Open Access Library Journal
2022, Volume 9, e8714
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Heterogeneity in the Elastic Half-Space (Deformations at Preparation of the Tectonic Earthquake)

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How to cite this paper: Dobrovolsky, I.P. (2022) Heterogeneity in the Elastic Half-Space (Deformations at Preparation of the Tectonic Earthquake). *Open Access Library Journal*, **9**: e8714.
https://doi.org/10.4236/oalib.1108714

Received: April 13, 2022
Accepted: July 9, 2022
Published: July 12, 2022

Abstract

With the help of Mindlin's tensor, the problem of heterogeneity in an elastic half-space comes down to the solution system of integrable equations. Different versions of the solution of such systems are considered. The main attention is paid to the solution of integrable equations with small parameters.

Subject Areas

Integral Equation

Keywords

Mindlin's Tensor, Series by Parameter

1. Introduction

With the help of Mindlin's tensor, the task of heterogeneity in an elastic half-space comes down to the solution system of integrable equations. The solution of such a system of integrable equations turns out simpler than the solution of the initial task in differential equations. The task about heterogeneity in the half-space is of particular importance for sciences about Earth and we see that work [1] remains relevant. Now there is the opportunity to receive the improved solution to the task of heterogeneity and the offered work is devoted to it.

2. Problem definition

Let's compare the two states of the environment. In the *first state*, the environment has some known properties. In the *second state*, the environment has the same properties except for area V in which properties will differ from values in the first state. Then area V is called *heterogeneity*.

We introduce the Cartesian system of coordinates x_i and we consider a half-space $x_3 \ge 0$. Designations are entered: *u*, *v* and *w* are displacements, *c* is the modulus of elasticity, and σ is stress. Normal rules of indexes are used: on the repeating indexes summing is made, and the comma in the inferior index designates differentiation on the corresponding coordinate.

The problem of the theory of elasticity contains equilibrium equations and communication between elastic stresses and strains.

$$\begin{aligned} &\sigma_{ij,j} + F_i = 0 \\ &\sigma_{ij} = c_{ijkl}\varepsilon_{kl} = c_{ijkl}u_{k,l} \end{aligned} \tag{2.1}$$

where $\varepsilon_{kl} = \left(u_{k,l} + u_{l,k}\right)/2$ are strains, F_i are bulk forces.

From Equation (2.1), we receive the system of equilibrium equations in movements for randomly inhomogeneous and non-isotropic environment.

$$\left(c_{ijkl}u_{k,l}\right)_{,j} + F_i = 0 \tag{2.2}$$

For isotropic environment, elastic modules have the appearance.

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{2.3}$$

where λ, μ are modules Lame, δ_{ij} are Kronecker's symbols.

The isotropic elastic medium has two independent modules, therefore, between different elastic modules, there are ratios.

$$\begin{aligned} &\mu = \frac{E}{2(1+\nu)}, \quad E = \frac{9\mu K}{3K+\mu}, \quad \nu = \frac{3K-2\mu}{2(3K+\mu)} = \frac{\lambda}{2(\lambda+\mu)}, \\ &K = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)} = \frac{2(1+\nu)}{3(1-2\nu)}\mu, \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = \frac{2\nu}{1-2\nu}\mu \end{aligned} \tag{2.4}$$

where K is the volume module, μ is shear modulus, E is Jung's module, ν is Poisson's coefficient.

Let's pass to problem definition about heterogeneity in the half-space. In the *first state*, the homogeneous isotropic medium is considered.

$$c^0_{ijkl}v_{k,lj} + F_i = 0, \quad \mathrm{B}\mathbf{v} = 0 \tag{2.5}$$

where $c^0_{ijkl} = \text{constant}$, B is operator of boundary conditions.

The operator of boundary conditions of B can have the different appearance, but the condition of lack of stresses on the surface $x_3 = 0$ surely enters it. It is supposed that the solution of this problem is known.

In the *second state* (with heterogeneity), we have system Equation (2.2).

$$c_{ijkl}u_{k,lj} + c_{ijkl,j}u_{k,l} + F_i = 0, \quad \mathrm{B}\mathbf{u} = 0 \tag{2.6}$$

In Equation (2.6), we accept

$$\mathbf{u} = \mathbf{v} + \mathbf{w}, \quad c_{ijkl} = c^0_{ijkl} + q\delta_V c'_{ijkl} \tag{2.7}$$

where q is numerical parameter, δ_V is characteristic function of area *V*, $\left|c'_{ijkl}\right| \le c^0_{ijkl}$.

Here $\mathbf{w}$ is the movement caused by emergence of heterogeneity. Substituting Equation (2.7) in Equation (2.6), we receive the system which solution is the solution of our problem.

$$c^0_{ijkl}w_{k,lj}+q\left(\delta_V c'_{ijkl}w_{k,l}\right)_{,j}+q\left(\delta_V c'_{ijkl}v_{k,l}\right)_{,j}=0,\quad \mathrm{B}\mathbf{w}=0 \tag{2.8}$$

We will offer two approaches to the solution of the problem Equation (2.8) below.

3. Mindlin's Tensor

For the solution of the problem Equation (2.8) Mindlin's tensor is used. Mindlin's tensor for the homogeneous isotropic half-space is formed from solutions for the single forces directed on Cartesian axes. Therefore, it is the solution of the system.

$$c^0_{ijkl}m_{sk,lj}+\delta_{si}\delta\left(\mathbf{x}-\boldsymbol{\xi}\right)=0,\quad \mathrm{B}m_{sk}=0 \tag{3.1}$$

where s is number of single force, δ is delta-function, ξ is point of application of single force.

Further, Mindlin's tensor is used in the following form.

$$\begin{aligned}
m_{si}&=A\big[4(1-\nu)n\delta_{si}-b_{,si}+4(1-\nu)\bar{n}\delta_{si}+2x_3^2\bar{n}_{,si}-(3-4\nu)\bar{b}_{,si}\\
&\quad-2x_3\bar{b}_{,si3}-4(1-\nu)(1-2\nu)\bar{p}_{,si}\big]\\
m_{s3}&=A\big[-b_{,s3}-4x_3(1-2\nu)\bar{n}_{,s}+2x_3^2\bar{n}_{,s3}+(3-4\nu)\bar{b}_{,s3}-2x_3\bar{b}_{,s33}\\
&\quad+4(1-\nu)(1-2\nu)\bar{l}_{,s}\big]\\
m_{3i}&=A\big[-b_{,3i}-8x_3(1-\nu)\bar{n}_{,i}-2x_3^2\bar{n}_{,3i}+(3-4\nu)\bar{b}_{,3i}+2x_3\bar{b}_{,33i}\\
&\quad-4(1-\nu)(1-2\nu)\bar{l}_{,i}\big]\\
m_{33}&=A\big[4(1-\nu)n-b_{,33}+8(1-\nu)^2\bar{n}-4x_3\bar{n}_{,3}-2x_3^2\bar{n}_{,33}\\
&\quad-(3-4\nu)\bar{b}_{,33}+2x_3\bar{b}_{,333}\big],\quad s,i=1,2,
\end{aligned} \tag{3.2}$$

where $r=\sqrt{\left(x_1-\xi_1\right)^2+\left(x_2-\xi_2\right)^2+\left(x_3-\xi_3\right)^2}$,
$\bar{r}=\sqrt{\left(x_1-\xi_1\right)^2+\left(x_2-\xi_2\right)^2+\left(x_3+\xi_3\right)^2}$, $n=1/r$, $b=r$, $\bar{n}=1/\bar{r}$, $\bar{b}=\bar{r}$,
$\bar{l}=\ln\left(\bar{r}+x_3+\xi_3\right)$, $\bar{p}=\left(x_3+\xi_3\right)\bar{l}-\bar{r}$, $A=1/\left(16\pi\mu(1-\nu)\right)$.

There are important ratios.

$$\Delta b=2n,\quad \Delta\bar{b}=2\bar{n},\quad \bar{p}_{,3}=\bar{l},\quad \bar{l}_{,3}=\bar{n} \tag{3.3}$$

where Δ is the operator of Laplace.

It is known that the solution of the problem.

$$c^0_{ijkl}u_{k,lj}+F_i=0,\quad \mathrm{B}\mathbf{u}=0 \tag{3.4}$$

has the form

$$u_i\left(\mathbf{x}\right)=\iiint m_{si}\left(\mathbf{x},\xi\right)F_s\left(\xi\right)\mathrm{d}V\left(\xi\right) \tag{3.5}$$

where integration is made on the half-space.

If to integrate functions $n,b,\bar{n},\bar{b},\bar{l},\bar{p}$ on ξ on the volume of V, then we will receive volume potentials $N,B,\bar{N},\bar{B},\bar{L},\bar{P}$ respectively. In the distant zone it is possible to use monopoly approximation of these potentials. If the center of volume V has coordinates (α, β, γ), it is defined by the formula.

$$\iiint\limits_V f\left(x_1,x_2,x_3,\xi_1,\xi_2,\xi_3\right)\mathrm{d}V\left(\xi\right)\approx Vf\left(x_1,x_2,x_3,\alpha,\beta,\gamma\right) \tag{3.6}$$

4. System of Integral Equations

The problem Equation (2.8) is equivalent to the system of integral equations which turns out as follows.

If in Equation (2.8) to consider the sum second and third composed as mass force, then according to Equation (3.4) and Equation (3.5), we receive the system of integral equations

$$w_i = q\iiint\left(\delta_V c'_{sjkl} w_{k,l}\right)_{,j} m_{si}\mathrm{d}V + q\iiint\left(\delta_V c'_{sjkl} v_{k,l}\right)_{,j} m_{si}\mathrm{d}V \tag{4.1}$$

We will apply Gauss-Ostrogradsky's formula for the product to integrals in Equation (4.1). In symbolical and component forms, it has the appearance.

$$\begin{gathered}\iiint \mathrm{div}\left(f\mathbf{g}\right)\mathrm{d}v = \iiint f\,\mathrm{div}\,\mathbf{g}\,\mathrm{d}v + \iiint \mathbf{g}\,\mathrm{grad}\,f\,\mathrm{d}v = \oiint\limits_S f\mathbf{g}\mathrm{d}\mathbf{s}\\ \iiint\left(fg_j\right)_{,j}\mathrm{d}v = \iiint fg_{j,j}\mathrm{d}v + \iiint f_{,j}g_j\mathrm{d}v = \oiint\limits_S fg_j n_j\mathrm{d}s\end{gathered} \tag{4.2}$$

In our case in Equation (4.2), triple integrals are calculated in the half-space and double integrals are calculated on the half-space surface. Follows from the second Equation (4.2).

$$\iiint fg_{j,j}\mathrm{d}v = -\iiint f_{,j}g_j\mathrm{d}v + \iint\limits_S fg_j n_j\mathrm{d}s \tag{4.3}$$

Appling Equation (4.3) to Equation (4.1), we receive

$$\begin{aligned}w_i = &-q\iiint \delta_V c'_{sjkl} w_{k,l} m_{si,j}\mathrm{d}V + \iint\limits_S m_{si}\delta_V c'_{sjkl} w_{k,l} n_j\mathrm{d}s\\ &-q\iiint \delta_V c'_{sjkl} v_{k,l} m_{si,j}\mathrm{d}V + \iint\limits_S m_{si}\delta_V c'_{sjkl} v_{k,l} n_j\mathrm{d}s\end{aligned} \tag{4.4}$$

We suppose that area V is in the half-space. Since δ_V is characteristic function of area V, double integrals in Equation (4.4) equal to zero and triple integrals are calculated on area V. As a result, we have the system

$$w_i + q\iiint\limits_V c'_{sjkl} w_{k,l} m_{si,j}\mathrm{d}V = -q\iiint\limits_V c'_{sjkl} v_{k,l} m_{si,j}\mathrm{d}V \tag{4.5}$$

Expression Equation (4.5) makes double sense: at $\mathbf{x}\in V$ it represents the system of integral equations for definition of w_i in area V, at $\mathbf{x}\notin V$, Equation (4.5) defines w_i out of area V. There are many methods of solution of integral equations [2].

5. Expansion in a Series

We will look for the solution of the system Equation (2.8) in shape

$$w_i = \sum_{n=0}^{\infty} q^n w_i^{(n)} \tag{5.1}$$

Substituting Equation (5.1) in Equation (2.8), we come to the system

$$\sum_{n=0}^{\infty} q^n \left(c_{ijkl}^0 w_{k,l}^{(n)}\right)_{,j} + \sum_{n=0}^{\infty} q^{n+1} \left(\delta_V c'_{ijkl} w_{k,l}^{(n)}\right)_{,j} + q\left(\delta_V c'_{ijkl} v_{k,l}\right)_{,j} = 0, \quad \mathbf{B}\mathrm{w}^{(n)} = 0 \tag{5.2}$$

From Equation (5.2) at identical degrees of q, we receive recurrent expressions

$$\begin{aligned} & w_i^{(0)} = 0, \\ & c_{ijkl}^0 w_{k,lj}^{(1)} + \left(\delta_V c'_{ijkl} v_{k,l}\right)_{,j} = 0, \\ & c_{ijkl}^0 w_{k,lj}^{(n)} + \left(\delta_V c'_{ijkl} w_{k,l}^{(n-1)}\right)_{,j} = 0, \quad n \geq 2 \end{aligned} \tag{5.3}$$

Equation (5.3) is similar to Equation (2.8). Applying to them the procedure of Section 4, we receive the solution

$$\begin{aligned} & w_i^{(1)} = \iiint m_{si}(x,\xi)\left(\delta_V c'_{ijkl} v_{k,l}\right)_{,j} \mathrm{d}V(\xi) = -\iiint_V m_{si,j} c'_{sjkl} v_{k,l} \mathrm{d}V(\xi) \\ & w_i^{(n)} = \iiint m_{si}(x,\xi)\left(\delta_V c'_{sjkl} w_{k,l}^{(n-1)}\right)_{,j} \mathrm{d}V(\xi) = -\iiint_V m_{si,j} c'_{sjkl} w_{k,l}^{(n-1)} \mathrm{d}V(\xi), \quad n \geq 2 \end{aligned} \tag{5.4}$$

6. Earthquake Preparation Calculation: A Special Case

Preparation of the earthquake is considered as emergence and development in crust of heterogeneity of properties. Let's accept that the aseismatic state (the first state) is described by the system Equation (2.5) without mass forces.

$$c_{ijkl}^0 v_{k,lj} = 0 \tag{6.1}$$

We accept the following boundary conditions.

$$\begin{aligned} & \sigma_{12}(\infty,\infty,x_3) = \sigma_{21} = \tau, \quad \sigma_{11}(\infty,\infty,x_3) = \sigma_{22}(\infty,\infty,x_3) = 0, \\ & \sigma_{3i}(x_1,x_2,0) = 0 \end{aligned} \tag{6.2}$$

Then the solution of the problem Equation (6.1), Equation (6.2) has the appearance

$$v_1 = \frac{\tau}{2\mu} x_2, \quad v_2 = \frac{\tau}{2\mu} x_1, \quad v_3 = 0 \tag{6.3}$$

In the second state, heterogeneity is characterized by size $qc'_{ijkl} v_{k,l}$. We will consider that heterogeneity is homogeneous and isotropic. Then it is characterized by two sizes: $K' \rightarrow$ is increment of the volume module and μ' is increment of shear modulus. In calculations we will accept $K' = 0$, $\mu' = \alpha\mu$ and $q = 1$.

$$qc'_{ijkl} v_{k,l} = K' \varepsilon \delta_{ij} + 2\mu' e_{ij} = 2\alpha\mu\varepsilon_{ij} = \alpha\tau k_{ij} \tag{6.4}$$

Here, the volume deformation $\varepsilon = \varepsilon_{ii}$, deviator of deformations $e_{ij} = \varepsilon_{ij} - \varepsilon\delta_{ij}/3$, ε_{kl} are the deformations calculated on Equation (6.3), $k_{12} = k_{21} = 1$ and rest $k_{ij} = 0$.

In Equation (6.4), we entered the new numerical parameter which makes clear physical sense: α is relative change of shear modulus in heterogeneity. Researches show that $\alpha \approx 0.1$ and we have the task with small parameter. In this case we take the solution in the form of the row and we keep only the first term in Equation (5.1) and Equation (5.4).

$$w_i = -\alpha\tau \iiint\limits_V \left(m_{1i,2} + m_{2i,1}\right) \mathrm{d}V(\xi) \tag{6.5}$$

We will open Equation (6.5) using Mindlin's tensor Equation (3.2). In the subsequent formulas, we will simplify designations: $w_1 = u$, $w_2 = v$, $w_3 = w$, $x_1 = x$, $x_2 = y$, $x_3 = z$ and inferior indexes will designate derivatives on the corresponding coordinates. As a result, we have.

Displacements

$$\begin{aligned} u = W\big[&4(1-\nu)N_y - 2B_{xxy} + 4(1-\nu)\bar{N}_y + 4z^2\bar{N}_{xxy} - 2(3-4\nu)\bar{B}_{xxy} \\ &- 4z\bar{B}_{xxyz} - 8(1-\nu)(1-2\nu)\bar{P}_{xxy}\big] \\ v = W\big[&4(1-\nu)N_x - 2B_{xyy} + 4(1-\nu)\bar{N}_x + 4z^2\bar{N}_{xyy} - 2(3-4\nu)\bar{B}_{xyy} \\ &- 4z\bar{B}_{xyyz} - 8(1-\nu)(1-2\nu)\bar{P}_{xyy}\big] \\ w = W\big[&-2B_{xyz} - 8z(1-2\nu)\bar{N}_{xy} + 4z^2\bar{N}_{xyz} + 2(3-4\nu)\bar{B}_{xyz} \\ &- 4z\bar{B}_{xyzz} + 8(1-\nu)(1-2\nu)\bar{L}_{xy}\big] \end{aligned} \tag{6.6}$$

Longitudinal strains

$$\begin{aligned} \varepsilon_{xx} = W\big[&4(1-\nu)N_{xy} - 2B_{xxxy} + 4(1-\nu)\bar{N}_{xy} + 4z^2\bar{N}_{xxxy} \\ &- 2(3-4\nu)\bar{B}_{xxxy} - 4z\bar{B}_{xxxyz} - 8(1-\nu)(1-2\nu)\bar{P}_{xxxy}\big] \\ \varepsilon_{yy} = W\big[&4(1-\nu)N_{xy} - 2B_{xyyy} + 4(1-\nu)\bar{N}_{xy} + 4z^2\bar{N}_{xyyy} \\ &- 2(3-4\nu)\bar{B}_{xyyy} - 4z\bar{B}_{xyyyz} - 8(1-\nu)(1-2\nu)\bar{P}_{xyyy}\big] \\ \varepsilon_{zz} = W\big[&-2B_{xyzz} - 8\nu(1-2\nu)\bar{N}_{xy} + 16z\nu\bar{N}_{xyz} + 4z^2\bar{N}_{xyzz} \\ &+ 2(1-4\nu)\bar{B}_{xyzz} - 4z\bar{B}_{xyzzz}\big] \end{aligned} \tag{6.7}$$

Volume strain

$$\varepsilon = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 4(1-2\nu)W\left[N_{xy} + (1-4\nu)\bar{N}_{xy} - 2z\bar{N}_{xyz} + 2\bar{B}_{xyzz}\right] \tag{6.8}$$

Shear strains

$$\begin{aligned} \varepsilon_{xy} = W\big[&-2(1-\nu)N_{zz} - 2B_{xxyy} - 2(1-\nu)\bar{N}_{zz} + 4z^2\bar{N}_{xxyy} \\ &- 2(3-4\nu)\bar{B}_{xxyy} - 4z\bar{B}_{xxyyz} - 8(1-\nu)(1-2\nu)\bar{P}_{xxyy}\big] \\ \varepsilon_{xz} = W\big[&2(1-\nu)N_{yz} - 2B_{xxyz} + 2(1-\nu)\bar{N}_{yz} - 2z(1-4\nu)\bar{N}_{xxy} \\ &+ 4z^2\bar{N}_{xxyz} - 2\bar{B}_{xxyz} - 4z\bar{B}_{xxyzz}\big] \\ \varepsilon_{yz} = W\big[&2(1-\nu)N_{xz} - 2B_{xyyz} + 2(1-\nu)\bar{N}_{xz} - 2z(1-4\nu)\bar{N}_{xyy} \\ &+ 4z^2\bar{N}_{xyyz} - 2\bar{B}_{xyyz} - 4z\bar{B}_{xyyzz}\big] \end{aligned} \tag{6.9}$$

In Equations (6.6)-(6.9): $W = \alpha\tau / \left(16\pi\mu(1-\nu)\right)$, potentials $N, B, \bar{N}, \bar{B}, \bar{P}, \bar{L}$ are determined in Section 3. For crust: $\nu = 1/4$ and the center of heterogeneity is in the point (0, 0, h). Having replaced potentials with monopoly approximation Equation (3.6), we will receive formulas for the distant zone.

7. Spherical Heterogeneity

Potentials of the homogeneous sphere of radius of R and volume of V in the

Cartesian coordinate system from the beginning in the center of the sphere have the form

$$B_e = \overline{B}_e = V\left(r + \frac{R^2}{5r}\right), \quad B_i = V\frac{15R^4 + 10R^2r^2 - r^4}{20R^3},$$
$$N_e = \overline{N}_e = \frac{V}{r}, \quad N_i = \frac{V}{2R}\left(3 - \frac{r^2}{R^2}\right), \tag{7.1}$$
$$\overline{L}_e = V\ln(r+z), \quad \overline{P}_e = V\left(z\ln(r+z) - r\right)$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and indexes e and i mean external and internal potentials respectively.

If the integrand depends on $z \pm \xi_3$, then at transfer of the center of the sphere to the point $(0,0,h)$ in Equation (7.1), we do replacement of $z \pm \xi_3$. As a result, we have.

Displacements

$$u = W\,y\left(F_1 + x^2F_2\right)$$
$$v = W\,x\left(F_1 + y^2F_2\right) \tag{7.2}$$
$$w = W\,xyF_3$$

where

$$F_1 = \left[\frac{4(5\nu-4)}{5R^3}\right]_i + \left[-\frac{2(1-2\nu)}{3r^3} - \frac{6R^2}{5r^5}\right]_e - \frac{12zh}{\overline{r}^5} + \frac{2(1-2\nu)}{\overline{r}^3}$$
$$-\frac{8(1-\nu)(1-2\nu)}{\overline{r}\left(\overline{r}+z+h\right)^2} + R^2\left(-\frac{6(3-4\nu)}{5\overline{r}^5} + \frac{12z(z+h)}{\overline{r}^7}\right)$$

$$F_2 = \left[-\frac{6}{r^5} + \frac{6R^2}{r^7}\right]_e + \frac{60zh}{\overline{r}^7} - \frac{6(3-4\nu)}{\overline{r}^5} + \frac{8(1-\nu)(1-2\nu)(3\overline{r}+z+h)}{\overline{r}^3\left(\overline{r}+z+h\right)^3}$$
$$+R^2\left(\frac{6(3-4\nu)}{\overline{r}^7} - \frac{84z(z+h)}{\overline{r}^9}\right)$$

$$F_3 = \left[\frac{6(z-h)\left(R^2-r^2\right)}{r^7}\right]_e + \frac{60zh(z+h)}{\overline{r}^7} - \frac{12z(3-4\nu)}{\overline{r}^5}$$
$$+\frac{6(3-4\nu)(z+h)}{\overline{r}^5} - \frac{8(1-\nu)(1-2\nu)(2\overline{r}+z+h)}{\overline{r}^3\left(\overline{r}+z+h\right)^2}$$
$$+R^2\left(\frac{12z}{\overline{r}^7} - \frac{6(3-4\nu)(z+h)}{\overline{r}^7} + \frac{84z(z+h)^2}{\overline{r}^9}\right)$$

Longitudinal strains

$$\varepsilon_{xx} = W\,xy\left(F_4 + x^2F_5\right),$$
$$\varepsilon_{yy} = W\,xy\left(F_4 + y^2F_5\right), \tag{7.3}$$
$$\varepsilon_{zz} = W\,xyF_6,$$

where

$$F_4 = \left[\frac{18R^2}{r^7} - \frac{6(1+2\nu)}{r^5}\right]_e + \frac{180zh}{\overline{r}^7} + \frac{6(10\nu-7)}{\overline{r}^5}$$
$$+\frac{24(1-\nu)(1-2\nu)(3\overline{r}+z+h)}{\overline{r}^3(\overline{r}+z+h)^3} + R^2\left(\frac{18(3-4\nu)}{\overline{r}^7} - \frac{252z(z+h)}{\overline{r}^9}\right)$$

$$F_5=\left[\frac{30}{r^7}-\frac{42R^2}{r^9}\right]_e-\frac{420zh}{\overline{r}^9}+\frac{30(3-4\nu)}{\overline{r}^7}$$
$$-\frac{24(1-\nu)(1-2\nu)\left(5\overline{r}^2+4\overline{r}(z+h)+(z+h)^2\right)}{\overline{r}^5(\overline{r}+z+h)^4}$$
$$+R^2\left(\frac{756z(z+h)}{\overline{r}^{11}}-\frac{42(3-4\nu)}{\overline{r}^9}\right)$$

$$F_6=\left[\frac{30(z-h)^2}{r^7}-\frac{6}{r^5}+R^2\left(\frac{6}{r^7}-\frac{42(z-h)^2}{r^9}\right)\right]_e-\frac{420zh(z+h)^2}{\overline{r}^9}-\frac{60z^2}{\overline{r}^7}$$
$$-\frac{60z(z+h)(3-4\nu)}{\overline{r}^7}-\frac{24\nu(1-2\nu)}{\overline{r}^5}+\frac{6(1-4\nu)}{\overline{r}^5}-\frac{30(1-4\nu)(z+h)^2}{\overline{r}^7}$$
$$+R^2\left(\frac{756z(z+h)^3}{\overline{r}^{11}}-\frac{252z(z+h)}{\overline{r}^9}-\frac{6(1-4\nu)}{\overline{r}^7}+\frac{42(1-4\nu)(z+h)^2}{\overline{r}^9}\right)$$

Volume strain

$$\varepsilon=4(1-2\nu)W\,xy\left(\left[\frac{3}{r^5}\right]_e+\frac{3(3-4\nu)}{\overline{r}^5}-\frac{30h(z+h)}{\overline{r}^7}+R^2\left(\frac{42(z+h)^2}{\overline{r}^9}-\frac{6}{\overline{r}^7}\right)\right)\quad(7.4)$$

Shear strains

$$\begin{aligned}\varepsilon_{xy}&=W\left(F_7+x^2y^2F_8\right)\\ \varepsilon_{xz}&=W\,y\left(F_9+x^2F_{10}\right)\\ \varepsilon_{yz}&=W\,x\left(F_9+y^2F_{10}\right)\end{aligned}\quad(7.5)$$

where

$$F_7=\left[\frac{11-10\nu}{5R^3}\right]_i+\left[-\frac{4}{r^3}+\frac{6\left(4R^2+5(z-h)^2\right)}{5r^5}-\frac{6R^2(z-h)^2}{r^7}\right]_e$$
$$+\frac{2(7\nu-5)}{\overline{r}^3}-\frac{6\left((3\nu-2)(z+h)^2-8zh\right)}{\overline{r}^5}-\frac{60zh(z+h)^2}{\overline{r}^7}$$
$$-\frac{8(1-\nu)(1-2\nu)\left((z+h)^2+2\overline{r}(z+h)-2\overline{r}^2\right)}{\overline{r}^3(\overline{r}+z+h)^2}$$
$$+R^2\left(\frac{24(3-4\nu)}{5\overline{r}^5}-\frac{6(z+h)\left(z(4\nu-15)-h(3-4\nu)\right)}{\overline{r}^7}+\frac{84z(z+h)^3}{\overline{r}^9}\right)$$

$$F_8=\left[\frac{30}{r^7}-\frac{42R^2}{r^9}\right]_e+\frac{30(3-4\nu)}{\overline{r}^7}-\frac{420zh}{\overline{r}^9}$$
$$-\frac{24(1-\nu)(1-2\nu)\left(5\overline{r}^2+4\overline{r}(z+h)+(z+h)^2\right)}{\overline{r}^5(\overline{r}+z+h)^4}$$
$$+R^2\left(-\frac{42(3-4\nu)}{\overline{r}^9}+\frac{756z(z+h)}{\overline{r}^{11}}\right)$$

$$F_9 = \left[-\frac{6\nu(z-h)}{r^5} + \frac{6R^2(z-h)}{r^7} \right]_e + \frac{6(\nu h + 3z(1-\nu))}{\bar{r}^5}$$

$$+ \frac{60zh(z+h)}{\bar{r}^7} + R^2 \left(\frac{3z+h}{\bar{r}^7} - \frac{14z(z+h)^2}{\bar{r}^9} \right)$$

$$F_{10} = \left[\frac{30(z-h)}{r^7} - \frac{42R^2(z-h)}{r^9} \right]_e + \frac{30(h + 4z(1-\nu))}{\bar{r}^7}$$

$$- \frac{420zh(z+h)}{\bar{r}^9} + R^2 \left(-\frac{42(3z+h)}{\bar{r}^9} + \frac{756z(z+h)^2}{\bar{r}^{11}} \right)$$

Tilts of the horizontal element

$$\gamma_x = \frac{\partial w}{\partial x} = Qy\left(F_{11} + x^2 F_{12}\right)$$
$$\gamma_y = \frac{\partial w}{\partial y} = Qx\left(F_{11} + y^2 F_{12}\right) \quad (7.6)$$

where

$$F_{11} = \left[-\frac{6(z-h)}{r^5} + \frac{6R^2(z-h)}{r^7} \right]_e - \frac{6(3-4\nu)(z-h)}{\bar{r}^5}$$

$$+ \frac{60zh(z+h)}{\bar{r}^7} - \frac{8(1-\nu)(1-2\nu)(2\bar{r} + z + h)}{\bar{r}^3(\bar{r} + z + h)^2}$$

$$+ R^2 \left(\frac{12z}{\bar{r}^7} - \frac{6(3-4\nu)(z+h)}{\bar{r}^7} - \frac{84z(z+h)}{\bar{r}^9} \right)$$

$$F_{12} = \left[-\frac{30(z-h)}{r^7} - \frac{42R^2(z-h)}{r^9} \right]_e + \frac{30(3-4\nu)(z-h)}{\bar{r}^7} - \frac{420zh(z+h)}{\bar{r}^9}$$

$$- \frac{8(1-\nu)(1-2\nu)\left(8\bar{r}^2 + 9\bar{r}(z+h) + 3(z+h)^2\right)}{\bar{r}^5(\bar{r} + z + h)^3}$$

$$- R^2 \left(\frac{42(3-4\nu)(z+h)}{\bar{r}^9} - \frac{84z}{\bar{r}^9} + \frac{756z(z+h)^2}{\bar{r}^{11}} \right)$$

In all formulas $r = \sqrt{x^2 + y^2 + (z-h)^2}$, $\bar{r} = \sqrt{x^2 + y^2 + (z+h)^2}$. In Equations (7.2)-(7.6), inferior index of "*e*" at square brackets means that these brackets work only out of area *V*, and index of "*i*" means that these brackets work only in area *V*. For crust $\nu = 1/4$.

From Equation (7.1) for the sphere follows that in external potentials of $N, \bar{N}, \bar{L}, \bar{P}$ monopoly approximations match exact values and they in an explicit form do not include R radius. Radius of R is included obviously into external potentials of B and $\bar{B}$ and we will receive monopoly approximation if in the exact formula, we accept $R = 0$. Therefore, we will receive monopoly approximation in Equations (7.2)-(7.6) if at calculations out of area V in all functions F_k and the Equation (7.4), we accept $R = 0$.

At the end of preparation before the earthquake

$$V_{\max} = 10^{1.242M + 4.534} \text{ m}^3 \tag{7.7}$$

where M is magnitude of earthquake.

8. Heterogeneity in the Space

The solution of the problem on heterogeneity in space is based on use of the Kelvin's tensor.

$$k_{si} = A\left[4(1-\nu)n\delta_{si} - b_{,si}\right], \quad s,i = 1,2,3 \tag{8.1}$$

Kelvin's tensor, of course, contains in Mindlin's tensor. We will receive Kelvin's tensor, if we accept $(\overline{n}, \overline{b}, \overline{p}, \overline{l}) = 0$ in Equation (3.2). Further, we will receive the solution of the problem for space if in all computations we replace m_{si} with k_{si}. In Equations (6.6)-(6.9) it is necessary to accept $(\overline{N}, \overline{B}, \overline{L}, \overline{P}) = 0$ and to again receive Equations (7.2)-(7.6).

Conflicts of Interest

The author declares no conflicts of interest.

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