Open Access Library Journal
2022, Volume 9, e4998
ISSN Online: 2333-9721
ISSN Print: 2333-9705

On Theta Transitivity in a Topological Space with Countable Base

Dana Mawlood Mohammed

Institute of Training and Educational Development in Sulaimani, Iraq
Email: Danamath82@gmail.com

How to cite this paper: Mohammed, D.M. (2022) On Theta Transitivity in a Topological Space with Countable Base. *Open Access Library Journal*, **9**: e4998.
https://doi.org/10.4236/oalib.1104998

Received: June 29, 2022
Accepted: August 5, 2022
Published: August 8, 2022

Abstract

In this paper, we have introduced some concepts about topological dynamical systems and proved some new corollary and theorems of transitivity of a theta irresolute function defined on topological space.

Subject Areas

Mathematical Analysis

Keywords

θ-Irresolute Function, Dynamics in Topological Spaces, Transitive Functions, θ-Adherent Points

1. Introduction

In this paper, we have investigated and introduced some new definitions of transitivity in topological space. To study the dynamics of a self-map $f: X \to X$ means to study the qualitative behavior of the sequences $\{f^n(x)\}$ as *n* goes to infinity when *x* varies in *X*, where f^n denotes the composition of *f* with itself *n* times:

By a topological system I mean a pair (X, f), where *X* is a locally compact Hausdorff topological space (the phase space), and $f: X \to X$ is a continuous function. The dynamics of the system is given by $x_{n+1} = f(x_n), x_0 \in X, n \in \mathbf{N}$ and the solution passing through x is the sequence $\{f(x_n)\}$ where $n \in \mathbf{N}$.

Let $x \in X$, then the set $\{x, f(x), f^2(x), \cdots\}$ is called an orbit of *x* under *f* and is denoted by $O_f(x)$, so $O_f(x)$ is the set of points which occur on the orbit of *x* at some positive time, and the sequence $x, f(x), f^2(x), \cdots$ is called the trajectory of *x*. Any point with dense orbit is called a transitive point. A point which is not transitive is called intransitive.

Topological dynamics is concerned with the behavior of iterations of a continuous map f from a space *X* into itself. Suppose for some $x \in X$, sequence $x, f(x), f^2(x), \cdots$ converges to some point say $x_0 \in X$, then we must have $f(x_0) = x_0$, because *f* is continuous. Such points we call as *fixed points*. We say that the point *x* is attracted by the fixed point x_0. The set of all points in *X attracted* by x_0 is called the *stable set* or the *basin of attraction of* the fixed point x_0 and is denoted by $W_f(x_0)$. A fixed point x_0 is said to be *attracting* if its stable set is a neighborhood of it.

A point $x \in X$ is said to be *periodic* if there exists a positive integer $n \in \mathbf{N}$ such that $f^n(x) = x$. The set of all periodic points of the map *f* is denoted by *per*(*f*).

A point $x \in X$ is called a θ-adherent point of *A* [1], if $A \cap Cl(U) \neq \phi$ for every open set *U* containing *x*. The set of all θ-adherent points of a subset *A* of *X* is called the θ-closure of *A* and is denoted by $Cl_\theta(A)$. A subset *A* of *X* is called θ-closed if $A = Cl_\theta(A)$. Dontchev and Maki [2] have shown that if *A* and *B* are subsets of a space *X*, then $Cl_\theta(A \cup B) = Cl_\theta(A) \cup Cl_\theta(B)$ and that $Cl_\theta(A \cap B) = Cl_\theta(A) \cap Cl_\theta(B)$. Recall that a space (*X*, τ) is Hausdorff if and only if every compact set is θ-closed. The complement of a θ-closed set is called a θ-open set. The family of all θ-open sets forms a topology on *X* and is denoted by τ^θ. This topology is coarser than τ and that a space (*X*, τ) is regular if and only if $\tau = \tau^\theta$ [3].

2. Basic Definition and Theorems

Definition 2.1 [4] By a topological system I mean a pair (X, f), where *X* is a locally compact Hausdorff topological space (the phase space), and $f: X \to X$ is a continuous function. The dynamics of the system is given by $x_{n+1} = f(x_n), x_0 \in X, n \in \mathbf{N}$ and the solution passing through x_0 is the sequence $\{f(x_n)\}$ where $n \in \mathbf{N}$.

Definition 2.2. 1) Let $x \in X$, then the set $\{x, f(x), f^2(x), \cdots\}$ is called an orbit of *x* under *f* and is denoted by $O_f(x)$, so $O_f(x)$ is the set of points which occur on the orbit of *x* at some positive time, and the sequence $x, f(x), f^2(x), \cdots$ is called the trajectory of *x*.

2) Let *X* be a topological space, $f: X \to X$, $\{f^n(x_0)\}_{n=0}^{\infty}$ be a sequence in *X*, and let $x \in X$. Then $\{f^n(x_0)\}$ converges to *x* if for all open sets *U* containing *x*, there exists an integer *N* such that $f^n(x_0) \in U$ for all *n* >*N*, Note that if this sequence is convergence then it converges to a fixed point, say *y*, *i.e.* $f(y) = y$.

Any point with dense orbit is called a transitive point. A point which is not transitive is called intransitive.

Definition 2.3. 1) (Transitivity) Let *X* be a topological space with no isolated point. Then the function $f: X \to X$ is said to be transitive if for any two open sets *U* and *V* in *X*, there is a point $x \in U$ and an $n > 0$ such that $f^n(x) \in V$. It is easily to show that if *f* is transitive then for every pair *U*, *V* of non-empty open sets, there exist a positive integer *n* such that $f^n(U)\cap V\neq\phi$.

2) Let *X* be a topological space, the function $f:X\to X$, is said to be *topologically mixing* if for every pair *U*, *V* of non-empty open sets, if there exist *N* such that $f^n(U)\cap V\neq\phi$ for all $n>N$.

Definition 2.4. (topological weak mixing) Let *X* has no isolated point. *g* is topologically *weakly mixing*, if the product of two functions $g\times g$ is topologically transitive.

Proposition 2.5. Every topological mixing function implies topological weak mixing. But the converse is no necessarily true.

Proof: It is easily to prove the foregoing theorem.

Definition 2.6. A map *f* is said to be transitive (resp., *θ-transitive* [5]) if for any non-empty open (resp., *θ*-open) sets *U* and *V* in *X*, there exists $n\in\mathbf{N}$ such that $f^n(U)\cap V\neq\phi$.

***Theorem* 2.7 [5].** Let *X* be a non-empty locally *θ*-compact Hausdorff space. Then the intersection of a countable collection of *θ*-open *θ*-dense subsets of *X* is *θ*-dense in *X*.

***Corollary* 2.8.** A subset A of a space (X,τ) is *θ*-dense if and only if $A\cap U\neq\phi$ for all $U\in\tau^\alpha$ other than $U=\phi$.

Two topological spaces (X,τ) and (Y,τ_1) are called homeomorphic if there exists a one-to-one onto function $f:(X,\tau)\to(Y,\tau_1)$ such that *f* and f^{-1} are both continuous.

Note that any homeomorphic spaces have the same dynamics, if we have any notion about first space then we have the same notion about the other one.

A map $h:X\to Y$ is a homeomorphism if it is continuous, bijective and has a continuous inverse.

A function $f:X\to X$ is called *θ*-irresolute [6] if the inverse image of each *θ*-open set is a *θ*-open set in *X*.

A map $h:X\to Y$ is *θr*-homeomorphism if it is bijective and thus invertible and both *h* and h^{-1} are *θ*-irresolute.

***Theorem* 2.9.** Let (X,f) be a topological system where *X* is a non-empty *θ*-compact topological space and $f:X\to X$ is *θ*-irresolute map and that *X* is separable. Suppose that *f* is topologically *θ*-transitive. Then there is an element $x\in X$ such that the orbit $O_f(x)=\{x,f(x),f^2(x),\cdots,f^n(x),\cdots\}$ is *θ*-dense in *X*.

Proof: Let $B=\{U_i\},i=1,2,3,\cdots$ be a countable basis for the *θ*-topology of *X*. For each *i*, let $O_i=\{x\in X: f^n(x)\in U_i \text{ for some } n\geq 0\}$

Then, clearly O_i is *θ*-open and *θ*-dense. It is *θ*-open since *f* is *θ*-irresolute, so, $O_i=\bigcup_{i=1}^{\infty}f^{-1}(U_i)$ is *θ*-open and *θ*-dense since *f* is topological *θ*-transitive map. Further, for every *θ*-open set *V*, there is a positive integer *n* such that $f^n(V)\cap U_i\neq\phi$, since *f* is *θ* transitive.

Now, apply theorem 2.7 to the countable *θ*-dense set $\{O_i\}$ to say that $\bigcap_{i=0}^{\infty}O_i$

is θ-dense and so non-empty. Let $y \in \bigcap_{i=0}^{\infty} O_i$. This means that, for each *i*, there is a positive integer *n* such that $f^n(y) \in U_i$ for every *i*. By Corollary 2.8 this implies that $O_f(x)$ is θ-dense in *X*.

Definition 2.10. The function $f : X \to X$, is strongly transitive [7] if for any nonempty open set $U \subset X$, $X = \bigcup_{k=0}^{s} f^k(U)$ for some *s* > 0. It is easily seen that $X = \bigcup_{k=0}^{\infty} f^k(U)$ for any nonempty open set $U \subset X$ if and only if $\bigcup_{k=0}^{\infty} f^{-k}(x)$ is dense in *X* for any $x \in X$.

We may consider that, the last statement of the foregoing definition as lemma, because we can use this statement to prove the following corollary.

Lemma 2.11. $X = \bigcup_{k=0}^{\infty} f^k(U)$ for any nonempty open set $U \subset X$ if and only if $\bigcup_{k=0}^{\infty} f^{-k}(x)$ is dense in *X* for any $x \in X$.

According to the definition 2.10 and lemma 2.11, we have the following important corollary.

Corollary 2.12. If $\bigcup_{k=0}^{\infty} f^{-k}(x)$ is dense in *X* for any $x \in X$, then the function $f : X \to X$, is strongly transitive.

3. Conclusion:

There are the following results:

Proposition 3.1. Every topological mixing function implies topological weak mixing. But the converse is no necessarily true.

***Theorem* 3.2.** Let (X, f) be a topological system where *X* is a non-empty θ-compact topological space and $f : X \to X$ is θ-irresolute map and that *X* is separable. Suppose that *f* is topologically θ-transitive. Then there is an element $x \in X$ such that the orbit $O_f(x) = \{x, f(x), f^2(x), \cdots, f^n(x), \cdots\}$ is θ-dense in *X*.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Velicko, N.V. (1968) H-Closed Topological Spaces. *American Mathematical Society Translations*, **78**, 102-118. https://doi.org/10.1090/trans2/078/05

[2] Dontchev, J. and Maki, H. (1998) Groups of θ-Generalized Homeomorphisms and the Digital Line. *Topology and Its Applications*, **20**, 1-16.

[3] Jankovic, D.S. (1986) θ-Regular Spaces. *International Journal of Mathematics and Mathematical Sciences*, **8**, 615-619. https://doi.org/10.1155/S0161171285000667

[4] Kaki, M.N.M. (2015) Chaos: Exact, Mixing and Weakly Mixing Maps. *Pure and Applied Mathematics Journal*, **4**, 39-42.
https://doi.org/10.11648/j.pamj.20150402.11

[5] Murad, M.N. (2012) Introduction to θ-Type Transitive Maps on Topological Spaces. *International Journal of Basic & Applied Sciences IJBAS-IJENS*, **12**, 104-108.

[6] Khedr, F.H. and Noiri, T. (1986) On θ-Irresolute Functions. *Indian Journal of Mathematics*, **3**, 211-217.

[7] Kameyama, A. (2002) Topological Transitivity and Strong Transitivity. *Acta Mathematica Universitatis Comeniana*, **LXXI**, 139-145.