $\mathbb{R}^3$, for u and ϕ defined above, we have the following lemmas.

Lemma 2.1. *For every* $u \in H^1(\Omega)$, *there exists a unique* $\phi = \phi_u \in \mathcal{D}^{1,2}(\Omega)$ *which solves the equation*:

$$-\Delta\phi + u^2\phi = -\omega u^2. \tag{2.3}$$

Moreover, the map $\Phi: u \in H^1(\Omega) \mapsto \phi_u \in \mathcal{D}^{1,2}(\Omega)$ *is continuously differentiable, and*

$$-\omega \le \phi_u \le 0 \quad \text{a.e. in } \Omega. \tag{2.4}$$

Proof. Its proof is similar to [19] [25]. □

Multiplying (2.3) by ϕ_u and integrating by parts, one has

$$\int_\Omega |\nabla\phi_u|^2 \,\mathrm{d}x = -\int_\Omega \omega\phi_u u^2 \mathrm{d}x - \int_\Omega \phi_u^2 u^2 \mathrm{d}x. \tag{2.5}$$

Using (2.2) and (2.5), the functional $I(u) := J(u,\phi)$ reduces to the following form

$$\begin{aligned} I(u) = {} & \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega V(x)u^2 \mathrm{d}x - \frac{1}{2}\int_\Omega \omega\phi_u u^2 \mathrm{d}x \\ & -\frac{1}{2}\int_\Omega u^2 \ln|u| \,\mathrm{d}x + \frac{1}{4}\int_\Omega u^2 \mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x, \end{aligned} \tag{2.6}$$

and we have for any $u, v \in E$

$$\begin{aligned} \langle I'(u), v\rangle = {} & \int_\Omega \nabla u \nabla v \mathrm{d}x + \int_\Omega V(x) uv \mathrm{d}x - \int_\Omega (2\omega + \phi_u)\phi_u uv \mathrm{d}x \\ & - \int_\Omega uv \ln|u| \,\mathrm{d}x - \int_\Omega |u|^{p-2} uv \mathrm{d}x. \end{aligned} \tag{2.7}$$

Then, $(u,\phi) \in E \times \mathcal{D}^{1,2}(\Omega)$ is a weak solution of (1.1) if, and only if, $\phi = \phi_u$ and $u \in E$ is a critical point of *I*, that is, a weak solution of

$$-\Delta u + V(x)u^2 - (2\omega + \phi_u)\phi_u u = u\ln|u| + |u|^{p-2} u, \tag{2.8}$$

for any $x \in \Omega$.

The second, to deal with logarithmic nonlinearity $u\ln|u|$. We shall also need a logarithmic Sobolev inequality [26] which holds for all $u \in H^1(\mathbb{R}^N)\setminus\{0\}$ and $b > 0$, we have

$$2\int_{\mathbb{R}^N} u^2 \ln\frac{|u|}{|u|_2} \mathrm{d}x + N(1+\ln b)|u|_2^2 \le \frac{b^2}{\pi}\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x. \tag{2.9}$$

For $u \in H_0^1(\Omega)$, we can define $u(x) = 0$ for $x \in \mathbb{R}^N \setminus \Omega$. Then it holds, for any positive number *b*,

$$2\int_\Omega u^2 \ln\frac{|u|}{|u|_2} \mathrm{d}x + N(1+\ln b)|u|_2^2 \le \frac{b^2}{\pi}\int_\Omega |\nabla u|^2 \,\mathrm{d}x. \tag{2.10}$$

Lemma 2.2. (Sobolev imbedding theorem) [[27], Theorem 1.8] *The following imbeddings are continuous*:

$$H^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N), \quad 2 \le p < \infty, N = 1,2$$
$$H^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N), \quad 2 \le p \le 2^*, N \ge 3$$
$$\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N), \quad N \ge 3$$

In particular, the Sobolev inequality holds:

$$S := \inf_{\substack{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \\ |u|_{2^*}=1}} |\nabla u|_2^2 > 0.$$

Lemma 2.3. [[27], Lemma 2.13] *Set* $N \ge 3$, $a \in L^{\frac{N}{2}}(\Omega)$, *then functional* $\chi : H_0^1(\Omega) \to \mathbb{R}$,

$$\chi(u) = \int_\Omega a u^2 \mathrm{d}x, \quad u \in H_0^1(\Omega), \tag{2.11}$$

is weakly continuous.

Lemma 2.4 (Hölder inequality) [28] *Assume* $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then if* $u \in L^p(\Omega), v \in L^q(\Omega)$, *we have*

$$\int_\Omega |uv| \mathrm{d}x \le |u|_p |v|_q. \tag{2.12}$$

3. Some Basic Lemmas

In this section, we prove that the functional I satisfy the Palais-Smale condition in the cases $4 < p < 6$ and $b \in (0, \sqrt{2\pi})$. First, we recall that a C^1 functional I satisfies the Palais-Smale condition at level c ($(PS)_c$ in short) if every sequence $\{u_n\}_n \subset H^1(\Omega)$ satisfying $I(u_n) \to c$ and $I(u_n) \to 0 \, (n \to \infty)$ has a convergent subsequence.

We first begin giving the following general mountain pass theorem (see [29]).

Theorem 3.1. *Let* X *is a real Banach space and* $I \in C^1(X, \mathbb{R})$, *with* $I(0) = 0$. *Assume that*

1) *there exist* $r, \alpha > 0$ *such that* $I(u) \ge \alpha$ *for all* $u \in X$, *with* $\|u\| = r$;

2) *there exist* $\|e\| > r$ *satisfying* $\|u\|_X > r$ *such that* $I(e) < 0$.

Define $\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\}$,

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \ge \alpha \tag{3.1}$$

and there exists a $(PS)_c$ sequence $\{u_n\} \in X$.

Next, we begin proving that I satisfies the assumptions of the mountain pass theorem.

Lemma 3.1. *Suppose that* $4 < p < 6$ *and* $b \in (0, \sqrt{2\pi})$ *are satisfied. Then the functional I satisfies the mountain pass geometry, that is*,

1) *there exist* $r, \alpha > 0$ *such that* $I(u) \ge \alpha$ *for any* $u \in E$ *such that* $\|u\| = r$;

2) *there exists* $e \in E$ *with* $\|u\| > r$ *such that* $I(e) < 0$.

Proof. By (V), (2.4) and (2.10), one has

$$\begin{aligned}
I(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega V(x)u^2dx - \frac{1}{2}\int_\Omega \omega\phi_u u^2dx \\
&\quad -\frac{1}{2}\int_\Omega u^2\ln|u|\,dx + \frac{1}{4}\int_\Omega u^2dx - \frac{1}{p}\int_\Omega |u|^p\,dx \\
&= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{4}\int_\Omega (2V(x)+1)u^2dx - \frac{1}{2}\int_\Omega \omega\phi_u u^2dx \\
&\quad -\frac{1}{2}\int_\Omega u^2\ln|u|\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx \\
&= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{4}\int_\Omega (2V(x)+1)u^2dx - \frac{1}{2}\int_\Omega \omega\phi_u u^2dx \\
&\quad -\frac{1}{2}\int_\Omega u^2\ln\frac{|u|}{|u|_2}dx - \frac{1}{2}\int_\Omega u^2\ln|u|_2\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx \\
&\geq \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{4}\int_\Omega (2V_0+1)u^2dx + \frac{3}{4}(1+\ln b)|u|_2^2 \\
&\quad -\frac{b^2}{4\pi}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\int_\Omega u^2\ln|u|_2\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx \\
&= \left(\frac{1}{2}-\frac{b^2}{4\pi}\right)\|u\|^2 + \frac{1}{4}\left(2V_0+4+3\ln b-2\ln|u|_2\right)|u|_2^2 - \frac{1}{p}|u|_p^p.
\end{aligned} \tag{3.2}$$

When $u\in E\setminus\{0\}$ and $|u|_2 \leq b^{\frac{3}{2}}e^{V_0+2}$, we get $2V_0+4+3\ln b-2\ln|u|_2 \geq 0$. Then, by Sobolev imbedding theorem, one has

$$I(u) \geq \left(\frac{1}{2}-\frac{b^2}{4\pi}\right)\|u\|^2 - \frac{1}{p}|u|_p^p \geq \left(\frac{1}{2}-\frac{b^2}{4\pi}\right)\|u\|^2 - \frac{S_p^{-\frac{p}{2}}}{p}\|u\|^p.$$

When $b\in\left(0,\sqrt{2\pi}\right)$, we can choose $r,\alpha>0$ such that $I(u)\geq\alpha$ for $\|u\|=r$.

On the other hand, let $u\in E\setminus\{0\}$ and $4<p<6$, using (2.4), we get

$$\begin{aligned}
I(tu) &= \frac{t^2}{2}\|u\|^2 - \frac{1}{2}\int_\Omega \omega\phi_{tu}(tu)^2\,dx + \frac{t^2}{2}\int_\Omega V(x)u^2dx \\
&\quad -\frac{t^2}{2}\int_\Omega u^2\ln|tu|\,dx + \frac{t^2}{4}\int_\Omega u^2dx - \frac{t^p}{p}\int_\Omega |u|^p\,dx \\
&\leq \frac{t^2}{2}\|u\|^2 + \frac{t^2\omega^2}{2}|u|^2 + \frac{t^2}{2}\int_\Omega V(x)u^2dx \\
&\quad -\frac{t^2\ln t}{2}|u|_2^2 - \frac{t^2}{2}\int_\Omega u^2\ln|u|\,dx + \frac{t^2}{4}|u|_2^2 - \frac{t^p}{p}|u|_p^p \\
&\to -\infty, \quad \text{as } t\to+\infty.
\end{aligned} \tag{3.3}$$

Thus, there exists $e\in E\setminus\{0\}$ such that $I(e)<0$. This completes the proof of Lemma 3.1.

Lemma 3.2. *Assume* (V) *and* $4<p<6$ *hold. Then I satisfies the* $(PS)_c$ *condition, that is, for any* $(PS)_c$ *sequence* $\{u_n\}\subset E$, *there admits a subsequence strongly convergent in E.*

Proof. First we show that $\{u_n\}$ is bounded. We assume that $\{u_n\}\subset E$ satisifies

$$\begin{aligned} c+o_n(1) &= I(u_n) \\ &= \frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx + \int_\Omega V(x)u_n^2 dx - \frac{1}{2}\int_\Omega \omega\phi_{u_n}u_n^2 dx \\ &\quad -\frac{1}{2}\int_\Omega u_n^2\ln|u_n|\,dx + \frac{1}{4}\int_\Omega u_n^2 dx - \frac{1}{p}\int_\Omega |u_n|^p\,dx \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} o_n(1)\|u_n\| &= \langle I'(u_n),u_n\rangle \\ &= \int_\Omega \nabla u_n^2 dx + \int_\Omega V(x)u_n^2 dx - \int_\Omega (2\omega+\phi_{u_n})\phi_{u_n}u_n^2 dx \\ &\quad -\int_\Omega u_n^2\ln|u_n|\,dx - \int_\Omega |u_n|^p\,dx. \end{aligned} \tag{3.5}$$

One has

$$\begin{aligned} c+o_n(1)\|u_n\| &= I(u_n) - \frac{1}{4}\langle I'(u_n),u_n\rangle \\ &= \frac{1}{4}\int_\Omega |\nabla u_n|^2\,dx + \frac{1}{4}\int_\Omega V(x)u_n^2 dx + \frac{1}{4}\int_\Omega \phi_{u_n}^2 u_n^2 dx \\ &\quad -\frac{1}{4}\int_\Omega u_n^2\ln|u_n|\,dx + \frac{1}{4}\int_\Omega u_n^2 dx + \left(\frac{1}{4}-\frac{1}{p}\right)\int_\Omega |u_n|^p\,dx \\ &\ge \frac{1}{4}\int_\Omega |\nabla u_n|^2\,dx + \frac{1}{4}\int_\Omega V(x)u_n^2 dx - \frac{1}{4}\int_\Omega u_n^2\ln|u_n|\,dx \\ &\quad +\frac{1}{4}\int_\Omega u_n^2 dx + \left(\frac{1}{4}-\frac{1}{p}\right)\int_\Omega |u_n|^p\,dx \\ &\ge \frac{1}{4}\|u_n\|^2 + \frac{1}{4}\int_\Omega V_0 u_n^2 dx - \frac{1}{4}\int_\Omega |u_n|^3\,dx + \left(\frac{1}{4}-\frac{1}{p}\right)\int_\Omega |u_n|^p\,dx \\ &\ge C_1\|u_n\|^2 - C_2\|u_n\|^3 + C_3\|u_n\|^p, \end{aligned} \tag{3.6}$$

where C_1, C_2, C_3 are positive constants. Since $p\in(4,6)$, then $\{u_n\}$ is bounded in *E*. Going if necessary to a subsequence (still denoted by $\{u_n\}$), we can assume that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } E, \\ &u_n \to u \quad \text{in } L^s(\Omega),\ s\in[2,6), \\ &u_n(x)\to u(x) \quad \text{a.e. in } \Omega. \end{aligned}$$

According to (2.7), one obtains

$$\begin{aligned} \langle I'(u_n),u_n-u\rangle &= \int_\Omega \nabla u_n\nabla(u_n-u) + \int_\Omega V(x)u_n(u_n-u)dx \\ &\quad -\int_\Omega (2\omega+\phi_{u_n})\phi_{u_n}u_n(u_n-u)dx \\ &\quad -\int_\Omega \ln|u_n|u_n(u_n-u)dx - \int_\Omega |u_n|^{p-2}u_n(u_n-u)dx. \end{aligned} \tag{3.7}$$

Similarly, one gets

$$\begin{aligned} \langle I'(u),u_n-u\rangle &= \int_\Omega \nabla u\nabla(u_n-u) + \int_\Omega V(x)u(u_n-u)dx \\ &\quad -\int_\Omega (2\omega+\phi_u)\phi_u u(u_n-u)dx \\ &\quad -\int_\Omega \ln|u|u(u_n-u)dx - \int_\Omega |u|^{p-2}u(u_n-u)dx. \end{aligned} \tag{3.8}$$

We easily get that

$$\begin{aligned}\left\langle I'(u_n)-I'(u),u_n-u\right\rangle &= \int_\Omega\left|\nabla(u_n-u)\right|^2+\int_\Omega V(x)(u_n-u)^2\,\mathrm{d}x\\ &\quad-2\omega\int_\Omega\left(\phi_{u_n}u_n-\phi_u u\right)(u_n-u)\,\mathrm{d}x\\ &\quad-\int_\Omega\left(\phi_{u_n}^2u_n-\phi_u^2u\right)(u_n-u)\,\mathrm{d}x\\ &\quad-\int_\Omega\left(u_n\ln|u_n|-u\ln|u|\right)(u_n-u)\,\mathrm{d}x\\ &\quad-\int_\Omega\left(|u_n|^{p-2}u_n-|u|^{p-2}u\right)(u_n-u)\,\mathrm{d}x.\end{aligned} \tag{3.9}$$

It is clear that

$$\left\langle I'(u_n)-I'(u),u_n-u\right\rangle\to 0,\quad \text{as } n\to\infty \tag{3.10}$$

In fact, by (2.5), we get

$$\begin{aligned}\left\|\phi_{u_n}\right\|^2&=-\int_\Omega\omega\phi_{u_n}u_n^2\mathrm{d}x-\int_\Omega\phi_{u_n}^2u_n^2\mathrm{d}x\\ &\le-\int_\Omega\omega\phi_u u^2\mathrm{d}x\le C\left\|\phi_{u_n}\right\|\left|u_n\right|_{\frac{12}{5}}^2\le C\left\|\phi_{u_n}\right\|\left\|u_n\right\|_{H^1}^2,\end{aligned} \tag{3.11}$$

So $\{\phi_{u_n}\}$ is bounded in $\mathcal{D}^{1,2}(\Omega)$. By the Hölder inequality and the Sobolev inequality, one has

$$\begin{aligned}\left|\int_\Omega\left(\phi_{u_n}-\phi_u\right)u_n(u_n-u)\,\mathrm{d}x\right|&\le\left|\phi_{u_n}-\phi_u\right|_6\left|u_n-u\right|_3\left|u_n\right|_2\\ &\le C\left\|\phi_{u_n}-\phi_u\right\|_6\left|u_n-u\right|_3\left|u_n\right|_2.\end{aligned} \tag{3.12}$$

Because $u_n\to u$ in $L^s(\Omega)$ for any $s\in[2,6)$, we have

$$\int_\Omega\left(\phi_{u_n}-\phi_u\right)u_n(u_n-u)\,\mathrm{d}x\to 0,\quad \text{as } n\to\infty, \tag{3.13}$$

and

$$\left|\int_\Omega\phi_u(u_n-u)^2\,\mathrm{d}x\right|\le\left|\phi_u\right|_6\left|u_n-u\right|_3\left|u_n-u\right|_2\to 0\quad \text{as } n\to\infty. \tag{3.14}$$

Thus, we get

$$\begin{aligned}&\int_\Omega\left(\phi_{u_n}u_n-\phi_u u\right)(u_n-u)\,\mathrm{d}x\\ &=\int_\Omega\left(\phi_{u_n}-\phi_u\right)u_n(u_n-u)\,\mathrm{d}x+\int_\Omega\phi_u(u_n-u)^2\,\mathrm{d}x\\ &\to 0,\quad \text{as } n\to\infty.\end{aligned} \tag{3.15}$$

Observe that the sequence $\{\phi_{u_n}^2u_n\}$ is bounded in $L^{\frac{3}{2}}(\Omega)$, since

$$\left|\phi_{u_n}^2u_n\right|_{\frac{3}{2}}\le\left|\phi_{u_n}\right|_6^2\left|u_n\right|_3, \tag{3.16}$$

so

$$\begin{aligned}&\left|\int_\Omega\left(\phi_{u_n}^2u_n-\phi_u^2u\right)(u_n-u)\,\mathrm{d}x\right|\\ &\le\left|\phi_{u_n}^2u_n-\phi_u^2u\right|_{\frac{3}{2}}\left|u_n-u\right|_3\\ &\le\left(\left|\phi_{u_n}^2u_n\right|_{\frac{3}{2}}+\left|\phi_u^2u\right|_{\frac{3}{2}}\right)\left|u_n-u\right|_3\\ &\to 0\quad \text{as } n\to\infty.\end{aligned} \tag{3.17}$$

Because $V(x) \in L^{\frac{3}{2}}(\Omega)$, by Lemma 2.3, we get

$$\int_{\Omega} V(x)(u_n - u)^2 \mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{3.18}$$

By $\left| t \ln |t| \right| \le C \left| 1 + t^2 \right|, t \in \mathbb{R}$, and Hölder inequality, one has

$$\begin{aligned} &\left| \int_{\Omega} (u_n - u)(u_n \ln |u_n| - u \ln |u|) \mathrm{d}x \right| \\ &\le \int_{\Omega} |u_n - u| \left| u_n \ln |u_n| \right| \mathrm{d}x + \int_{\Omega} |u_n - u| \left| u \ln |u| \right| \mathrm{d}x \\ &\le C \int_{\Omega} |u_n - u| \left| 1 + u_n^2 \right| \mathrm{d}x + C \int_{\Omega} |u_n - u| \left| 1 + u^2 \right| \mathrm{d}x \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.19}$$

Moreover, by the Hölder inequality, we have that

$$\begin{aligned} &\left| \int_{\Omega} \left(|u_n|^{p-2} u_n - |u|^{p-2} u \right)(u_n - u) \mathrm{d}x \right| \\ &\le \left| |u_n|^{p-2} u_n - |u|^{p-2} u \right|_{\frac{p}{p-1}} |u_n - u|_p \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.20}$$

Therefore, according to (3.10)-(3.20), we obtain that

$$\|u_n - u\|^2 \to 0 \quad \text{as } n \to \infty. \tag{3.21}$$

Thus $\{u_n\}$ has a strongly convergent subsequence in *E*.

4. Proof of Theorem 1.1

Next, we only need to prove that $u \ne 0$. Suppose by contradiction that $u = 0$, and hence $\phi_u = 0$. Since as $n \to \infty$, $\langle I'(u_n), u_n \rangle \to 0$, $u_n \to 0$ in $L^p(\Omega)\left(p \in (2, 2^*)\right)$. Thus we get

$$\int_{\Omega} V(x) u_n^2 \mathrm{d}x \to 0,$$

$$\int_{\Omega} \phi_{u_n}^2 u_n^2 \mathrm{d}x + 2\omega \int_{\Omega} \phi_{u_n} u_n^2 \mathrm{d}x \to 0,$$

$$\int_{\Omega} u_n^2 \ln |u_n| \mathrm{d}x \to 0,$$

$$\int_{\Omega} |u_n|^p \mathrm{d}x \to 0.$$

We may assume

$$\int_{\Omega} |\nabla u_n|^2 \mathrm{d}x = \|u_n\|^2 \to l, \quad l \ge 0.$$

Obviously, $l = 0 \Leftrightarrow u_n \to 0$ in *E*. As a consequence we obtain that

$$I(u_n) \to \frac{l}{2}, \quad \text{as } n \to \infty.$$

According to $I(u_n) \to c > 0$, we get

$$c = \frac{l}{2} > 0,$$

which implies that $l = 0$ is impossible, this is, which contradicts with $u = 0$. Therefore, *u* is a nontrivial solution of system (1.1). We have completed the proof of Theorem 1.1.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Benci, V. and Fortunato, D. (2001) The Nonlinear Klein-Gordon Equation Coupled with the Maxwell Equations. *Nonlinear Analysis: Theory, Methods & Applications*, **47**, 6065-6072. https://doi.org/10.1016/S0362-546X(01)00688-5

[2] Benci, V. and Fortunato, D. (2002) Solitary Waves of the Nonlinear Klein-Gordon Equation Coupled with the Maxwell Equations. *Reviews in Mathematical Physics*, **14**, 409-420. https://doi.org/10.1142/S0129055X02001168

[3] D'Aprile, T. and Mugnai, D. (2004) Solitary Waves for Nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell Equations. *Proceedings of the Royal Society of Edinburgh Section A*, **134**, 893-906. https://doi.org/10.1017/S030821050000353X

[4] D'Aprile, T. and Mugnai, D. (2004) Non-Existence Results for the Coupled Klein-Gordon-Maxwell Equations. *Advanced Nonlinear Studies*, **4**, 307-322. https://doi.org/10.1515/ans-2004-0305

[5] Azzollini, A. and Pomponio, A. (2010) Ground State Solutions for the Nonlinear Klein-Gordon-Maxwell Equations. *Topological Methods in Nonlinear Analysis*, **35**, 33-42.

[6] Wang, F.Z. (2011) Ground-State Solutions for the Electrostatic Nonlinear Klein-Gordon-Maxwell System. *Nonlinear Analysis: Theory, Methods & Applications*, **74**, 4796-4803. https://doi.org/10.1016/j.na.2011.04.050

[7] Cassani, D. (2004) Existence and Non-Existence of Solitary Waves for the Critical Klein-Gordon Equation Coupled with Maxwell's Equations. *Nonlinear Analysis*, **58**, 733-747. https://doi.org/10.1016/j.na.2003.05.001

[8] Wang, F.Z. (2011) Solitary Waves for the Klein-Gordon-Maxwell System with Critical Exponent. *Nonlinear Analysis: Theory, Methods & Applications*, **74**, 827-835. https://doi.org/10.1016/j.na.2010.09.033

[9] Cunha, P.L. (2012) Subcritical and Supercritical Klein-Gordon-Maxwell Equations without Ambrosetti-Rabinowitz Condition.

[10] Liu, X.-Q. and Tang, C.-L. (2022) Infinitely Many Solutions and Concentration of Ground State Solutions for the Klein-Gordon-Maxwell System. *Journal of Mathematical Analysis and Applications*, **505**, Article ID: 125521. https://doi.org/10.1016/j.jmaa.2021.125521

[11] Gan, C.L., Xiao, T. and Zhang, Q.F. (2020) Existence Result for Fractional Klein-Gordon-Maxwell System with Quasicritical Potential Vanishing at Infinity. *Journal of Applied Mathematics and Physics*, **8**, 1318-1327. https://doi.org/10.4236/jamp.2020.87101

[12] Carriao, P.C., Cunha, P.L. and Miyagaki, O.H. (2012) Positive Ground State Solutions for the Critical Klein-Gordon-Maxwell System with Potentials. *Nonlinear Analysis: Theory, Methods & Applications*, **75**, 4068-4078. https://doi.org/10.1016/j.na.2012.02.023

[13] Che, G.F. and Chen, H.B. (2017) Existence and Multiplicity of Nontrivial Solutions for Klein-Gordon-Maxwell System with a Parameter. *Journal of the Korean Mathematical Society*, **54**, 1015-1030. https://doi.org/10.4134/JKMS.j160344

[14] Chen, S.T. and Tang, X.H. (2018) Infinitely Many Solutions and Least Energy Solu-

tions for Klein-Gordon-Maxwell Systems with General Superlinear Nonlinearity. *Computers & Mathematics with Applications*, **75**, 3358-3366. https://doi.org/10.1016/j.camwa.2018.02.004

[15] Ding, L. and Li, L. (2014) Infinitely Many Standing Wave Solutions for the Nonlinear Klein-Gordon-Maxwell System with Sign-Changing Potential. *Computers & Mathematics with Applications*, **68**, 589-595. https://doi.org/10.1016/j.camwa.2014.07.001

[16] Tang, X.H., Wen, L.X. and Chen, S.T. (2020) On Critical Klein-Gordon-Maxwell Systems with Super-Linear Nonlinearities. *Nonlinear Analysis*, **196**, Article ID: 111771. https://doi.org/10.1016/j.na.2020.111771

[17] Wang, L.X., Wang, X.M. and Zhang, L.Y. (2019) Ground State Solutions for the Critical Klein-Gordon-Maxwell System. *Acta Mathematica Scientia*, **39**, 1451-1460. https://doi.org/10.1007/s10473-019-0521-y

[18] d'Avenia, P., Pisani, L. and Siciliano, G. (2008) Klein-Gordon-Maxwell System in a Bounded Domain. *Discrete and Continuous Dynamical Systems*, **26**, 135-149.

[19] Wu, Y.H., Ge, B. and Miyagaki, O.H. (2019) Existence Results for the Klein-Gordon-Maxwell System in Rotationally Symmetric Bounded Domains. *Zeitschrift für Analysis und ihre Anwendungen*, **38**, 209-229. https://doi.org/10.4171/ZAA/1635

[20] Guo, L.F., Sun, Y. and Shi, G.N. (2022) Ground States for Fractional Nonlocal Equations with Logarithmic Nonlinearity. *Opuscula Mathematica*, **42**, 157-178. https://doi.org/10.7494/OpMath.2022.42.2.157

[21] Ji, C. and Szulkin, A. (2016) A Logarithmic Schrödinger Equation with Asymptotic Conditions on the Potential. *Journal of Mathematical Analysis and Applications*, **437**, 241-254. https://doi.org/10.1016/j.jmaa.2015.11.071

[22] Peng, L.Y., Suo, H.M., Wu, D.K., Feng, H.X. and Lei, C.Y. (2021) Multiple Positive Solutions for a Logarithmic Schrödinger-Poisson System with Singular Nonelinearity. *Electronic Journal of Qualitative Theory of Differential Equations*, **2021**, 1-15. https://doi.org/10.14232/ejqtde.2021.1.90

[23] Squassina, M. and Szulkin, A. (2015) Multiple Solutions to Logarithmic Schrödinger Equations with Periodic Potential. *Calculus of Variations and Partial Differential Equations*, **54**, 585-597. https://doi.org/10.1007/s00526-014-0796-8

[24] Zhang, Y.Y., Yang, Y. and Liang, S.H. (2022) Least Energy Sign-Changing Solution for n-Laplacian Problem with Logarithmic and Exponential Nonlinearities. *Journal of Mathematical Analysis and Applications*, **505**, Article ID: 125432. https://doi.org/10.1016/j.jmaa.2021.125432

[25] d'Avenia, P., Pisani, L. and Siciliano, G. (2009) Dirichlet and Neumann Problems for Klein-Gordon-Maxwell Systems. *Nonlinear Analysis: Theory, Methods & Applications*, **71**, e1985-e1995. https://doi.org/10.1016/j.na.2009.02.111

[26] Gross, L. (1975) Logarithmic Sobolev Inequalities. *American Journal of Mathematics*, **97**, 1061-1083. https://doi.org/10.2307/2373688

[27] Willem, M. (1996) Minimax Theorems. Progress in Nonlinear Differential Equations and Their Applications, 24. Birkhäuser Inc., Boston.

[28] Evans, L.C. (2010) Partial Differential Equations, Volume 19 of Graduate Studies in Mathematics. 2nd Edition, American Mathematical Society, Providence.

[29] Ambrosetti, A. and Rabinowitz, P.H. (1973) Dual Variational Methods in Critical Point Theory and Applications. *Journal of Functional Analysis*, **14**, 349-381. https://doi.org/10.1016/0022-1236(73)90051-7