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On Weak Nil Clean Rings

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Abstract

If a ring R is called weak nil clean if every element in R can be expressed as the sum or difference of nilpotent element and idempotent, if further the idempotent element and nilpotent element commute the ring is called weak* nil clean. The purpose of this paper is to give some characterization and basic properties of weak nil clean rings. The main results of this work are: 1) Let R be a ring, then R is weak nil clean if and only if $R/P(R)$ is weak nil clean; 2) In a commutative ring R, if x is weak nil clean element, then x^m is a weak nil clean element if $(x-y)^m = \sum_{k=0}^{m}(-1)^{2k}\binom{n}{k}x^k y^{m-k}$ $x, y \in R$ (2); 3) Let R be a ring with $\mathrm{Idem}(R)=\{0,1\}$, then R is weak nil clean if and only if R is local ring and $J(R)$ is Nil ideal.

Subject Areas

Algebra

Keywords

Clean Rings, Nil Clean Rings, Weak Nil Clean and Strongly π-Regular Rings

1. Introduction

If an element a in an associative unital ring can be represented as a sum $a = u + e$, where e is an idempotent element and u is a unit [1] [2] [3] [4], it is called clean. Similarly, diesel [3] developed the nil clean ring, which defined an element a in a ring as nil clean (strongly nil clean), $a = b + e$ ($a = b + e$ and $be = be$).

$a = b + e$ for e is an idempotent element and b is a nilpotent element. If each member of a ring R is nil clean (Strongly nil clean), then a ring R is nil clean (Strongly nil clean) [5]. A ring is considered to be von Neumann regular ring (or simply regular) if and only if for each a in R there is an element c in R, such that,

$a = aca$ [2]. If there is a positive integer n for all a in a ring R, it is said to be strongly π-regular. And an element c in R such that $a^n = a^{n+1}c$ [6].

A ring R is referred to as a local ring, if it contains a unique maximal ideal [7]. For a ring R, we will use $U(R)$, $N(R)$, reg(R), $J(R)$, Idem(R), $C(R)$ and $P(R)$, denote to group of units, the set of nilpotent elements, the set of regular elements, the Jacobson radical, the set of idempotent elements of R, the centre of R and ($P(R)$) the prime of (R).

2. Weak Nil Clean Rings

This section is devoted to defining weakly nil clean rings, as well as some of their characterization and basic features.

2.1. Definition 2.1. [8]

If $a = b + e$ or $a = b - e$, for some $b \in N(R)$ or $e \in \text{Idem}(R)$, an element a is said to be a weak nil clean element of the ring, and a ring is said to be weak nil clean if each of its elements is weak nil clean.

2.2. Maintaining the Integrity of the Specifications

Example:

1) All nil clean rings are weak nil clean rings, but the convers is not true, $\mathbb{Z}_6$ is not a nil clean ring since 2' and 5' cannot be expressed as a sum of idempotent and nilpotent of $\mathbb{Z}_6$.

2) $\mathbb{Z}_6$ is weak nil clean rings.

3) $\mathbb{Z}_{3^k}$, $k \in \mathbb{N}$, is weak nil clean but not nil clean.

4) Consider R be a commutative ring and M a left R-module. The idealization of R and M is the ring $R(M) = R \oplus M$, where $\oplus$ the direct sum, with product defined as $(r,m)(r',m') = (rr', rm' + r'm)$ and sum as $(r,m) + (r',m') = (r + r', m + m')$ for $(r,m),(r',m') \in R(M)$.

Now, we give the most important results of this section.

Proposition 2.1 [8].

Any weak nil clean rings homomorphic image is weak nil clean. The opposite is not true; for example, $\mathbb{Z}_6 \cong \mathbb{Z}/(6)$ is a weak nil clean ring, but $\mathbb{Z}$ is not a weak nil clean.

Proposition 2.2 [8].

Let R be a commutative ring. Then R is weak nil clean if and only if $R/\text{Nil}(R)$ is weak nil clean, when the idempotents can be lifted modulo Nil(R).

Proposition 2.3 [8].

Let R be a weak nil clean ring, then,

$$J(R) \subseteq N(R) \tag{1}$$

Proposition 2.4.

Let R be a ring. Then R is weak nil clean if and only if, $R/P(R)$ is weak nil clean.

Proof:

Assume that the ring R is a weak nil clean ring. Then, according to proposition 2.1, we have $R/P(R)$ is weak nil clean. Now, assume that $R/P(R)$ is weak nil-clean consider $a \in R$, So $a+P(R)=(b+P(R))+(e+P(R))$ or $a+P(R)=(b+P(R))-(e+P(R))$. That is, $a+P(R)=(b+e)+P(R)$ or $a+P(R)=(b-e)+P(R)$ implies that, $(a-(b+e)) \in P(R)$ or $(a-(b-e)) \in P(R)$, $((a-e)-b) \in P(R) \subset N(R)$ or $((a+e)-b) \in P(R) \subset N(R)$. Thus, $(a-e) \in N(R)$ or $(a+e) \in N(R)$. Hence, $a-e=b$ or $a+e=b$, therefore R is weak nil clean ring.

Proposition 2.5.

Let R be abelian weak nil clean ring. Then $C(R)$ is weak nil clean.

Proof:

Assume that R is weak nil clean and let $a \in C(R)$.

Now write, $a=b+e$ or $a=b-e$ for some $e \in \text{Idem}(R)$ and $b \in N(R)$, since e is central, Then $e \in C(R)$, Hence $a-e=b \in C(R)$ or $a+e=b \in C(R)$. Therefore $C(R)$ is weak nil clean.

Proposition 2.6.

Let R be a ring. Then for any $x \in R$.

1) If x is weak nil clean, then $(1-x)$ is clean element.

2) If $(1-x)$ is weak nil clean, then x is clean element.

Proof:

1) Let x be weak nil clean element. Then $x=e+b$ or $x=b-e$. where, $e \in \text{Idem}(R)$ and $b \in N(R)$. Thus, $(1-x)=(1-e)-b$ that is, $(1-x)$ is nil clean and therefore, $(1-x)$ is clean by ([3] proposition 3.4) or $(1-x)=(1-b)+e$. Now since, $b \in N(R)$ that is, $b^n=0$ for some $b \in \mathbb{Z}^+$, then, $1-b^n=(1-b)(1+b+b^2+\cdots+b^{n-1})$, Thus $(1-b) \in U(R)$ and $e \in \text{Idem}(R)$ Therefore, $(1-x)$ is clean element.

2) Let $(1-x)$ be weak nil-clean element, Then $(1-x)=b+e$ or $(1-x)=b-e$. Thus, $x=(1-e)-b$ or $x=e+(1-b)$ That is x is nil clean then, by ([3] proposition 3.4) x is clean element.

Proposition 2.7.

In a commutative ring R, if x is weak nil clean element, then x^m is weak nil clean element

$$\text{if } (x-y)^m=\sum_{k=0}^{m}(-1)^{2k}\binom{n}{k}x^k y^{m-k} \quad x,y \in R \tag{2}$$

Proof:

Let x be a weak nil clean such that $x=b+e$ or $x=b-e$ where $b \in N(R)$ and $e \in \text{Idem}(R)$, Now we must prove x^m by induction in the number of m. When $m=2$ then, $x^2=(b+e)^2$ or $x^2=(b-e)^2$ then, $x^2=b^2+2eb+e^2=b(b+2e)+e$ where, $e \in \text{Idem}(R)$ and $b(b+2e) \in N(R)$ or $x^2=(b^2-2eb+e^2)=b(b-2e)+e$; $e \in \text{Idem}(R)$ and $b(b-2e) \in N(R)$ is true.

Now, suppose that x^{m-1} is weak nil clean, that is

$x^{m-1}=(b+e)^{m-1}=\sum_{k=0}^{m-1}\binom{m}{k}b^k e^{m-k}$ or

$x^{m-1}=(b-e)^{m-1}=\sum_{k=0}^{m-1}(-1)^{2k}\binom{m}{k}b^k e^{m-k}$ if, $x^{m-1}=\sum_{k=0}^{m-1}\binom{m}{k}b^k e^{m-k}$, then,

$x^{m-1}=b^{m-1}+e(e-1)b^{m-2}+\frac{e(e-1)(e-2)}{2!}b^{m-3}e+\cdots+e^{m-2}$, Thus

$$x^m=xx^{m-1}=(b^m+e)\left(b^{m-1}+e(e-1)b^{m-2}+\frac{e(e-1)(e-2)}{2!}b^{m-3}e+\cdots+e^{m-2}\right),$$

Now, $x^m=xx^{m-1}=b^{m-1}+e\left(b^{m-1}+\frac{e(e-1)}{2!}b^{m-2}e+\cdots+e^{m-1}\right)=b^m+e^m$.

Thus, $b^m\in N(R)$ and $e^m\in \mathrm{Idem}(R)$.

Or if,

$$x^{m-1}=(b-e)^{m-1}$$
$$=b^{m-1}+e\left((-1)^2(e-1)b^{m-2}+(-1)^4\frac{e(e-1)(e-2)}{2!}b^{m-3}e+\cdots+(-1)^{2m}e^{m-2}\right)$$

Now,

$$x^m=xx^{m-1}$$
$$=(b^m+e)\left(b^{m-1}+e\left((-1)^2(e-1)b^{m-2}+(-1)^4\frac{e(e-1)(e-2)}{2!}b^{m-3}e+\cdots+(-1)^{2m}e^{m-2}\right)\right)$$
$$=b^m+e^m$$

Hence, $b^m\in N(R)$ and $e^m\in \mathrm{Idem}(R)$ Therefore, x^m is weak nil clean element.

Lemma 2.8.

Let R be a ring with $\mathrm{Idem}(R)=\{0\}$ if $x\in J(R)$, Then x is weak nil clean.

Proof:

Let $x\in J(R)$, then $(1-x)\in U(R)$, that is $1-x=u+0$, hence $x=(1-u)+0$ or $x=(1-u)-0$, $(1-u)\in N(R)$ and $0\in \mathrm{Idem}(R)$ therefore x is weak nil clean element.

Proposition 2.9.

Let R be a weakly Nil clean ring and let $a\in R$. If aR contains no non-zero idem potent. Then, a is the sum of a nilpotent and a right unit.

Proof:

Suppose aR contains no non- zero idempotent choose, $e\in \mathrm{Idem}(R)$ and $b\in N(R)$.

Such that, $a-1=e+b$ or $a-1=b-e$. Then, $a=e+(b+1)$ or $a=b+(1-e)$, $a=e+(b+1)$ or $-a=-b-1+e$ or $-a=-(b+1)+e$, such that, $a=e+(b+1)$ or $-a=e-(b+1)$. Now, $(b+1)\in U(R)$ and $e=e^2$ since, $a(1)e=e+(b+1)(1)e$ or $-a(1)e=e-(b+1)(1)e$. Then,
$a(1-e)=(b+1)(1-e)$ or $-a(1-e)=(b+1)$, So,
$a(1-e)(b+1)^{-1}=(b+1)(1-e)(b+1)^{-1}$ or
$-a=(1-e)(b+1)^{-1}=(b+1)(e-1)(b+1)\in aR$, Clearly $(b+1)(1-e)(b+1)^{-1}$

and $(b+1)(e-1)(b+1)^{-1}$ are idempotent in aR then by assumption, $(b+1)(1-e)(b+1)^{-1}=0$ or $(b+1)(e-1)(b+1)^{-1}=0$ Hence, $1-e=0$ and then, $e=1$ Therefore, a is the sum of a nilpotent and a right unit.

Proposition 2.10.

Let R be abelian and weakly nil clean ring in which, $2\in U(R)$. Then every element of R can be written as a sum of nilpotent two unit elements.

Proof:

Let $x\in R$, Then $x=b+e$ or $x=b-e$ where $b\in N(R)$ and $e\in \text{Idem}(R)$, Now let $v=2e-1$ then, $v^2=(2e-1)^2=4e^2-4e+1=1$. Thus, $(2e-1)(2e-1)^{-1}=1$, $2e=v+1$, Since $2\in U(R)$, then $e=2^{-1}v+2^{-1}$, Therefore $a=b+u_1+u_2$ now, $a=b+2^{-1}v+2^{-1}$ or $a=b-\left(2^{-1}v+2^{-1}\right)$.

3. The Relationship between the Weak Nil Clean Ring and Other Rings

Throughout this section let us study the relationship between weakly nil clean ring and local ring, strongly π-regular ring and clean ring.

Proposition 3.1.

Let R be a ring with $\text{Idem}(R)=\{0,1\}$. Then R is weak nil clean if and only if R is local ring and $J(R)$ is Nil ideal.

Proof:

Suppose R be a weak nil-clean ring and let $a\in R$, then there exist $e\in \text{Idem}(R)$ and $b\in N(R)$, Such that $a=b+e$ or $a=b-e$, Now if $e=0$, then $a=b$ is nilpotent element. If $e=1$ that is, $e\in U(R)$, then $a=b+1$ or $a=b+(-1)$. Hence a is a sum of nilpotent element and unit element which is commute. Therefore either a or $1-a$ is unit that is R is local ring. Now let $a\in J(R)$, then similarly either a is a nilpotent or a unit, in the second case is impossible, Thus $J(R)$ is nil and write $a=b+e$ or $a=b-e$ for some $e\in \text{Idem}(R)$ and $b\in N(R)$. If $r=0$, Then $a=b$ is nilpotent, If $r=1\in U(R)$ then, $a=b+1$ or $a=b-1$ then, $a=b+(-1)$, that is a is a sum of a nilpotent or a unit, where is commute. Therefore $a\in U(R)$ (Contradiction)!

Now, suppose R is a local ring with $J(R)$ is nil. Then, for any $a\in R$, either $a\in J(R)$ or $(a-1)\in J(R)$. In first case, a is nilpotent ($J(R)$ is nil). and $a=a+0$, $e=0$ or $a=a-0$.

In the second case $(a-1)$ is nilpotent that is, $a=1+(a-1)$.

Thus, a is a weak nil clean element. therefore, R is a weak nil clean ring.

Proposition 3.2.

Every weak nil clean ring is clean.

Proof:

Let R be weak nil clean ring and let $a\in R$, then $a=b+e$ or $a=b-e$ where $b\in N(R)$ and $e\in \text{Idem}(R)$, $a-1=b+e$, thus $a=e+(b+1)$, since $(b+1)$ is unit, then $a=e+u$ is clean, or $a=b-e$ then $a=(1-2e+b)-(1-e)$ where $(1-2e+b)\in U(R)$ and $(1-e)\in \text{Idem}(R)$

Therefore, a is clean.

Proposition 3.3.

If a is weak *nil clean element, then a is a strongly π-regular.

Proof:

Suppose a is weak* nil clean, then $a=b+e$ or $a=b-e$ and $eb=be$, if $a=b+e$, then, $a=(2e-1+b)+(1-e)$ is clean where, $2e-1+b\in U(R)$ and $(1-e)\in \text{Idem}(R)$, hence, a is strongly clean, thus strongly nil clean. or $a=b-e=(1-2e+b)+(1-e)$ where, $(1-2e+b)\in N(R)$ and $(1-e)\in \text{Idem}(R)$, thus a is strongly nil clean Therefore, a is strongly π-regular element by ([3], corollary 3.11).

4. Conclusion

From the study on characterization and properties of weak nil clean rings, we obtain the following results:

1) Let R be a ring, then R is weak nil clean if and only if $R/P(R)$ is weak nil clean.

2) Let R be abelian weak nil clean ring, then $C(R)$ is weak nil clean.

3) Let R be a ringm, then for any $x\in R$.

a) If x is weak nil clean, then $(1-x)$ is a clean element.

b) If $(1-x)$ is weak nil clean, then x is a clean element.

4) In a commutative ring R, if x is a weak nil clean element, then x^m is a weak nil clean element

$$\text{if } (x-y)^m=\sum_{k=0}^{m}(-1)^{2k}\binom{n}{k}x^k y^{m-k} \quad x,y\in R \tag{2}$$

5) Let R be a weakly Nil clean ring and let $a\in R$. If aR contains no non-zero idem potent, then a is the sum of a nilpotent and a right unit.

6) Let R be abelian and weakly nil clean ring in which $2\in U(R)$, then every element of R can be written as a sum of nilpotent two unit elements.

Conflicts of Interest

The authors declare no conflicts of interest.

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