Retraction Notice

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A Literature Review of Random Greedy Kaczmarz

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Abstract

The problem of solving large-scale linear equations widely exists in various research fields. There are many methods to solve the problem. Most of them study the Kaczmarz iterative algorithm, which can be applied to overdetermined or underdetermined situations. In recent years, the improvement of the algorithm has been a research hotspot in the iterative algorithm. This paper summarizes the literature on the combination of Kaczmarz algorithm and greedy algorithm at home and abroad, analyzes the development trend of the combination idea, and sorts out the innovation points, operating conditions and numerical results of various improved algorithms, hoping to provide reference for the improvement research of the algorithm in the future.

Subject Areas

Information Communication Theory and Algorithm

Keywords

Kaczmarz, Greedy Algorithm, Large Linear Equations, Overdetermined Compatible System

1. Introduction

In the research of computer, medicine and other fields, it is often necessary to solve multiple unknown linear equations. The more complex the problem is, the more unknowns are contained in the equation. Therefore, an effective method for solving linear equations came into being. At present, the main methods for solving large-scale linear equations are algebraic reconstruction (ART), synchronous image reconstruction (SIRT), continuous over relaxation (SOR), conjugate gradient method (CG), generalized minimum residual method (GMRES) and double conjugate gradient stability method (BiCGSTAB). Among them, Kaczmarz method in ART is widely used to solve large-scale linear equations, and there is more and more research on related algorithms.

This paper discusses a stochastic iterative method for solving large-scale linear systems. The following is the general expression of linear equations,

$$Ax = b. \tag{1}$$

where matrix $A \in R^{m \times n}$, the vector $x \in R^n$ is unknown, and the vector $b \in R^m$ is given. When the system is overdetermined ($m > n$) compatible, the linear system corresponding to the full column rank matrix A has the minimum norm solution. When the linear system is overdetermined inconsistent, the linear system corresponding to the full column rank matrix A has the least square solution. For different types of matrices (dense or sparse), the approximate solutions of the equations are different, and the methods selected are also different.

The classical Kaczmarz method can quickly solve the approximate solution regardless of the number of equations. Its main method is to orthogonally project the initial point onto the hyperplane, and take the iterative point as the starting point of the next iteration. Through each row of the cyclic coefficient matrix *A*, an approximate solution infinitely close to the exact solution is finally obtained. The specific iterative process is as follows: x_0 is the initial vector, α_k is the step size and the value is 1. According to the order of each row of matrix *A*, the current approximate vector is orthogonally projected onto the hyperplane of the solution, *i.e.* $\{x \mid \langle A(i), x \rangle = b_i\}$, and the obtained vector is put into the next iteration. The approximate solution of the $(k+1)$ iteration is:

$$x_{k+1} := x_k + \alpha_k \frac{b_i - \langle A^i, x_k \rangle}{\left\| A^{(i)} \right\|_2^2} A^{(i)},$$

where $\|\cdot\|_2$ is Euclidean norm; $\langle \cdot, \cdot \rangle$ is Euclidean inner product. Once this algorithm was proposed, many improved algorithms based on it were derived.

In 2009, Strohmer and Vershynin [1] proposed an exponentially convergent stochastic Kaczmarz method (RK). This method is improved on the basis of the classical Kaczmarz method. The improved algorithm is suitable for large-scale linear systems that are overdetermined. Through theory, it is demonstrated that the algorithm converges at the expected exponential rate, which is the first convergence rate that does not depend on the size of the system of equations. It does not even need to know the entire linear system, only the random iterative part of it to operate. The difference between RK algorithm and Kaczmarz algorithm is that the algorithm no longer iterates all the rows of the matrix *A*, but randomly selects those working rows according to the probability, and only performs iterative operations on the row vectors in the selected subscript set. This can greatly save computing time and be more efficient. The specific probability operation method is as follows: take the ratio of the 2-norm of each row of matrix *A* to the F-norm of matrix *A* as the row selection probability, and randomly select the row,

$$x_{k+1} := x_k + \frac{b_{r(i)} - \left\langle A_{r(i)}, x_k \right\rangle}{\left\| A_{r(i)} \right\|_2^2} A_{r(i)},$$

where $r_{(i)}$ is from the set $\{1, 2, \cdots, m\}$, in proportion to the norm $\left\| a_{r(i)} \right\|_2^2$ filtered out. When matrix A is a non quantitative matrix, RK algorithm has obvious advantages over conjugate gradient method and classical Kaczmarz.

However, when the coefficient matrix of RK method is a quantity matrix, that is, when the norm value of each row is the same, the probability of selecting active rows according to the above method is the same. In this case, RK becomes the classical Kaczmarz method. For this shortcoming, many improved algorithms aiming at this shortcoming of RK have emerged, and iterative algorithms combining greedy algorithm, coordinate descent method, CGLS and Kaczmarz have been derived. How to select work lines randomly to accelerate the stochastic Kaczmarz method is still a problem worthy of study. In recent years, there are many improved algorithms about the combination of greedy algorithm and Kaczmarz algorithm. They all combine the advantages of Kaczmarz and greedy algorithm to speed up the convergence speed by adding blocks and using coordinate descent method.

This paper summarizes the algorithm of combining greedy and Kaczmarz, summarizes the research status of the combination of Kaczmarz and greedy algorithm at home and abroad, and grasps the iterative process of various improved algorithms, which lays a foundation for further improving and improving greedy Kaczmarz algorithm. This paper discusses the development trend of the algorithm, analyzes the applicable conditions, improvement process and numerical example comparison results of each iterative algorithm, finds out the advantages and disadvantages of various methods through the overview of greedy Kaczmarz algorithm, combined with different conditions and environments, and puts forward constructive suggestions for its advantages and disadvantages, hoping to be helpful to the improved algorithm in the future.

2. Domestic and International Researches

In 2018, Bai and Wu [2] proposed to combine greedy algorithm with Kaczmarz to obtain greedy stochastic Kaczmarz (GRK) algorithm, which is suitable for solving large-scale compatible sparse linear equations, and the approximate solution converges to the minimum norm. Based on RK, GRK uses greedy method to select work lines. The main idea of greedy algorithm is to filter out the larger components in the residual vector r to form a subscript set about the matrix row, bring these subscript sets into the iteration, and confirm these behavior active row vectors. Among these selected row vectors, the ratio proportional to the norm of residual vector r is taken as the row selection probability, and the row or sub-matrix with higher probability in the working row is selected to enter the iteration first. In this way, the method of giving priority to the larger components is greedy algorithm. The advantage of greedy algorithm is that it has fast speed, fast convergence and reaches the optimal solution first. GRK uses the advantages of greedy algorithm to make up for the shortcomings of RK. It allows the larger components in the residual vector r to be selected and eliminated first, the lines corresponding to each component enter the iteration first, and sets the subscript set of the active line to U_k. Define constant ϵ_k. Those subscript sets i_k that will satisfy the inequality is respectively substituted into the probability criterion for residuals, $P=\frac{|\hat{r}_k(i_k)|^2}{\|\hat{r}_k\|_2^2}$, those lines with high probability enter the iteration first. Theoretically, each iteration of GRK requires $7m+2(n+1)$ floating-point operations, while each iteration of RK requires $4n+1$ floating-point operations. Therefore, when $m>\frac{1}{7}(2n-1)$, the total computational complexity of greedy algorithm is greater than that of RK. In numerical examples, the paper discusses both full rank and non rank matrices. The convergence speed of GRK is faster than RK. When the relative error $RSE<10^{-6}$, or when the number of iterative steps exceeds 200,000, the iteration ends. At the end of the paper, the author puts forward the prospect that if the matrix product AA^* can be calculated, GRK will further improve the computational efficiency. In the same year, Bai and Wu [3] added relaxation parameters to the GRK algorithm $\theta(\theta\in[0,1])$, so as to adjust the convergence rate through the relaxation parameters on the basis of GRK, and improved the relaxation greedy random Kaczmarz (RGRK), also suitable for solving large systems of uniformly sparse linear equations. The computational complexity of RGRK and GRK algorithms is the same in each iteration, but RGRK algorithm can achieve constant by changing the relaxation parameters ϵ_k, and then accelerate the convergence rate of the algorithm. Among them, ϵ_k is selected from $\left[\frac{1}{\|A\|_F^2},\frac{1}{\|b-Ax^k\|^2}\max\left(\frac{\left|b_j-\left\langle a^j,x^{(k)}\right\rangle\right|^2}{\|a^j\|^2}\right)\right]$. In the theoretical part, it is proved that the upper bound of RGRK convergence is less than GRK. RGRK in $\theta\geq\frac{1}{2}$, the convergence does not change much, when $\theta=\frac{1}{2}$, RGRK becomes partially random, but no matter what the value of θ is, when the parameters are properly selected, RGRK is better than GRK in CPU and IT. In the numerical example part, the relaxation parameters may change under different properties of the matrix. The article compares the changes of CPU and IT of RGEK under different relaxation parameters to find the most suitable parameter value. It can be seen that the RGRK algorithm is better than GRK when the appropriate parameters are selected.

In 2019, Zhang [4] proposed a new greedy Kaczmarz algorithm (GK) for large-scale overdetermined linear compatible systems. The algorithm adds adaptive relaxation parameters on the basis of GRK algorithm $\alpha_k\subset(0,2)$, here in RK, $\alpha_k=1$, new definition $\hat{\alpha}_k$ adds step size in the iteration α_k. Through the demonstration of numerical examples for matrices with different dimensions ($m > n$), it is concluded that whether matrix A is dense or sparse, algorithm GK is better than algorithm GRK.

In 2019, Bai and Wu [5] published an article on combining the greedy algorithm with the coordinate descent method, and proposed the greedy random coordinate descent method (GRCD), which added the greedy algorithm to the random coordinate descent method (RCD). The algorithm is suitable for solving the least squares problem of overdetermined linear algebraic equations ($m \geq n$) with full rank. The coordinate descent method is an iterative method that can effectively solve the linear least squares problem. The greedy algorithm can let the residual vector r_k and the orthogonal space of the working column $A_{(j_k)}$ in the next iteration x_{k+1}. The included angle is relatively large. In addition, for the selected columns, we also hope that the probability of being selected into the iteration is high. Through the demonstration of some lemmas and theorems, the article shows that the convergence factor of GRCD is smaller, that is, the upper bound of convergence is smaller. When n is larger, the constant factor β is closer to 1, but these methods are affected by many factors, such as the sparse structure of the matrix, the condition number and the algebraic properties of the matrix in the linear least squares problem. In the numerical example part of the article, the author compares the convergence of the two algorithms from the aspects of CPU and IT, and discusses the compatible and incompatible cases. The termination condition is set as $RSE < 10^{-6}$. From the experimental data, it can be seen that GRCD is better than RCD regardless of whether it is consistent or inconsistent. GRCD combines the advantages of the random coordinate descent method and the greedy algorithm, and can more effectively select the working column vector of the coefficient matrix A, thereby speeding up the convergence rate.

In 2020, Zhang and Guo [6] added relaxation parameters based on the GRCD algorithm, and proposed the maximum distance coordinate descent method. On the basis of GRCD, improvements are made to make the convergence speed faster under suitable conditions.

In 2020, Niu and Zheng [7] proposed the greedy block Kaczmarz algorithm (GBK) on the basis of GRK, which is suitable for large linearly consistent systems, for any consistent real matrix ($m \geq n$), whether or not ill-conditioned, all are applicable. When the minimum singular value of matrix A is larger, the convergence speed is faster. GRBK is more efficient than GRK when choosing a suitable parameter η. GRBK defines ϵ_k differently from GRK, iterating with Moore-Penrose pseudo inverse of matrix A. In the numerical example, in terms of CPU and floating-point operations, GBK requires less floating-point operations, so the iteration time is shorter, and it is concluded that GBK is significantly better than GRK. The geometric properties of the matrix and its sub-matrices control the convergence speed. Because the convergence factor of GBK is smaller than that of GRK, although GRK has many advantages, GBK will be better than GRK algorithm in terms of convergence speed. Termination rule: relative error $RE < 10^{-10}$, through numerical examples, the advantages of GBK are strongly demonstrated. In sparse or dense systems, although GBK requires more complexity than GRK, GBK requires fewer iterations.

In 2020, Zhang Xuanjing [8] and others published an article on the algorithm research based on the greedy random Kaczmarz method, and proposed the Kaczmarz algorithm to accelerate the greedy random expansion. The algorithm is based on the GRK and Nesterov acceleration strategies, combined and improved. The algorithm combines the advantages of the greedy algorithm and the acceleration strategy, and can solve the linear equation system more quickly. In the numerical example part, the overdetermined equations ($m > n$) and the well-posed equations ($m = n$) are discussed and studied separately. The termination rule of the iteration is the relative error of the iteration $RSE < 10^{-6}$, or has been The maximum number of iterations is reached. Through five numerical experiments, it can be concluded that AGREK is significantly better than GRK and AREK algorithms under suitable conditions.

In 2020, Du Yishu [9] and others published an article on the greedy distance stochastic Kaczmarz method (GDRK) for solving large-scale sparse linear equations. Taking advantage of the advantages of the GRK algorithm, they proposed an improved algorithm GDRK. GDRK is based on the GRK algorithm. Improve the algorithm, improve the subscript set selection conditions and row selection probability of GRK, select the row with the largest distance from the current solution to the hyperplane formed by each row of the current coefficient matrix, as the working row of this iteration, and use the distance The ratio of the norm to the norm of the matrix row is used as the criterion for the probability of selecting a row. The algorithm is suitable for a consistent system of sparse linear equations. The article analyzes the convergence speed of the GDRK algorithm from the size of the convergence factor, and theoretically concludes that the convergence factor of the GDRK algorithm is smaller, that is, the convergence speed is faster, and the algorithm requires $7m + 2(n+1)$ floating point Therefore, when $m > \frac{2n-1}{7}$, the computational cost of the greedy distance random Kaczmarz method is smaller than that of the random Kaczmarz method. In the numerical example part of the article, the termination rule of the algorithm is set to the relative error $RSE < 10^{-6}$ or the number of iteration steps exceeds 200,000 steps. The full-rank matrix and the rank deficient matrix are compared. It can be seen that under the same conditions, GDRK is better than RK and GRK algorithms in terms of operation time and number of iterations.

In 2021, Liu and Gu [10] applied greedy algorithm in block T based on RBK algorithm $T = \{\tau_1, \tau_2, \cdots, \tau_p\}$ The blocks with large residuals are selected as the sub matrix of each iteration. In this method, greedy random block algorithm (GRBK) is proposed. Firstly, the row of matrix A is arbitrarily divided into blocks, and the greedy algorithm is run in each block to obtain GRBK algorithm. The improvement of GRBK on RBK algorithm is reflected in: modifying the greedy probability criterion to the almost maximum residual principle, constructing the target index set, so that almost all large items in the residual vector can be indexed. It is no longer the mechanical random selection of the divided blocks for iteration, but through the screening of the blocks divided by the residual vector, select those residual blocks with large values for iteration, so as to continuously reduce the residual vector and increase the feasibility of the algorithm. In terms of the calculation time of GRBK algorithm, this paper also proposes another improved algorithm, almost maximum residual block Kaczmarz method (BEM). This algorithm can also be regarded as an extension of Motzkin. It combines blocking and greedy algorithm, but it is better than GRBK and RBK. The difference between BEM algorithm and GRBK algorithm is that a rule is defined according to greedy algorithm to filter out the largest lines in the component of residual vector, divide the filtered lines into a block, run RBK algorithm on the block, and select all the blocks with the largest residual to eliminate. For the overdetermined inconsistent system $Ax = b$, the RBK algorithm was proposed in 2014. The algorithm performs block iteration on matrix A to improve the algorithm. In 2018, greedy algorithm and Kaczmarz algorithm were combined to minimize the residual vector r, and GRK algorithm was obtained. In 2021, the author proposed two improved algorithms, GRBK and BEM. On the basis of RBK and GRK, the former greatly reduces the number of iterations and operation time, and the latter greatly reduces the size of residual vector and improves the accuracy. In the numerical example part of this paper, four different types of dense matrix and full rank sparse matrix are used to compare the operation time and iteration times of algorithms RBK, GRBK and BEM respectively. The data show that BEM method has greater advantages under appropriate parameters. It makes the residual vector smaller, that is, estimate the solution x_k is closer to the exact solution x. Comprehensive analysis shows that the improved algorithm is more efficient than other methods in iterative speed.

In 2020, Yang [11] proposed geometric probabilistic random Kaczmarz algorithm (GPRK), established a new subscript set, defined probability supervision criteria based on subscript set and geometric properties, and obtained a convergence upper bound smaller than the upper bound of RK and GRK.

3. Conclusions

At present, there are many articles about Kaczmarz solving linear equations in different systems. Most of them are based on the classical algorithms REK, RK, RBK [12], add greedy random, relaxation parameters, Gaussian, coordinate descent, transform pseudo-inverse, add Methods such as double-block, partially random, select working rows, or combine with other methods of solving least squares. Among them, the literature on the greedy algorithm is the majority, and the focus of improvement is mainly on the selection strategy of the active row, and the continuous acceleration is based on the iteration time, the number of iterations, the computational complexity, etc., compared with the original algorithm to judge whether the improved algorithm is feasible. However, the method for solving large linear equations is not limited to the Kaczmarz method, but can also be studied in combination with other domain knowledge, and many new methods will appear in the future.

In fact, the greedy algorithm is very advantageous when it is used to filter the matrix subscript set, which can achieve the effect of accelerating the convergence rate. On the basis of greed, it can be faster to iterate in blocks. In the future algorithm research, the greedy algorithm can be combined with other more algorithms to generalize the problem of solving linear equations to tensors, and combine the Kaczamrz method to solve some ill-posed problems. I believe it will be more meaningful.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Strohmer, T. and Vershynin, R. (2009) A Randomized Kaczmarz Algorithm with Exponential Convergence. *Journal of Fourier Analysis & Applications*, **15**, Article number: 262. https://doi.org/10.1007/s00041-008-9030-4

[2] Bai, Z.Z. and Wu, W.T. (2018) On Greedy Randomized Kaczmarz Method for Solving Large Sparse Linear Systems. *SIAM Journal on Scientific Computing*, **40**, A592-A606. https://doi.org/10.1137/17M1137747

[3] Bai, Z.Z. and Wu, W.T. (2018) On Relaxed Greedy Randomized Kaczmarz Methods for Solving Large Sparse Linear Systems. *Applied Mathematics Letters*, **83**, 21-26. https://doi.org/10.1016/j.aml.2018.03.008

[4] Zhang, J.J. (2019) A New Greedy Kaczmarz Algorithm for the Solution of Very Large Linear Systems. *Applied Mathematics Letters*, **91**, 207-212. https://doi.org/10.1016/j.aml.2018.12.022

[5] Bai, Z. and Wu, W. (2019) On Greedy Randomized Coordinate Descent Methods for Solving Large Linear Least-Squares Problems. *Numerical Linear Algebra with Applications*, **26**, No. 4. https://doi.org/10.1002/nla.2237

[6] Zhang, J. and Guo, J. (2020) On Relaxed Greedy Randomized Coordinate Descent Methods for Solving Large Linear Least-Squares Problems. *Applied Numerical Mathematics*, **157**, 372-384. https://doi.org/10.1016/j.apnum.2020.06.014

[7] Niu, Y.Q. and Zheng, B. (2020) A Greedy Block Kaczmarz Algorithm for Solving Large-Scale Linear Systems. *Applied Mathematics Letters*, **104**, Article ID: 106294. https://doi.org/10.1016/j.aml.2020.106294

[8] Zhang, X.J. and He, Q.L. (2020) Algorithm Research Based on Greedy Stochastic Kaczmarz Method. *Mathematics Learning and Research*, No. 3, 2.

[9] Du, Y.S. and Yin, J.F. (2020) Greedy Randomized-Distance Kaczmarz Method for Solving Large Sparse Linear Systems. *Journal of Tongji University* (*Natural Science*), **48**, 1224-1231+1240.

[10] Liu, Y. and Gu, C.Q. (2021) On Greedy Randomized Block Kaczmarz Method for Consistent Linear Systems. *Linear Algebra and Its Applications*, **616**, 178-200. https://doi.org/10.1016/j.laa.2021.01.024

[11] Yang, X. (2020) A Geometric Probability Randomized Kaczmarz Method for Large Scale Linear Systems. *Applied Numerical Mathematics*, **164**, 139-160. https://doi.org/10.1016/j.apnum.2020.10.016

[12] Needell, D., Zhao, R. and Zouzias, A. (2015) Randomized Block Kaczmarz Method with Projection for Solving Least Squares. *Linear Algebra & Its Applications*, **484**, 322-343. https://doi.org/10.1016/j.laa.2015.06.027