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Further Study of the Shape of the Numbers and More Calculation Formulas

Ji Peng

Department of Electronic Information, Nanjing University, Nanjing, China
Email: mcfroo@sina.com

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Abstract

The core of Shape of numbers is formal calculation, which has three forms. This paper proves the equivalence of these forms and extends the formula to the general case. Some properties of the coefficients are summarized and some new conclusions are drawn. The coefficient matrix is studied and the corresponding results are obtained. Using the formal method, the calculation formula of $\sum_{n=0}^{N-1}\prod_{i=1}^{M}\left(K_i+D_iq^n\right)$ is obtained. The key is to use the Gaussian coefficient, which shows its new scope of application. Using the derivation in this paper, the calculation formula of $\sum_{n=0}^{N-1}q^n\binom{n+M}{M}$ is obtained. By introducing a new number: $A_q^M=\sum_{k=0}^{M}(1-q)^{M-k}q^KS_2(M,k)k$, this paper obtains the formula of $\sum_{n=0}^{N-1}q^nn^M$, at the same time, find the other three expressions of A_q^M.

Subject Areas

Discrete Mathematics

Keywords

Shape of Numbers, Calculation Formula, Combinatorics, Congruence, Gaussian Coefficient, Stirling Number

1. Introduction

Peng, J. has introduced Shape of numbers and three forms of calculation in [1] [2] [3] [4] [5]:

$K_i, D_i \in$ Commutative Ring.

M series: $Serie_i=\left\{K_i, K_i+D_i, K_i+2D_i, \cdots, K_i+(N-1)D_i\right\}$, $i\in[1,M]$.

Use $PS=[K_1:D_1,\cdots,K_M:D_M]$ to represent thems.

$[K_1:1,\cdots,K_M:1]$ is abbreviated as $[K_1,\cdots,K_M]$.

An item $=(I_1,I_2,\cdots,I_M)$, I_i comes from *Serie$_i$*. A product = $\prod_{i=1}^{M} I_i$.

Use $PT=[T_1=1,T_2,\cdots,T_M]$ to indicate the item's range:

$$I_i=K_i+a_iD_i,\begin{cases}T_{i+1}-T_i=1,\text{ means } a_i=a_{i+1}, \text{continuity}\\ T_{i+1}-T_i=2,\text{ means } a_i\le a_{i+1}, \text{discontinuity.}\end{cases}$$

PS and *PT* are defined as Shape of numbers, they indicate some items.

$$SUM(N,PS,PT)=\sum_{\text{all items}}\text{product}.$$

PB(*PT*) = count of discontinuity in *PT*, *PM*(*PT*) = count of factors in *PT*.

By default, the following uses:

$$PS=[K_1:D_1,\cdots,K_M:D_M],\quad PT=[T_1,\cdots,T_M].$$

H(*q*) is short for *H*(*PS*, *PT*, *q*), *SUM*(*N*) is short for *SUM*(*N*, *PS*, *PT*).

The Form: $(T_1+K_1)(T_2+K_2)\cdots(T_M+K_M)=\sum\prod_{i=1}^{M}X_M$, $X_i=T_i$ or K_i.

Don't swap the factors. Each $\prod X_i$ corresponds to one expression in *SUM*(…).

$$X(T)=\text{count of }\{X_1,\cdots,X_M\}\in\{T_1,\cdots,T_M\},$$

$$X_{K-1}=\text{count of }\{X_1,\cdots,X_{i-1}\}\in\{K_1,\cdots,K_M\}.$$

1.1) $q=X(T)$, $PM=PM(PT)$,

$SUM(N)=$

$$\xrightarrow{\text{Form}_1}\sum_{q=0}^{PM}H_1(q)\binom{N+T_M-PM}{N-1-q},B_i=\begin{cases}(T_i-X_{K-1})D_i;X_i=T_i\\ K_i+X_{T-1}D_i;X_i=K_i\end{cases}$$

$$\xrightarrow{\text{Form}_2}\sum_{q=0}^{PM}H_2(q)\binom{N+T_M-PM+q}{N-1},B_i=\begin{cases}(T_i-X_{K-1})D_i;X_i=T_i\\ K_i+(X_{K-1}-T_i)D_i;X_i=K_i\end{cases}$$

$$\xrightarrow{\text{Form}_3}\sum_{q=0}^{PM}H_3(q)\binom{N+T_M-q}{N-1-q},B_i=\begin{cases}-K_i+(T_i-X_{T-1})D_i;X_i=T_i\\ K_i+X_{T-1}D_i;X_i=K_i\end{cases}$$

$$H(q)=H(PS,PT,q)=\sum_{\text{all of the }\prod X_i\text{ with }X(T)=q}\prod_{i=1}^{M}B_i.$$

In particular:

1.2) $SUM(N,[1,2,\cdots,M],[1,3,\cdots,2M-1])=S_1(N+M,N)$, unsigned Stirling number.

1.3) $SUM(N,[1,1,\cdots,1],[1,3,\cdots,2M-1])=S_2(N+M,N)$, Stirling number of the second kind.

1.4) $SUM(N,[1,1,\cdots,1],[1,2,\cdots,M])=1^M+2^M+\cdots+N^M$.

2. Equivalence of Three Forms

The following change T_i's domain to $\mathbb{Z}$ and $T_{i+1}-T_i$ is not restricted.

$PS1=[PS,K_{M+1}:D_{M+1}]$, $PT1=[PT,T_{M+1}]$, use $H(PS1,q)=H(PS1,PT1,q)$.

2.0) Recurrence relation

① $H_1(PS1,q)=H_1(q-1)\left(T_{M+1}-\left[M-(q-1)\right]\right)D_{M+1}+H_1(q)\left(K_{M+1}+qD_{M+1}\right)$

② $$H_2(PS1,q)=H_2(q-1)\left(T_{M+1}-\left[M-(q-1)\right]\right)D_{M+1}+H_2(q)\left(K_{M+1}+\left[-T_{M+1}+M-q\right]D_{M+1}\right)$$

③ $$H_3(PS1,q)=H_3(q-1)\left(-K_{M+1}+\left[T_{M+1}-(q-1)\right]D_{M+1}\right)+H_3(q)\left(K_{M+1}+qD_{M+1}\right)$$

2.1) ① $H_1(q)=\sum_{x=q}^{M}H_2(x)\binom{x}{q}=\sum_{x=0}^{q}H_3(x)\binom{M-x}{M-q}$

② $H_2(q)=\sum_{x=q}^{M}(-1)^{x+q}H_1(x)\binom{x}{q}, H_3(q)=\sum_{x=0}^{q}(-1)^{x+q}H_1(x)\binom{M-x}{M-q}$

[Proof]

Suppose $H_1(q)=\sum_{x=q}^{M}H_2(x)\binom{x}{q}=\sum_{x=0}^{M}H_2(x)\binom{x}{q}$, $y=T_{M+1}-M-1$.

$$\begin{aligned}
&\mathrm{A}: H_1(PS1,q)\\
&=\left(T_{M+1}-M-1+q\right)D_{M+1}H_1(q-1)+\left(K_{M+1}+qD_{M+1}\right)H_1(q)\\
&=(y+q)D_{M+1}\sum_{x=0}^{M}H_2(x)\left\{\binom{x+1}{q}-\binom{x}{q}\right\}+\left(K_{M+1}+qD_{M+1}\right)\sum_{k=0}^{M}H_2(x)\binom{x}{q}\\
&=(y+q)D_{M+1}\sum_{x=0}^{M}H_2(x)\binom{x+1}{q}+\left(K_{M+1}-yD_{M+1}\right)\sum_{k=0}^{M}H_2(x)\binom{x}{q}
\end{aligned}$$

$$\begin{aligned}
&\mathrm{B}: \sum_{x=0}^{M+1}H_2(PS1,x)\binom{x}{q}\\
&=\sum_{x=0}^{M+1}\left\{H_2(x-1)(y+x)D_{M+1}\binom{x}{q}+H_2(x)\left(K_{M+1}+(-y-1-x)D_{M+1}\right)\binom{x}{q}\right\}\\
&=\sum_{x=0}^{M}\left\{H_2(x)(y+x+1)D_{M+1}\binom{x+1}{q}+H_2(x)\left(K_{M+1}+(-y-1-x)D_{M+1}\right)\binom{x}{q}\right\}
\end{aligned}$$

$$\begin{aligned}
&[\mathrm{A}-\mathrm{B}]\\
&=\sum_{x=0}^{M}H_2(x)(q-1-x)D_{M+1}\binom{x+1}{q}+\sum_{x=0}^{M}H_2(x)(1+x)D_{M+1}\binom{x}{q}\\
&=D_{M+1}\sum_{x=0}^{M}H_2(x)q\binom{x+1}{q}-D_{M+1}\sum_{x=0}^{M}H_2(x)(1+x)\binom{x}{q-1}=0
\end{aligned}$$

$$\rightarrow H_1(q)=\sum_{x=q}^{M}H_2(x)\binom{x}{q}$$

$$\xrightarrow{\text{same way}}H_1(q)=\sum_{x=0}^{q}H_3(x)\binom{M-x}{M-q}\xrightarrow{\text{Inversion}} ②$$

q.e.d.

In particular:

2.2) ① $H_1(0)=H_3(0)=\sum_{x=0}^{M}H_2(x)=\prod_{i=1}^{M}K_i$;

② $H_1(M)=H_2(M)=\sum_{x=0}^{M}H_3(x)=\prod_{i=1}^{M}T_iD_i$;

③ $H_2(0)=(-1)^{M}H_3(M)=\sum_{x=0}^{M}(-1)^{x}H_1(x)$;

④ $H_1(1)=\sum_{x=1}^{M}H_2(x)x=MH_3(0)+H_3(1)$.

Calculation with 2.1):

2.3) $\sum_{q=0}^{M} H_1(q) = \sum_{q=0}^{M} H_2(q) 2^q = \sum_{q=0}^{M} H_3(q) 2^{M-q}$.

Use **2.1)** $\to \text{Form}_1 = \text{Form}_2 = \text{Form}_3$.

2.4) ① $\sum_{q=0}^{M} H_1(q)\binom{A}{B-q} = \sum_{q=0}^{M} H_2(q)\binom{A+q}{B} = \sum_{q=0}^{M} H_3(q)\binom{A+M-q}{B-q}$

② $\sum_{q=0}^{M} H_1(q)\binom{A}{q} = \sum_{q=0}^{M} H_2(q)\binom{A+q}{q} = \sum_{q=0}^{M} H_3(q)\binom{A+M-q}{M}$

2.5)

$$\sum_{q=0}^{M} H_1(q) q \binom{A+1}{B-q} = \sum_{q=0}^{M} H_2(q) q \binom{A+q}{B-1}$$

$$= \sum_{q=0}^{M} \left\{ H_3(q) q \binom{A+M-q}{B-q} + M H_3(q) \binom{A+M-q}{B-1-q} \right\}$$

[Proof]

Suppose it's holds when *M*, Let $y = T_{M+1} - M - 1$.

$$\text{A} : \sum_{q=0}^{M+1} H_1(PS1, q)\binom{A}{B-q}$$

$$= \sum_{q=0}^{M+1} \left\{ (y+q) D_{M+1} H_1(q-1) + (K_{M+1} + q D_{M+1}) H_1(q) \right\} \binom{A}{B-q}$$

$$= \sum_{q=0}^{M} \left\{ (y+q+1) D_{M+1} H_1(q) \binom{A}{B-1-q} + (K_{M+1} + q D_{M+1}) H_1(q) \binom{A}{B-q} \right\}$$

$$= (y+1) D_{M+1} \sum_{q=0}^{M} H_1(q) \binom{A}{B-1-q} + K_{M+1} \sum_{q=0}^{M} H_1(q) \binom{A}{B-q}$$

$$+ D_{M+1} \sum_{q=0}^{M} H_1(q) q \binom{A+1}{B-q}$$

$$\text{B} : \sum_{q=0}^{M+1} H_2(PS1, q)\binom{A+q}{B}$$

$$= \sum_{q=0}^{M+1} \left\{ (y+q) D_{M+1} H_2(q-1) + (K_{M+1} + (-y-1-q) D_{M+1}) H_2(q) \right\} \binom{A+q}{B}$$

$$= (y+1) D_{M+1} \sum_{q=0}^{M} H_2(q) \binom{A+q}{B-1} + K_{M+1} \sum_{q=0}^{M} H_2(q) \binom{A+q}{B}$$

$$+ D_{M+1} \sum_{q=0}^{M} H_2(q) q \binom{A+q}{B-1}$$

$$\xrightarrow{2.4)} \text{A} = \text{B}$$

$$\xrightarrow{2.4)} \sum H_1(q)\binom{A}{B-1-q} = \sum H_2(q)\binom{A+q}{B-1},$$

$$\sum H_1(q)\binom{A}{B-q} = \sum H_2(q)\binom{A+q}{B} \to \text{left}$$

q.e.d.

Example 2.1

M = 1:

$$\text{Form}_1 = T_1 D_1 \binom{A}{B-1} + K_1 \binom{A}{B};$$

$$\text{Form}_2 = T_1 D_1 \binom{A+1}{B} + (K_1 - T_1 D_1)\binom{A}{B};$$

$$\text{Form}_3 = (-K_1 + T_1 D_1)\binom{A}{B-1} + K_1 \binom{A+1}{B};$$

$M = 2$:

$$\text{Form}_1 = T_1 T_2 D_1 D_2 \binom{A}{B-2} + \left[K_1 (T_2 - 1) D_2 + T_1 D_1 (K_2 + D_2)\right]\binom{A}{B-1} + K_1 K_2 \binom{A}{B};$$

$$\text{Form}_2 = T_1 T_2 D_1 D_2 \binom{A+2}{B} + \left[(K_1 - T_1 D_1)(T_2 - 1) D_2 + T_1 D_1 (K_2 - T_2 D_2)\right]\binom{A+1}{B} + (K_1 - T_1 D_1)(K_2 - T_2 D_2 + D_2)\binom{A}{B};$$

$$\text{Form}_3 = (-K_1 + T_1 D_1)(-K_2 + T_2 D_2 - D_2)\binom{A}{B-2} + \left[K_1(-K_2 + T_2 D_2) + (-K_1 + T_1 D_1)(K_2 + D_2)\right]\binom{A+1}{B-1} + K_1 K_2 \binom{A+2}{B}.$$

3. Generalization of Calculation Formula

If $f(n) = \sum A_i \binom{N_i}{m_i}$, m_i is not changed with n, then define

$$\nabla^p f(n) = \sum A_i \binom{N_i - p}{m_i - p}, P \in \mathbb{Z}$$

$\nabla f(n) = f(n) - f(n-1)$, this is a little different from the difference

$$\nabla^0 f(n) = f(n), \nabla^{-1} f(N) = \sum_{n=0}^{N-1} f(n)$$

$$\text{Eg: } \nabla \binom{n+1}{n-1} = \nabla \binom{n+1}{2} = \binom{n}{1} \ne \binom{n}{n-2}.$$

In [1], 1.1) is proved by

$$(*)\quad \sum_{n=0}^{N-1} n \binom{n+K}{M} = (M+1)\binom{N+K}{M+2} + (M-K)\binom{N+K}{M+1}$$

$$SUM\left(N, PS1, [PT, T_M + 1]\right) = \sum_{n=0}^{N-1} (K_{M+1} + n \times D_{M+1}) \times \nabla SUM(n+1)$$

$$SUM\left(N, PS1, [PT, T_M + 2]\right) = \sum_{n=0}^{N-1} (K_{M+1} + n \times D_{M+1}) \times \nabla^0 SUM(n+1)$$

$$PS1 = [PS, K_{M+1} : D_{M+1}], PT1 = [PT, T_{M+1} = T_M + 2 - p].$$

$$\text{Define: } SUM(N, PS1, PT1) = \sum_{n=0}^{N-1} (K_{M+1} + n \times D_{M+1}) \times \nabla^p SUM(n+1)$$

$SUM(N, PS1, PT1)$ can be calculated using the same method of 1.1).

The Form: $(T_1 + K_1)(T_2 + K_2)\cdots(T_M + K_M) = \sum \prod_{i=1}^{M} X_M$, $X_i = T_i$ or K_i.

3.1) $q = X(T)$, $PM = PM(PT)$,

$SUM(N) =$

$$\xrightarrow{\text{Form}_1} \sum_{q=0}^{PM} H_1(q)\binom{N+T_M-PM}{N-1-q}, B_i = \begin{cases}(T_i - X_{K-1})D_i; X_i = T_i \\ K_i + X_{T-1}D_i; X_i = K_i\end{cases}$$

$$\xrightarrow{\text{Form}_2} \sum_{q=0}^{PM} H_2(q)\binom{N+T_M-PM+q}{N-1}, B_i = \begin{cases}(T_i - X_{K-1})D_i; X_i = T_i \\ K_i + (X_{K-1} - T_i)D_i; X_i = K_i\end{cases}$$

$$\xrightarrow{\text{Form}_3} \sum_{q=0}^{PM} H_3(q)\binom{N+T_M-q}{N-1-q}, B_i = \begin{cases}-K_i + (T_i - X_{T-1})D_i; X_i = T_i \\ K_i + X_{T-1}D_i; X_i = K_i\end{cases}$$

$$H(q) = H(PS, PT, q) = \sum_{\text{all of the } \prod X_i \text{ with } X(T)=q} \prod_{i=1}^{M} B_i .$$

[Proof]

$$SUM(N) \xrightarrow{\text{Form}_1} \sum_{q=0}^{M} H_1(q)\binom{N+T_M-M}{N-1-q} = \sum_{q=0}^{M} H_1(q)\binom{N+T_M-M}{T_M-M+1+q}$$

$$\sum_{n=0}^{N-1}(K_{M+1} + n \times D_{M+1}) \times \nabla^p SUM(n+1)$$

$$= \sum_{n=0}^{N-1}(K_{M+1} + n \times D_{M+1}) \times \sum_{q=0}^{M} H_1(q)\binom{n+1+T_M-M-p}{T_M-M+1+q-p} \xrightarrow{(*)}$$

$$= \sum_{q=0}^{M}(T_M - M + 2 + q - p)D_{M+1} \times H_1(q)\binom{N+1+T_M-M-p}{T_M-M+3+q-p}$$

$$+ \sum_{q=0}^{M}(K_{M+1} + qD_{M+1}) \times H_1(q)\binom{N+1+T_M-M-p}{T_M-M+2+q-p}$$

$$= \sum_{q=0}^{M}(T_{M+1} - [M-q])D_{M+1} \times H_1(q)\binom{N+T_{M+1}-(M+1)}{N-1-(q+1)}$$

$$+ \sum_{q=0}^{M}(K_{M+1} + qD_{M+1}) \times H_1(q)\binom{N+T_{M+1}-(M+1)}{N-1-q}$$

$$= \sum_{q=0}^{M+1} H_1(PS1, PT1, q)\binom{N+T_{M+1}-(M+1)}{N-1-q} \xrightarrow{2.4)} \text{three forms}$$

q.e.d.

3.2) $PS1 = [D \times A : D, PS]$, $PT1 = [A, PT]$

① $H_1(PS1, q) = D \times A \times [H_1(q) + H_1(q-1)]$;

② $H_2(PS1, 0) = 0, H_2(PS1, q) = D \times A \times H_2(q-1)$;

③ $H_3(PS1, M+1) = 0, H_3(PS1, q) = D \times A \times H_3(q)$;

④ $H_1([D \times T_1 : D, \cdots, D \times T_M : D], [T_1, \cdots, T_M], q) = D^M \binom{M}{q} \prod_{i=1}^{M} T_i$;

⑤ $SUM(N, PT, PT) = \prod_{i=1}^{M} T_i \binom{N+T_M}{T_M+1}$;

⑥ $SUM(N, [L_1, \cdots, L_P, PS], [L_1, \cdots, L_P, PT]) = \prod_{i=1}^{P} L_i SUM(N)$.

These are conclusions of [1] and can be extended to the new *PT*.

3.3) $PS1 = [1, 1, \cdots, 1, PS]$, $PT1 = [1, 1, \cdots, 1, PT]$,

$SUM(N, PS1, PT1) = SUM(N)$.

$P =$ Count of 1 added

$$\to \text{ expands } \sum_{q=0}^{M} H_1(q)\binom{A+P}{B-q} \text{ to } \sum_{q=0}^{M+P} H_1(PS1,q)\binom{A}{B-q}.$$

Now PT's domain is extended to $\mathbb{N}$ and $T_{i+1}-T_i$ is not restricted.

If $T_M \le 0$, $SUM(N)=\sum_{q=0}^{M} H_1(q)\binom{A}{B}$, $B<0$, the formula has no meaning when regardless of the actual meaning, $\text{Form}_1=\text{Form}_2=\text{Form}_3$ still established.

PT's domain can be extended to $\mathbb{C}$.

4. Properties of Coefficients

Define

$$F_M^{N+M-1}=\sum_{1\le I_i\le I_{i+1}\le N+M-1}\prod_{i=1}^{M} I_i = S_1(N+M,N)$$

$$E_M^N=\sum_{\lambda_1+\lambda_2+\cdots+\lambda_N=M} 1^{\lambda_1}2^{\lambda_2}\cdots N^{\lambda_N}=S_2(N+M,N)$$

$$E_p^{q+1}\odot(PT,K)=\sum_{\lambda_1+\lambda_2+\cdots+\lambda_{q+1}=p} 1^{\lambda_1}2^{\lambda_2}\cdots(q+1)^{\lambda_{q+1}}$$

$$\times(T_1+\lambda_1K)(T_2+\lambda_1K+\lambda_2K)\cdots(T_q+\lambda_1K+\lambda_2K+\cdots+\lambda_qK),\lambda_i\ge 0$$

$$PT \text{ in section 1: } T_1=1,\ \begin{cases} T_{i+1}-T_i=1: \text{means continuity} \\ T_{i+1}-T_i=2, \text{means discontinuity.} \end{cases}$$

[1] call them Basic Shapes and define:

$$PB(PT)=\text{count of discontinuity}, MIN(PT)=\prod_{i=1}^{M} T_iD_i\ .$$

Expand the definition:

$$PS=[K_1:D_1,K_2:D_2,\cdots],\text{Item}=\{K_1+\lambda_1D_1,K_2+\lambda_2D_2,\cdots\}$$

$$\text{Specify } \lambda_1=0,\ \begin{cases} \lambda_i=\lambda_{i+1}: \text{means continuity} \\ \lambda_i+1=\lambda_{i+1}, \text{means discontinuity} \end{cases}$$

PB (item of PS) = count of discontinuity in an item, value $\in[0,M-1]$.

$$MIN(\text{item of } PS)=\prod_{i=1}^{M}(K_i+\lambda_iD_i)$$

$$MIN_q(PS)=\sum_{PB(\ )=q} MIN(\text{item of } PS), MIN_q \text{ is short for } MIN_q([1,\cdots,M]).$$

By definition →

4.1) ①

$$H_1(q)=(-1)^{M-q}H_2([-K_i+T_iD_i-(i-1)D_i],PT)$$

$$=(-1)^q H_3\left(PS,\left[\frac{K_i}{D_i}-T_i+(i-1)\right]\right);$$

② $H_2(q)=(-1)^{M-q}H_1([-K_i+T_iD_i-(i-1)D_i],PT);$

③ $H_3(q)=(-1)^q H_1\left(PS,\left[\frac{K_i}{D_i}-T_i+(i-1)\right]\right).$

4.2) $PS=[1,1,\cdots],PT=[T_i=T_1+(i-1)(K+1)]$

① $H_1(q) = E_{M-q}^{q+1} \odot (PT, K)$;
② $H_3(q) = E_{M-q}^{q+1} \odot ([T_i = (T_1 - 1) + (i-1)K], K+1)$;
③ $E_{M-q}^{q+1} \odot (PT, K) = E_{M-1-q}^{q+1} \odot ([K + T_i], K) + T_1 E_{M-1-q}^{q+1} \odot ([K + T_i], K)$.

[Proof]

$$H_1(q) = \sum \prod (X \in T) \prod (X \in K) \xrightarrow{\text{def}} \sum \prod (X \in K) = E_{M-q}^{q+1}$$

$$\prod X = 1^{\lambda_1} \left\{ \prod_{x=1}^{A-1} X_{\lambda_1 + x} \right\} A^{\lambda_A} \left\{ \prod_{x=1}^{B-A} X_{\lambda_1 + A - 1 + \lambda_A + x} \right\} B^{\lambda_B} \cdots$$

$$\prod_{x=1}^{A-1} X_{\lambda_1 + x} = \prod_{x=1}^{A-1} (T_1 + [\lambda_1 + x - 1][K+1] - \lambda_1)$$

$$= \prod_{x=1}^{A-1} (T_X + \lambda_1 K) \xrightarrow{\lambda_2, \lambda_3, \cdots = 0} = (T_1 + \lambda_1 K) \cdots (T_{A-1} + \cdots + \lambda_{A-1} K)$$

$$\xrightarrow{\text{same way}} \prod_{x=1}^{B-A} X_{\lambda_1 + A - 1 + \lambda_A + x} = (T_A + \lambda_1 K + \cdots) \cdots (T_{B-1} + \cdots) \to ①$$

q.e.d.

4.3) $$\begin{cases} ① \ PS1 = [K_2 : D_2, \cdots, K_M : D_M], PT1 = \left[\frac{K_2}{D_2} + 1, \cdots, \frac{K_M}{D_M} + M - 1 \right] \\ ② \ PS1 = [D_2 : D_2, \cdots, D_M : D_M], PT1 = \left[\frac{K_2}{D_2} + 1, \cdots, \frac{K_M}{D_M} + M - 1 \right] \to \\ ③ \ PS1 = [K_2 : D_2, \cdots, K_M : D_M], PT1 = [-1, \cdots, -1] \end{cases}$$

$$\begin{cases} K_1 \times H_1(PS1, q) = MIN_q(PS), H_1\left(PS, \left[\frac{K_1}{D_1}, PT1 \right], q \right) = MIN_q + MIN_{q-1} \\ K_1 \times H_2(PS1, q) = (-1)^{M-1-q} MIN_q(PS) \\ K_1 \times H_3(PS1, q) = (-1)^q MIN_q(PS) \end{cases}$$

Example 4.1: $H_1(q), H_2(q), H_3(q)$ are equal to:

① $PS = [1, 1, \cdots], PT = [1, 1, \cdots]$ $$\begin{cases} \binom{M}{q} = E_{M-q}^{q+1} \odot (PT, -1) \\ H_2(M) = 1, H_2(q < M) = 0 \\ H_3(0) = 1, H_3(q > 0) = 0 \end{cases}$$

② $$PS = [1, 2, \cdots, M], PT = [1, 2, \cdots, M] = \begin{cases} M! \binom{M}{q} \\ H_2(M) = M!, H_2(q < M) = 0 \\ H_3(0) = M!, H_3(q > 0) = 0 \end{cases}$$

③ $PS = [1, 1, \cdots]$, $PT = [1, 2, \cdots, M]$,

$$\left\langle \begin{matrix} M \\ q \end{matrix} \right\rangle = \text{Eulerian number} : N^M = \sum_{q=0}^{M-1} \left\langle \begin{matrix} M \\ q \end{matrix} \right\rangle \binom{N+q}{M}$$

$$\begin{cases} q! E_{M-q}^{q+1} = q! S_2(M+1, q+1) \\ \xrightarrow{\text{def}} (-1)^{M-q} q! E_{M-q}^{q} = (-1)^{M-q} q! S_2(M, q) \\ \left\langle \begin{matrix} M \\ M-1-q \end{matrix} \right\rangle = \left\langle \begin{matrix} M \\ q \end{matrix} \right\rangle = E_{M-q}^{q+1} \odot ([0, 0, \cdots], 1) = E_{M-1-q}^{q+1} \odot ([1, 1, \cdots], 1) \end{cases}$$

$$\xrightarrow{2.2)} H_1(M) = \sum_{q=0}^{M} H_3(q) \to \sum_{q=0}^{M} \left\langle \begin{matrix} M \\ q \end{matrix} \right\rangle = M!$$

$$\xrightarrow{2.2)} H_1(1)=\sum_{q=0}^{M} H_2(q)q \to \sum_{q=0}^{M}(-1)^{M-q} q\times q!S_2(M,q)=2^M-1$$

④ $PS=[1,1,\cdots], PT=[2,3,\cdots,M]$

$$\begin{cases}(q+1)!E_{M-1-q}^{q+1}=(q+1)!S_2(M,q+1)\\(-1)^{M-1-q}(q+1)!E_{M-1-q}^{q+1}=(-1)^{M-1-q}(q+1)!S_2(M,q+1)\\\left\langle \begin{matrix} M \\ M-1-q\end{matrix}\right\rangle=\left\langle \begin{matrix} M \\ q\end{matrix}\right\rangle=E_{M-1-q}^{q+1}\odot([1,1,\cdots],1)\end{cases}$$

⑤ $$PS=[1,1,\cdots], PT=[1,3,\cdots,2M-1]=\begin{cases}E_{M-q}^{q+1}\odot(PT,1)\\(-1)^{M-q}MIN_{q-1}\\E_{M-q}^{q+1}\odot([0,1,2,\cdots],2)\end{cases}$$

⑥ $$PS=[1,1,\cdots], PT=[3,5,\cdots,2M-1]=\begin{cases}E_{M-1-q}^{q+1}\odot(PT,1)\\(-1)^{M-1-q}MIN_{q}\\E_{M-1-q}^{q+1}\odot([2,3,4,\cdots],2)\end{cases}$$

⑦ $PS=[1,2,\cdots,M]$,

$$PT=[1,3,\cdots,2M-1]=\begin{cases}MIN_q+MIN_{q-1}\\\xrightarrow{\text{def}}1\times(-1)^{M-q}E_{M-q}^{q}\odot([3,5,\cdots],1)\\\xrightarrow{\text{def}}1\times E_{q}^{M-q}\odot([2,3,4,\cdots],2)\end{cases}$$

⑧ $PS=[2,3,\cdots,M]$,

$$PT=[3,5,\cdots,2M-1]=\begin{cases}MIN_q\\\xrightarrow{\text{def}}(-1)^{M-1-q}E_{M-1-q}^{q+1}\odot([3,5,\cdots],1)\\\xrightarrow{\text{def}}E_{q}^{M-q}\odot([2,3,4,\cdots],2)\end{cases}$$

③ ④, ⑤ ⑥, ⑦ ⑧ are in pairs, they can verify 3.2).

5. Continuity and Discontinuity

MIN_q appears in $SUM(N,[1,1,\cdots],[3,5,\cdots])$ and $SUM(N,[2,3,\cdots],[3,5,\cdots])$. It's easy to write out their items directly by continuity and discontinuity.

[1] has proved: $\sum_{q=0}^{M-1}(-1)^{M-1-q}MIN_q=1$. Extand it:

$$\xrightarrow{2.2)} K_1\times H_1(0)=K_1\sum_{q=0}^{M-1}H_2(q)\xrightarrow{4.3)}=\sum_{q=0}^{M-1}(-1)^{M-1-q}MIN_q(PS)\to$$

5.1) $\sum_{q=0}^{M-1}(-1)^{M-1-q}MIN_q(PS)=K_1\prod_{i=2}^{M}D_i$

Example 5.1:

Basic Shape, $M=3$:

$$1\times3\times5-(1\times3\times4+1\times2\times4)+1\times2\times3=1.$$

Basic Shape, $M=4$:

$$1\times3\times5\times7-(1\times3\times5+1\times3\times4+1\times2\times4)\times6$$
$$+(1\times2\times3+1\times3\times4+1\times2\times4)\times5-1\times2\times3\times4=1.$$

$$PS=[5,10:10,2:3,3:10]:$$

$$5\times 20\times 8\times 33-(5\times 10\times 5+5\times 20\times 5+5\times 20\times 8)\times 23$$
$$+(5\times 10\times 2+5\times 10\times 5+5\times 20\times 5)\times 13-5\times 10\times 2\times 3$$
$$=1500=5\times 10\times 3\times 10$$

$$PS=[1,1,\cdots],PT=[2,\cdots,M]\to \sum_{q=1}^{M}(-1)^{M-q}q!S_2(M,q)=1.$$

5.2) ① $MIN_{M-2}=\frac{2(M-1)}{3}(2M-1)!!$

② $2\times MIN_{M-1}+MIN_{M-2}\equiv 0\ MOD\ (M+2)^2$, $M>3$, M is odd

[Proof]

$$PS=[2,3,\cdots,M],PT=[3,5,\cdots,2M-1]$$

$$H_1(M-2)=MIN_{M-2}=\sum_{i=1}^{M-1}\left(\prod X\in T\right)\times 2i$$

$$H_2(M-2)=-\sum_{i=1}^{M-1}\left(\prod X\in T\right)\times i=-\frac{1}{2}H_1(M-2)$$

$$\xrightarrow{2.1)}H_2(M-2)=-(M-1)H_1(M-1)+H_1(M-2)\to ①$$

$$2\times MIN_{M-1}+MIN_{M-2}=\frac{2(M+2)}{3}\times(2M-1)!!\to ②$$

q.e.d.

Example 5.2:

$$2\times(1\times 3\times 5)+(1\times 2\times 4+1\times 3\times 4)=50\equiv 0\ MOD\ 25$$

$$2\times(1\times 3\times 5\times 7\times 9)+(1\times 3\times 5\times 7+1\times 3\times 5\times 6+1\times 3\times 4\times 6+1\times 2\times 4\times 6)\times 8$$
$$=4410\equiv 0\ MOD\ 49$$

when PT is Basic Shape, items in SUM can be classified by continuity and discontinuity.

Eg: use A for continuity, B for discontinuity

$$[1,2,3]=\text{AA},[1,2,4]=\text{AB},[1,3,4]=\text{BA},[1,3,5]=\text{BB}.$$

Products of F_M^{N+M-1} can be divided into 2^{M-1} categories.

It's easy to write them intuitively. Eg: $1\times 2\times 4$, $2\times 3\times 5$, $1\times 2\times 5$, $\cdots$, $I_1+1=I_2$, $I_2+1<I_3\in[1,2,4]$.

Each category has a simple formula $\xrightarrow{3.2)-⑤}$

$$SUM(N-PB(PT),PT,PT)=MIN(PT)\binom{N+M}{T_M+1}.$$

This is the promotion of $\sum_{n=0}^{N-1}\binom{n}{M}=\binom{N}{M+1}$

$$\xrightarrow{\text{Traverse}}F_M^{N+M-1}=SUM(N,[2,3,\cdots],[3,5,\cdots])=\sum_{q=0}^{M-1}MIN_q\binom{N+M}{M+1+q}.$$

Similarly: for Basic PT, arbitrarily PS can use the classification.

Example 5.3:

$$SUM(N,[1,1,1],[1,3,5])$$
$$= SUM(N,[1,1,1],[1,2,3]) + SUM(N-1,[1,1,2],[1,2,4])$$
$$+ SUM(N-1,[1,2,2],[1,3,4]) + SUM(N-2,[1,2,3],[1,3,5])$$

$$SUM(N,[1,1,1],[1,3,4])$$
$$= SUM(N,[1,1,1],[1,2,3]) + SUM(N-1,[1,2,2],[1,3,4])$$

The pairs of PSx and PTx compare with PS and $[1,2,\cdots,M]$, $PB(PSx) = PB(PTx)$, and the discontinuity at the same position. They are called having the same shape.

5.3) For Basic PT, $PS1 = [1,1,\cdots,PS]$, $PT1 = [1,1,\cdots,PT]$, count of 1 added = $PB(PT)$

$$H_1(PS1,PT1,q) = \sum_{A+B=q,PB(PSx)=PB(PTx)=A,\text{same shape}} H_1(PSx,PTx,B).$$

P is Prime, $P > 2$, [1] has proved:

5.4) $MIN_q \equiv 0\ MOD\ P$, $q > 0$, $q + M = P - 1$

5.5) $MIN_q \equiv 0\ MOD\ P$, $\begin{cases} ① M = P, q \ge 0 \\ ② M = P-1, q > 0 \\ ③ M < P-1, q+M \ge P-1 \end{cases}$

[Proof]

For ③: $q + M = P - 1 \rightarrow$ proved by 5.4)

$$q + M = P : P = \text{Max factor of } MIN_q \rightarrow \text{holds}$$

$$q + M > P : MIN_q = (\text{Items has } P) + (\text{Items has no } P)$$

$$(\text{Items has no } P) = \sum\prod(\text{factors} \ge P+1) \times \left\{\sum\prod(\text{factors} \le P-1)\right\}$$

$\left\{\sum\prod(\text{factors} \le P-1)\right\}$ is a MIN that match the conditions of 5.4)

q.e.d.

5.6) $P = M + 2$, ① $E_M^N \equiv$ ② $F_M^{M+N-1} \equiv \begin{cases} 1, N \equiv 1\ MOD\ P \\ 0, N \not\equiv 1\ MOD\ P \end{cases} MOD\ P$

[Proof]

$$\xrightarrow{\text{Example 4.1-⑥}} E_M^N = \sum_{q=0}^{M-1} (-1)^{M-1-q} MIN_q \binom{N+M+q}{N-1} \xrightarrow{5.5)-③}$$

$$\equiv MIN_0 \binom{N+M}{N-1} \equiv (P-2)! \binom{N+P-2}{N-1} \equiv \binom{P+N-2}{N-1} MOD\ P \rightarrow ①$$

$$\xrightarrow{\text{Example 4.1-⑧}} F_M^{M+N-1} = \sum_{q=0}^{M-1} MIN_q \binom{N+M}{N-1-q}$$

$$\equiv MIN_0 \binom{N+P-2}{P-1} MOD\ P \rightarrow ②$$

q.e.d.

5.7) $P = M + N$, ① $F_M^{M+N-1} \equiv$ ② $E_M^N \equiv 0\ MOD\ P$, $0 < M < P - 1$

[Proof]

$$F_M^{M+N-1} = \sum_{q=0}^{M-1} MIN_q \binom{P}{M+1+q}$$

$$\begin{cases} 2M < P \rightarrow \binom{P}{M+1+q} \equiv 0 \ MOD \ P, 0 \le q \le M-1 \\ 2M > P, M+q \ge P-1 \xrightarrow{5.5)} MIN_q \equiv 0 \ MOD \ P \ \ \rightarrow ① \\ 2M > P, M+q < P-1 \rightarrow \binom{P}{M+1+q} \equiv 0 \ MOD \ P \end{cases}$$

q.e.d.

$$N+M=P \xrightarrow{\text{def}} F_M^{P-1} = (P-1) F_{M-1}^{P-2} + F_M^{P-2}; E_M^N = N E_{M-1}^N + E_M^{N-1} \rightarrow$$

(1) $S_1(P,N) = (P-1) S_1(P-1,N) + S_1(P-1,N-1)$

(2) $S_2(P,N) = N S_2(P-1,N) + S_2(P-1,N-1)$

5.8) ① $S_1(P-1,q) \equiv 1 \ MOD \ P, 1 \le q \le P-1$

② $q! S_2(P-1,q) \equiv (-1)^{q+1} \ MOD \ P, 1 \le q \le P-1$

[Proof]

For ①: $q=1$, $S_1(P-1,q)=1$, holds.

If q holds, $S_1(P,q+1) \equiv 0 \ MOD \ P$

$$S_1(P,q+1) = (P-1) S_1(P-1,q+1) + S_1(P-1,q)$$
$$\equiv -S_1(P-1,q+1) + 1 \equiv 0 \ MOD \ P \rightarrow ①$$

For ②: $q=1$, $q! S_2(P-1,q) = 1$, holds.

If q holds, $S_2(P,q+1) \equiv 0 \ MOD \ P$, $q! S_2(P,q+1) \equiv 0 \ MOD \ P$

$$q! S_2(P,q+1) = (q+1)! S_2(P-1,q+1) + q! S_2(P-1,q) \equiv 0 \ MOD \ P \rightarrow ②$$

q.e.d.

Chart of 5.6), $F_M^{M+N-1} = S_1(N+M,N)$, $E_M^N = S_2(N+M,N)$

M \ k	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	1	1							
3	0	2	3	1						
4	0	6	11	6	1					
5	0	24	50	35	10	1				
6	0	120	274	225	85	15	1			
7	0	720	1764	1624	735	175	21	1		
8	0	5040	13068	13132	6769	1960	322	28	1	
9	0	40320	109584	118124	67284	22449	4536	546	36	1

M \ k	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	1	1							
3	0	1	3	1						
4	0	1	7	6	1					
5	0	1	15	25	10	1				
6	0	1	31	90	65	15	1			
7	0	1	63	301	350	140	21	1		
8	0	1	127	966	1701	1050	266	28	1	
9	0	1	255	3025	7770	6951	2646	462	36	1

5.9) $\sum_{n=1}^{P-1} n^{P-2} \equiv 0 \ MOD \ P^2, P > 3$

[Proof]

[1] has obtained this, but its proof is incorrect.

$$P^{P-2}+\sum_{n=1}^{P-1}n^{P-2}=\sum_{n=1}^{P-1}n^{P-2}=SUM\left(P,[1,\cdots,1],[1,\cdots,P-2]\right)$$

$$=\sum_{q=0}^{P-2}q!S_2\left(P-1,q+1\right)\binom{P}{q+1}=\sum_{q=1}^{P-1}(q-1)!S_2\left(P-1,q\right)\binom{P}{q}$$

$$=\sum_{q=1}^{\frac{P-1}{2}}\left\{(q-1)!S_2\left(P-1,q\right)\binom{P}{q}+(p-q-1)!S_2\left(P-1,p-q\right)\binom{P}{p-q}\right\}$$

$$\xrightarrow{5.8)-②}\equiv\sum_{q=1}^{\frac{P-1}{2}}\left[(-1)^{q+1}+(-1)^{P-q+1}\right]\binom{P}{q}$$, this step of [1] is wrong

$$\sum_{n=1}^{P}n^{P-2}=\sum_{q=1}^{P-1}(q-1)!S_2\left(P-1,q\right)\binom{P}{q}$$

$$=p\sum_{q=1}^{P-1}\frac{q!S_2\left(P-1,q\right)}{q^2}\binom{P-1}{q-1}\xrightarrow{5.8)-②}$$

$$(q-1)!S_2\left(P-1,q\right)\binom{P}{q}\in\mathbb{N}\rightarrow\frac{q!S_2\left(P-1,q\right)}{q^2}\binom{P-1}{q-1}\in\mathbb{N}$$

$$\frac{\sum_{n=1}^{P}n^{P-2}}{p}\equiv\sum_{q=1}^{P-1}\frac{(-1)^{q+1}}{q^2}\binom{P-1}{q-1}\xrightarrow{\binom{P-1}{q-1}\equiv(-1)^{q-1}}$$

$$\equiv\sum_{q=1}^{P-1}\frac{1}{q^2}\equiv\sum_{q=1}^{P-1}q^2\equiv 0\ MOD\ P$$

q.e.d.

6. Coefficient Matrix

$$SUM\left(N,PS,PT\right)=\sum_{q=0}^{M}H_1\left(q\right)\binom{N+T_M-M}{N-1-q}=\sum_{q=0}^{M}H_1\left(q\right)\binom{N+T_M-M}{T_M-M+1+q}$$

N starts from x to $x+M$, taking $H_1(q)$ as variables, then get a linear equations.

Let $P=x+T_M-M, Q=x-1$, each row from left to right, Q is from small to large

$$\xrightarrow{\text{Form}_1}A_1\left(P,Q,M\right)=\begin{bmatrix}\binom{P}{Q} & \cdots & \binom{P}{Q-M}\\ \vdots & \ddots & \vdots\\ \binom{P+M}{Q+M} & \cdots & \binom{P+M}{Q}\end{bmatrix}$$

$$\xrightarrow{\text{Form}_2}A_2\left(P,Q,M\right)=\begin{bmatrix}\binom{P}{Q} & \cdots & \binom{P+M}{Q}\\ \vdots & \ddots & \vdots\\ \binom{P+M}{Q+M} & \cdots & \binom{P+2M}{Q+M}\end{bmatrix}$$

$$\xrightarrow{\text{Form}_3}A_3\left(P,Q,M\right)=\begin{bmatrix}\binom{P+M}{Q} & \cdots & \binom{P}{Q-M}\\ \vdots & \ddots & \vdots\\ \binom{P+2M}{Q+M} & \cdots & \binom{P+M}{Q}\end{bmatrix}$$

They are $(M+1)\times(M+1)$ matrices

6.1) ① $\|A_2(P,Q,M)\| = \dfrac{\binom{P}{Q}\binom{P+2}{Q+1}\binom{P+4}{Q+2}\cdots\binom{P+2M}{Q+M}}{\binom{P}{0}\binom{P+2}{1}\binom{P+4}{2}\cdots\binom{P+2M}{M}} = \|A_2(P,P-Q,M)\|$

② Upper triangle: col_q of $A_2(1,0,M) = \left[\binom{q}{0},\binom{q}{1},\binom{q}{2},\cdots,\binom{q}{q},\cdots\right]^{\mathrm{T}}$

[Proof]

$$\mathrm{Row}_{M+1} := \mathrm{Row}_{M+1} - \mathrm{Row}_M \frac{P+M}{Q+M}$$

$$= \left[\begin{matrix} 0 & \dfrac{\binom{P+M+1}{Q+M}}{P+M+1} & \dfrac{2\binom{P+M+2}{Q+M}}{P+M+2} & \cdots & \dfrac{M\binom{P+2M}{Q+M}}{P+2M} \end{matrix}\right]$$

$$\mathrm{Row}_M := \mathrm{Row}_M - \mathrm{Row}_{M-1} \frac{P+M-1}{Q+M-1}$$

$$= \left[\begin{matrix} 0 & \dfrac{\binom{P+M}{Q+M-1}}{P+M} & \dfrac{2\binom{P+M+1}{Q+M-1}}{P+M+1} & \cdots & \dfrac{M\binom{P+2M-1}{Q+M-1}}{P+2M-1} \end{matrix}\right]$$

$$\cdots$$

$$\mathrm{Row}_2 := \mathrm{Row}_2 - \mathrm{Row}_1 \frac{P+1}{Q+1} = \left[\begin{matrix} 0 & \dfrac{\binom{P+2}{Q+1}}{P+2} & \dfrac{2\binom{P+3}{Q+1}}{P+3} & \cdots & \dfrac{M\binom{P+M+1}{Q+1}}{P+M+1} \end{matrix}\right]$$

Repeat the above process and change it into upper triangle.

$$\frac{\binom{P}{Q}\binom{P+1}{Q}\binom{P+2}{Q}\binom{P+3}{Q}\cdots\binom{P+M}{Q}}{\binom{P}{0}\binom{P+1}{0}\binom{P+2}{0}\binom{P+3}{0}\cdots\binom{P+M}{0}}$$

$$\frac{\binom{P+2}{Q+1}}{\frac{p+2}{1}}\frac{\binom{P+3}{Q+1}}{\frac{p+3}{2}}\frac{\binom{P+4}{Q+1}}{\frac{p+4}{3}}\cdots\frac{\binom{P+M+1}{Q+1}}{\frac{p+M+1}{M}}$$

$$\frac{\binom{P+4}{Q+2}}{\frac{(p+4)(p+3)}{2!}}\frac{\binom{P+5}{Q+2}}{\frac{(p+5)(p+4)}{3\times 2}}\cdots\frac{\binom{P+M+2}{Q+2}}{\frac{(p+M+2)(p+M+1)}{M(M-1)}}$$

$$\cdots$$

when the original matrix is transformed into an upper triangular matrix,

$$\text{Row}_{K+1} := \text{Row}_{K+1} - \text{Row}_K \frac{P+K}{Q+K}, \text{ repeat the operation } K \text{ times.}$$

q.e.d.

6.2) $\left\|A_1(P,Q,M)\right\| = \left\|A_2(P,Q,M)\right\| = \left\|A_3(P,Q,M)\right\|$

[Proof]

$$A_2 \xrightarrow{\text{Row}_{i+1}:=\text{Row}_{i+1}-\text{Row}_i \text{ and repeat}} \text{change}(\text{row}_1, \text{col}_1) \text{ to } \binom{P}{Q}$$

$$\begin{bmatrix} \binom{P}{Q} & \cdots & \binom{P+M}{Q} \\ \vdots & \ddots & \vdots \\ \binom{P}{Q+M} & \cdots & \binom{P+M}{Q+M} \end{bmatrix} \xrightarrow{\text{transpose}}$$

$$\begin{bmatrix} \binom{P}{Q} & \cdots & \binom{P}{Q+M} \\ \vdots & \ddots & \vdots \\ \binom{P+M}{Q} & \cdots & \binom{P+M}{Q+M} \end{bmatrix} = A_1(P, P-Q, M)$$

$$A_1 \xrightarrow{\text{col}_i:=\text{col}_i+\text{col}_{i+1} \text{ and repeat}\rightarrow} \text{change}(\text{row}_1, \text{col}_1) \text{ to } \binom{P+M}{Q} = A_3$$

q.e.d.

Use $\|A\|$ for $\|A_{1,2,3}\|$.

6.3) $\left\|A(P,0,M)\right\| = 1$,

$$\left\|A(P,1,M)\right\| = \binom{P+M}{1+M}, \quad \left\|A(P,Q>0,M)\right\| = \prod_{q=0}^{Q-1} \frac{\binom{P+M-q}{1+M}}{\binom{1+M+q}{1+M}}$$

[Proof]

$$\xrightarrow{6.1)} \left\|A(P,1,M)\right\| = \frac{P}{1}\frac{P+1}{2}\frac{P+2}{3}\cdots\frac{P+M}{M+1} = \binom{P+M}{1+M}$$

q.e.d.

$T_M \ge M$, $N \in [1, M+1]$, then

Matrix of $SUM(N) = A(1+T_M-M, 0, M)$, Matrix of $\nabla SUM(N) = A(T_M - M, 0, M)$

Use Cramer's law, let $y(n) = SUM(N)$ or $\nabla SUM(N)$

$$H_1(q) = \left\|A_1(P,0,M) \xrightarrow{\text{replace col}_{q+1}} \left[y(1), \cdots, y(M+1)\right]^{\text{T}}\right\|$$

$$A_1(P,0,M) = \begin{bmatrix} \binom{P}{0} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \binom{P+M}{M} & \cdots & \binom{P+M}{0} \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \binom{P+M}{M} & \cdots & 1 \end{bmatrix}$$

when col_{q+1} replace with $\left[y(1),\cdots\right]^{\mathrm{T}}$, calculate from col_{q+1}, only $\left\{y(1),\cdots,y(q+1)\right\}$ work.

From algebraic cofactor, $y(k)$ corresponds to $1^{K-1}\times\left\|A_{Tmp}\right\|\times 1^{M-q}$ count of rows of $A_{Tmp}=(M+1)-(K-1)-(M-q)-1=q-K+1$

$$\left(\text{row}_k,\text{col}_{q+1}\right)=\binom{P+K-1}{K-1-q},(q+1)-(q-k+1)-1=k-1$$

$$\left(\text{row}_0,\text{col}_0\right)\text{ of }A_{Tmp}=\left(\text{row}_{k+1},\text{col}_{k-1}\right)=\binom{P+K}{1}$$

$$\left\|A_{Tmp}\right\|=\left\|A_1(P+k,1,q-k)\right\|=\binom{P+q}{q+1-k}\to$$

6.4) $T_M\ge M$, $H_1(q)=\begin{cases}\sum_{k=1}^{q+1}(-1)^{q+1+k}\binom{1+T_M-M+q}{q+1-k}SUM(k)\\ \sum_{k=1}^{q+1}(-1)^{q+1+k}\binom{T_M-M+q}{q+1-k}\nabla SUM(k)\end{cases}$

$$\nabla SUM\left(N,[1,\cdots,1],[2,\cdots,M]\right)=N^M\to A_1(1,0,M-1)$$

$$(q+1)!S_2(M,q+1)=\sum_{k=1}^{q+1}(-1)^{q+1+k}\binom{q+1}{q+1-k}k^M$$

$$\to q!S_2(M,q)=\sum_{k=0}^{q}(-1)^{q+k}\binom{q}{q-k}k^M$$

$$x=q-k\to q!S_2(M,q)=\sum_{x=0}^{q}(-1)^x\binom{q}{x}(q-x)^M,$$

this is a known formula.

$$SUM\left(N,[1,\cdots,M],[1,\cdots,M]\right)=\binom{M+N}{M+1}$$

$$\to\binom{M}{q}=\sum_{k=0}^{q}(-1)^{k+q}\binom{q}{k}\binom{M+k}{M+1}$$

$$SUM\left(N,[2,3,\cdots,M],[3,5,\cdots,2M-1]\right)$$

$$\to MIN_{q-1}=\sum_{k=0}^{q}(-1)^{k+q}\binom{M+q}{q-k}S_1(k+M,k)$$

Similarly,

$$A_3(P,0,M)=\begin{bmatrix}\binom{P+M}{0}&\cdots&0\\ \vdots&\ddots&\vdots\\ \binom{P+2M}{M}&\cdots&\binom{P+M}{0}\end{bmatrix}=\begin{bmatrix}1&\cdots&0\\ \vdots&\ddots&\vdots\\ \binom{P+2M}{M}&\cdots&1\end{bmatrix}\to$$

$y(k)$ corresponds to $1^{K-1}\times\left\|A_{Tmp}\right\|\times 1^{M-q}$, count of rows of $A_{Tmp}=q-k+1$

$$\left(\text{row}_k,\text{col}_{q+1}\right)=\binom{P+M+K-1-q}{K-1-q}$$

$$\left(\text{row}_0,\text{col}_0\right)\text{ of }A_{Tmp}=\left(\text{row}_{k+1},\text{col}_{k-1}\right)=\binom{P+M+1}{1}$$

$$P+M+1-(q-k)=P+M+1-q+k$$

$$\left\|A_{Tmp}\right\|=\left\|A_3\left(P+M+1-q+K,1,q-k\right)\right\|=\binom{P+M+1}{q+1-k}\rightarrow$$

6.5) $T_M\geq M$, $H_3(q)=\begin{cases}\sum_{k=1}^{q+1}(-1)^{q+1+k}\binom{2+T_M}{q+1-k}SUM(k)\\ \sum_{k=1}^{q+1}(-1)^{q+1+k}\binom{1+T_M}{q+1-k}\nabla SUM(k)\end{cases}$

$$\nabla SUM\left(N,[1,\cdots,1],[2,\cdots,M]\right)=N^M\rightarrow A_3(1,0,M-1)\rightarrow$$

$$\left\langle{M \atop q}\right\rangle=\sum_{k=1}^{q+1}(-1)^{q+k-1}\binom{M+1}{q+1-k}k^M$$

$$\xrightarrow{x=q+1-k}\sum_{x=0}^{q}(-1)^x\binom{M+1}{x}(q+1-x)^M$$

This is a known formula too.

$$\xrightarrow{6.1)}A_2(1,0,M)=\begin{bmatrix}\binom{0}{0}&\cdots&\binom{M}{0}\\ \vdots&\ddots&\vdots\\ 0&\cdots&\binom{M}{M}\end{bmatrix}$$

In the algebraic cofactor, $\left(\text{row}_{q+1},\text{row}_{q+2},\cdots,\text{row}_{M+1}\right)$ will work.

Row_K corresponds to $1^q\times\left\|A_{Tmp}\right\|\times 1^{M+1-K}$ count of rows of $A_{Tmp}=(M+1)-q-(M+1-k)-1=k-q-1$

$$\left(\text{row}_k,\text{col}_{q+1}\right)=\binom{q}{k-1},k-(k-q-1)=q+1$$

$$\left(\text{row}_0,\text{col}_0\right)\text{ of }A_{Tmp}=\left(\text{row}_{q+1},\text{col}_{q+2}\right)=\binom{q+1}{q}$$

$$\left\|A_{Tmp}\right\|=\left\|A_2\left(q+1,q,k-q-2\right)\right\|=\left\|A_2\left(q+1,1,k-q-2\right)\right\|=\binom{k-1}{q}$$

It can be concluded by induction:

$$\xrightarrow{6.1)}y(k)\text{ will change to }z(k)=\sum_{x=1}^{k}(-1)^{k-x}y(x)\binom{K}{x}$$

$$H_2(q)=\sum_{k=q+1}^{M+1}(-1)^{q+k-1}\binom{k-1}{q}z(k)$$

$$\nabla SUM\left(N,[1,\cdots,1],[2,\cdots,M]\right)=N^M\rightarrow Z(k)=k!S_2(M,k)\rightarrow$$

$q!S_2(M,q)=\sum_{k=q}^{M}(-1)^{M+k}\binom{k-1}{q-1}k!S_2(M,k)$, this matches 2.1)-②

7. $\sum_{n=0}^{N-1}\prod_{i=1}^{M}\left(K_i+D_iq^n\right)$

We need an expression similar to $\begin{pmatrix}N\\M\end{pmatrix}$, which is Gaussian coefficient $\begin{bmatrix}N\\M\end{bmatrix}_q$

$$\begin{bmatrix}N\\M\end{bmatrix}_q=\frac{\left(q^N-1\right)\left(q^{N-1}-1\right)\cdots\left(q^{N-(M-1)}-1\right)}{\left(q^M-1\right)\left(q^{M-1}-1\right)\cdots(q-1)},\ q\neq 1,\begin{bmatrix}N\\0\end{bmatrix}_q=1,\text{ Abbreviated as }\begin{bmatrix}N\\M\end{bmatrix}$$

1) $\begin{bmatrix}N\\M\end{bmatrix}=\begin{bmatrix}N\\M-q\end{bmatrix}$

2) $\begin{bmatrix}N\\M\end{bmatrix}=\begin{bmatrix}N-1\\M-1\end{bmatrix}+q^M\begin{bmatrix}N-1\\M\end{bmatrix}$

7.1) ① $\sum_{n=0}^{N-1}q^n\begin{bmatrix}n+M\\M\end{bmatrix}=\begin{bmatrix}N+M\\M+1\end{bmatrix}$

② $\sum_{n=0}^{N-1}q^n\begin{bmatrix}n+K\\M\end{bmatrix}=q^{M-K}\begin{bmatrix}N+K\\M+1\end{bmatrix};\sum_{n=0}^{N-1}q^n\begin{bmatrix}n\\M\end{bmatrix}=q^M\begin{bmatrix}N\\M+1\end{bmatrix}$

[Proof]

When $n = 0$, ① is obviously true. Suppose it holds when $N-1$,

$$\sum_{n=0}^{N}q^n\begin{bmatrix}n+M\\M\end{bmatrix}=\begin{bmatrix}N+M\\M+1\end{bmatrix}+q^N\begin{bmatrix}N+M\\M\end{bmatrix}$$

$$=\frac{\left(q^{N+M}-1\right)\left(q^{N+M-1}-1\right)\cdots\left(q^N-1\right)}{\left(q^{M+1}-1\right)\left(q^M-1\right)\left(q^{M-1}-1\right)\cdots(q-1)}+q^N\frac{\left(q^{N+M}-1\right)\left(q^{N+M-1}-1\right)\cdots\left(q^{N+1}-1\right)}{\left(q^M-1\right)\left(q^{M-1}-1\right)\cdots(q-1)}$$

$$=\begin{bmatrix}N+M+1\\M+1\end{bmatrix}$$

$$\sum_{n=0}^{N-1}q^n\begin{bmatrix}n+K\\M\end{bmatrix}\xrightarrow{x=n+K-M}q^{M-K}\sum_{x=0}^{N-1+K-M}q^x\begin{bmatrix}x+M\\M\end{bmatrix}=q^{M-K}\begin{bmatrix}N+K\\M+1\end{bmatrix}$$

q.e.d.

$$\sum_{n=0}^{N-1}q^n=\frac{q^N-1}{q-1}=\begin{bmatrix}N\\1\end{bmatrix}$$

$$\sum_{n=0}^{N-1}q^n\left(K_1+D_1q^n\right)=\sum_{n=0}^{N-1}\left(q^n\left(q^n-1\right)D_1+\left(K_1+D_1\right)q^n\right)$$

$$=\sum_{n=0}^{N-1}(q-1)q^n\begin{bmatrix}N\\1\end{bmatrix}D_1+\left(K_1+D_1\right)q^n\begin{bmatrix}N\\0\end{bmatrix}$$

$$=q(q-1)D_1\begin{bmatrix}N\\2\end{bmatrix}+\left(K_1+D_1\right)\begin{bmatrix}N\\1\end{bmatrix}$$

$$\sum_{n=0}^{N-1}q^n\left(K_1+D_1q^n\right)\left(K_2+D_2q^n\right)$$

$$=\sum_{n=0}^{N-1}\left(D_1D_2q^{3n}+\left(K_1D_2+K_2D_1\right)q^{2n}+K_1K_2q^n\right)$$

$$=\sum_{n=0}^{N-1}\left(D_1D_2q^{2n}\left(q^n-1\right)+\left(K_1D_2+K_2D_1+D_1D_2\right)q^{2n}+K_1K_2q^n\right)$$

$$=\sum_{n=0}^{N-1}\left[D_1D_2q^{n+1}\left(q^n-1\right)\left(q^{n-1}-1\right)+\left(K_1D_2+K_2D_1+D_1D_2\right.\right.$$

$$\left.\left.+qD_1D_2\right)q^n\left(q^n-1\right)+\left(K_1K_2+K_1D_2+K_2D_1+D_1D_2\right)q^n\right]$$

$$= D_1 D_2 q^3 \left(q^2-1\right)(q-1)\begin{bmatrix} N \\ 3 \end{bmatrix} + \left(K_1 D_2 + K_2 D_1 + D_1 D_2\right.$$
$$\left. + qD_1 D_2\right) q(q-1)\begin{bmatrix} N \\ 2 \end{bmatrix} + \left(K_1 K_2 + K_1 D_2 + K_2 D_1 + D_1 D_2\right)\begin{bmatrix} N \\ 1 \end{bmatrix}$$
$$= q(q-1) D_1 \times q^2\left(q^2-1\right) D_2 \begin{bmatrix} N \\ 3 \end{bmatrix} + q(q-1)\left\{\left(K_1 + D_1\right) D_2\right.$$
$$\left. + D_1\left(K_2 + qD_2\right)\right\}\begin{bmatrix} N \\ 2 \end{bmatrix} + \left(K_1 + D_1\right)\left(K_2 + D_2\right)\begin{bmatrix} N \\ 1 \end{bmatrix}$$

T_i is arbitrary, use the Form $\prod_{i=1}^{M}\left(K_i + T_i\right)$, X_T = count of $\left\{X_1, \cdots, X_i\right\} \in T$

7.2) $\sum_{n=0}^{N-1} q^n \prod_{i=1}^{M}\left(K_i + D_i q^n\right) = \sum_{g=0}^{M} G(M,g)\begin{bmatrix} N \\ g+1 \end{bmatrix}$

$$G(M,g) = \sum_{\prod X_i \text{ with } X(T)=g} \prod_{i=1}^{M} f\left(X_i\right)$$
$$f\left(X_i\right) = \begin{cases} q^{X_T}\left(q^{X_T}-1\right) D_i ; X \in T \\ K_i + q^{X_{T-1}} D_i ; X \in K \end{cases}$$

[Proof]

When M = 1, 2, it's true. Let $G(q) = G(M,q)$

Suppose $F(N) = \sum_{n=0}^{N-1} q^n \prod_{i=1}^{M}\left(K_i + D_i q^n\right) = \sum_{g=0}^{M} G(g)\begin{bmatrix} N \\ g+1 \end{bmatrix} \rightarrow$

$$q^n \prod_{i=1}^{M}\left(K_i + D_i q^n\right) = F(n+1) - F(n)$$
$$= \sum_{g=0}^{M} G(g)\left\{\begin{bmatrix} n+1 \\ g+1 \end{bmatrix} - \begin{bmatrix} n \\ g+1 \end{bmatrix}\right\}$$
$$= \sum_{g=0}^{M} G(g)\left\{\left(q^{g+1}-1\right)\begin{bmatrix} n \\ g+1 \end{bmatrix} + \begin{bmatrix} n \\ g \end{bmatrix}\right\}$$

$$\sum_{n=0}^{N-1} q^n \prod_{i=1}^{M+1}\left(K_i + D_i q^n\right)$$
$$= K_{M+1} \sum_{n=0}^{N-1} q^n \prod_{i=1}^{M}\left(K_i + D_i q^n\right) + D_{M+1} \sum_{n=0}^{N-1} q^n \prod_{i=1}^{M+1}\left(K_i + D_i q^n\right) q^n$$
$$= K_{M+1} \sum_{g=0}^{M} G(g)\begin{bmatrix} N \\ g+1 \end{bmatrix} + D_{M+1} \sum_{g=0}^{M} G(g) \sum_{n=0}^{N-1} q^n \left\{\left(q^{g+1}-1\right)\begin{bmatrix} n \\ g+1 \end{bmatrix} + \begin{bmatrix} n \\ g \end{bmatrix}\right\}$$
$$\xrightarrow{7.1)} \sum_{g=0}^{M} G(g)\left(K_{M+1} + q^g D_{M+1}\right)\begin{bmatrix} N \\ g+1 \end{bmatrix}$$
$$+ \sum_{g=0}^{M} G(g) q^{g+1}\left(q^{g+1}-1\right) D_{M+1}\begin{bmatrix} N \\ g+2 \end{bmatrix}$$
$$= \sum_{g=0}^{M+1} G(M+1, g)\begin{bmatrix} N \\ g+1 \end{bmatrix}$$

q.e.d.

In the same way, use the Form = $\left(T_1 + K_1\right) \cdots \left(T_M + K_M\right)$:

7.3) $\sum_{n=0}^{N-1} \prod_{i=1}^{M}\left(K_i + D_i q^n\right) = \sum_{g=1}^{M} G(M,g)\begin{bmatrix} N \\ g \end{bmatrix} + N \prod_{i=1}^{M} K_i$

$$G(M,g)=\sum_{\prod X_i \text{ with } X(T)=g}\prod_{i=1}^{M} f(X_i)$$

$$f(X_i)=\begin{cases}1;X\in T,X_{T-1}=0\\ q^{X_{T-1}}\left(q^{X_{T-1}}-1\right)D_i;X\in T,X_{T-1}>0\\ K_i;X\in K,X_{T-1}=0\\ K_i+q^{X_{T-1}-1}D_i;X\in K,X_{T-1}>0\end{cases}$$

7.4) $\dfrac{q^{MN}-1}{q^{M}-1}=\sum_{g=1}^{M}\left(\prod_{i=1}^{g-1}q^{i}\left(q^{i}-1\right)\right)\begin{bmatrix}N\\g\end{bmatrix}\begin{bmatrix}M-1\\M-g\end{bmatrix}$

[Proof]

$$\sum_{n=0}^{N-1}\prod_{i=1}^{M}\left(0+q^{n}\right)=\sum_{n=0}^{N-1}q^{Mn}=\frac{q^{MN}-1}{q^{M}-1}=\sum_{g=1}^{M}G(M,g)\begin{bmatrix}N\\g\end{bmatrix}$$

In $G(M,g)$, $\prod(X\in T)=\prod_{i=1}^{g-1}q^{i}\left(q^{i}-1\right)$, X_1 must be T_1, count of $(X\in K)=M-g$, $M-1$ positions can be placed.

In 1916 MacMahon [6] observed that $\begin{bmatrix}N\\K\end{bmatrix}=\sum_{w\in\Omega(N,K)}q^{inv(w)}$, $\Omega(N,K)$ denotes all permutations of the multiset $\left\{0^{N-K},1^{K}\right\}$, that is, all words $w=w_1,\cdots,w_n$ with n - k zeroes and k ones, and inv(·) denotes the inversion statistic defined by $inv(w_1,\cdots,w_n)=\left|\left\{(i,j):1\le i<j\le n,w_i>w_j\right\}\right|$.

So in $G(M,g)$, $\sum\prod(X\in K)=\begin{bmatrix}M-1\\M-g\end{bmatrix}$

q.e.d.

7.5) ① $(K+D)^{M}=K^{M}+D\sum_{g=0}^{M-1}k^{g}(K+D)^{M-g}$

② $(K+q)^{M}=K^{M}+q\sum_{g=0}^{M-1}k^{g}(K+1)^{M-1-g}$
$+q(q-1)\sum_{a+b+c=M-2,a,b,c\ge0}k^{a}(K+1)^{b}(K+q)^{c}$

[Proof]

$$\sum_{n=0}^{0}\prod_{i=1}^{M}\left(K+Dq^{n}\right)=K^{M}+G(M,1)\begin{bmatrix}1\\1\end{bmatrix}\to①$$

$$\begin{aligned}&\sum_{n=0}^{1}\prod_{i=1}^{M}\left(K+q^{n}\right)-\sum_{n=0}^{0}\prod_{i=1}^{M}\left(K+q^{n}\right)\\&=2K^{M}+G(M,1)\begin{bmatrix}2\\1\end{bmatrix}+G(M,2)\begin{bmatrix}2\\2\end{bmatrix}-\left(K^{M}+G(M,1)\begin{bmatrix}1\\1\end{bmatrix}\right)\\&=K^{M}+G(M,1)\frac{\left(q^{2}-1\right)}{(q-1)}+G(M,2)-\left(K^{M}+G(M,1)\right)\to②\end{aligned}$$

q.e.d.

7.2) can be understood as use the Form = $(T_1+K_1)\cdots(T_M+K_M)$, $PT=[1,2,\cdots,M]$

$$f(X_i)=\begin{cases}q^{T_i-X_{K-1}}\left(q^{T_i-X_{K-1}}-1\right)D_i;X_i=T_i\\ K_i+q^{X_{T-1}}D_i;X_i=K_i\end{cases}$$

But it can not be simply extended to something like 3.1).

8. $\sum_{n=0}^{N-1} q^n \binom{n+M}{M}$

8.1) $\sum_{n=0}^{N-1} q^n \binom{n+M}{M} = q^N \sum_{g=0}^{M} (-1)^g \frac{\binom{N+M-1-g}{M-g}}{(q-1)^{g+1}} + \frac{(-1)^{M+1}}{(q-1)^{M+1}}$

[Proof]

$$M=0, \sum_{n=0}^{N-1} q^n \binom{n}{0} = \frac{q^N-1}{q-1} = \frac{q^N\binom{N-1}{0}}{q-1} - \frac{1}{q-1}, \text{ holds}$$

$$M=1, \sum_{n=0}^{N-1} q^n \binom{n+1}{1} = \sum_{n=0}^{N-1} q^n (n+1)$$

$$= \sum_{n=0}^{N-1} q^n + \sum_{n=1}^{N-1} q^n + \cdots + \sum_{n=N-1}^{N-1} q^n$$

$$= N\sum_{n=0}^{N-1} q^n - \left(\sum_{n=0}^{0} q^n + \sum_{n=0}^{1} q^n + \cdots + \sum_{n=0}^{N-2} q^n\right)$$

$$= N\frac{q^N-1}{q-1} - \left(\frac{q-1}{q-1} + \frac{q^2-1}{q-1} + \cdots + \frac{q^{N-1}-1}{q-1}\right)$$

$$= N\frac{q^N-1}{q-1} - \frac{\left(1+q+q^2+\cdots+q^{N-1}\right)-N}{q-1}$$

$$= \frac{q^N N}{q-1} - \frac{q^N-1}{(q-1)^2} = \frac{q^N\binom{N}{1}}{q-1} - \frac{q^N\binom{N}{0}}{(q-1)^2} + \frac{1}{(q-1)^2}, \text{ holds}$$

Suppose it holds at $M-1$; When M and $N=1$, $\sum_{n=0}^{N-1} q^n \binom{n+M}{M} = 1$

$$A = \sum_{g=0}^{M} (-1)^g \frac{q^N\binom{N+M-1-g}{M-g}}{(q-1)^{g+1}} + \frac{(-1)^{M+1}}{(q-1)^{M+1}}$$

$$= \frac{q}{(q-1)^1} + \cdots + (-1)^M \frac{q}{(q-1)^{M+1}} + \frac{(-1)^{M+1}}{(q-1)^{M+1}}$$

$$(q-1)^{M+1} A = q\left\{(q-1)^M - (q-1)^{M-1} + \cdots + (-1)^{M-1}\right\} + (-1)^{M+1}$$

$\xrightarrow{7.5)-②, K=-1} = (q-1)^{M+1} \rightarrow A=1$; It holds when M and $N=1$.

Suppose it holds at M and N

$$\sum_{n=0}^{N} q^n \binom{n+M}{M} = \sum_{n=0}^{N} q^n \left\{\binom{n+M-1}{M} + \binom{n+M-1}{M-1}\right\}$$

$$= q\sum_{n=0}^{N-1} q^n \binom{n+M}{M} + \sum_{n=0}^{N} q^n \binom{n+M-1}{M-1}$$

$$= \sum_{g=0}^{M} (-1)^g \frac{q^{N+1}\binom{N+M-1-g}{M-g}}{(q-1)^{g+1}} + \frac{q(-1)^{M+1}}{(q-1)^{M+1}}$$

$$+\sum_{g=0}^{M-1}(-1)^g \frac{q^{N+1}\binom{N+M-1-g}{M-1-g}}{(q-1)^{g+1}}+\frac{(-1)^M}{(q-1)^M}$$

$$=\sum_{g=0}^{M}(-1)^g \frac{q^{N+1}\binom{(N+1)+M-1-g}{M-g}}{(q-1)^{g+1}}+\frac{(-1)^{M+1}}{(q-1)^{M+1}}$$

It holds when M and $N+1$.

q.e.d.

Example 8.1

$$\sum_{n=0}^{N-1} q^n \binom{n}{0} = \frac{q^N}{q-1}-\frac{1}{q-1}$$

$$\sum_{n=0}^{N-1} q^n \binom{n+1}{1} = \frac{q^N\binom{N}{1}}{q-1}-\frac{q^N}{(q-1)^2}+\frac{1}{(q-1)^2}$$

$$\sum_{n=0}^{N-1} q^n \binom{n+2}{2} = \frac{q^N\binom{N+1}{2}}{q-1}-\frac{q^N\binom{N}{1}}{(q-1)^2}+\frac{q^N}{(q-1)^3}-\frac{1}{(q-1)^3}$$

8.2) $\sum_{n=0}^{N-1} q^n \binom{n+M-K}{M} = q^N\sum_{g=0}^{M}(-1)^g \frac{\binom{N-K+M-1-g}{M-g}}{(q-1)^{g+1}}+\frac{(-1)^{M+1}q^K}{(q-1)^{M+1}}$,

$N \ge 1+K$

[Proof]

$$\sum_{n=0}^{N-1} q^n \binom{n+M-K}{M} = \sum_{n=k}^{N-1} q^n \binom{n+M-K}{M} = q^K\sum_{n=0}^{N-1-K} q^n \binom{n+M}{M}$$

$$=q^K\left\{q^{N-K}\sum_{g=0}^{M}(-1)^g \frac{\binom{N-K+M-1-g}{M-g}}{(q-1)^{g+1}}+\frac{(-1)^{M+1}}{(q-1)^{M+1}}\right\}$$

q.e.d.

$$\to \sum_{n=0}^{N-1} q^n \binom{n}{M} = q^N\sum_{g=0}^{M}(-1)^g \frac{\binom{N-1-g}{M-g}}{(q-1)^{g+1}}+\frac{(-1)^{M+1}q^M}{(q-1)^{M+1}}, N \ge 1+M$$

$$\to \sum_{n=0}^{N-1} q^n \binom{n+M-1}{M} = q^N\sum_{g=0}^{M}(-1)^g \frac{\binom{N+M-2-g}{M-g}}{(q-1)^{g+1}}+\frac{(-1)^{M+1}q}{(q-1)^{M+1}}, N \ge 2$$

$$\to \sum_{n=0}^{N-1} q^n \binom{n-1}{M} = q^N\sum_{g=0}^{M}(-1)^g \frac{\binom{N-2-g}{M-g}}{(q-1)^{g+1}}+\frac{(-1)^{M+1}q^{M+1}}{(q-1)^{M+1}}, N \ge 2+M$$

9. $\sum_{n=0}^{N-1} q^n n^M$

Define $A_q^M = \sum_{k=0}^{M} (1-q)^{M-k} q^K S_2(M,k) k!,\ q \neq 0,1,\ M \geq 0$

$A_2^M = 1,2,6,26,150,1082,9366,\cdots$ http://oeis.org/A000629

$A_3^M = 1,3,12,66,480,4368,47712,\cdots$ http://oeis.org/A123227

$A_4^M = 1,4,20,132,1140,12324,160020,\cdots$ http://oeis.org/A201355

$$A_q^0 = 1,\ A_q^1 = q$$

$$n^M = \nabla SUM(n,[1,\cdots,1][2,\cdots,M]) \xrightarrow{\text{Form}_1} \sum_{K=0}^{M} k! S_2(M,k) \binom{n}{K}$$

$$\nabla^g (N-1)^M = \sum_{K=g}^{M} k! S_2(M,k) \binom{N-1-g}{K-g}.$$

9.1) $\sum_{n=0}^{N-1} q^n n^M = \frac{q^N}{(q-1)^{M+1}} \sum_{g=0}^{M} (q-1)^{M-g} (-1)^g \nabla^g (N-1)^M + \frac{(-1)^{M+1} A_q^M}{(q-1)^{M+1}}$

[Proof]

$$\sum_{n=0}^{N-1} q^n n^0 = \frac{q^N}{q-1} - \frac{1}{q-1}, \text{ holds}$$

$$\sum_{n=0}^{N-1} q^n n^1 = \sum_{n=0}^{N-1} q^n \binom{n}{1} = q^N \sum_{g=0}^{1} (-1)^g \frac{\binom{N-1-g}{1-g}}{(q-1)^{g+1}} + \frac{q}{(q-1)^2}$$

$$= \frac{q^N (N-1)}{(q-1)^1} - \frac{q^N}{(q-1)^2} + \frac{q}{(q-1)^2}$$

$$= \frac{q^N}{(q-1)^2}\left((q-1)(N-1)^1 - 1\right) + \frac{A_q^1}{(q-1)^2}, \text{ holds}$$

$$\sum_{n=0}^{N-1} q^n n^M = \sum_{n=0}^{N-1} q^n \sum_{K=0}^{M} k! S_2(M,k) \binom{n}{K}$$

$$= \sum_{K=0}^{M} K! S_2(M,K) \left\{ q^N \sum_{g=0}^{k} (-1)^g \frac{\binom{N-1-g}{k-g}}{(q-1)^{g+1}} + \frac{(-1)^{k+1} q^k}{(q-1)^{k+1}} \right\}$$

$$\sum_{K=0}^{M} K! S_2(M,K) q^k \frac{(-1)^{K+1}}{(q-1)^{K+1}}$$

$$= \frac{(-1)^{M+1}}{(q-1)^{M+1}} \sum_{K=0}^{M} K! S_2(M,K) q^k (q-1)^{M-K} (-1)^{-(M-K)} = \frac{(-1)^{M+1} A_q^M}{(q-1)^{M+1}}$$

$$\sum_{K=0}^{M} K! S_2(M,K) q^N \sum_{g=0}^{k} (-1)^g \frac{\binom{N-1-g}{k-g}}{(q-1)^{g+1}} = \frac{q^N (\ldots)}{(q-1)^{M+1}}$$

$$= \frac{q^N}{(q-1)^{M+1}} \sum_{K=0}^{M} k! S_2(M,k) \sum_{g=0}^{k} (-1)^g \binom{N-1-g}{K-g} (q-1)^{M-g}$$

$$(\ldots)=K!S_2(M,M)\left\{\binom{N-1}{M}(q-1)^M-\binom{N-2}{M-1}(q-1)^{M-1}+\cdots+(-1)^M\binom{N-1-M}{0}\right\}$$

$$+K!S_2(M,M-1)\left\{\binom{N-1}{M-1}(q-1)^M-\binom{N-2}{M-2}(q-1)^{M-1}+\cdots\right\}+\cdots$$

$$+K!S_2(M,0)\left\{\binom{N-1}{0}(q-1)^M\right\}$$

$$\text{Vertical calculation}=(q-1)^M\nabla^0(N-1)^M-(q-1)^{M-1}\nabla^1(N-1)^M\cdots$$

q.e.d.

9.2) $A_q^M=q\sum_{k=0}^{M}(q-1)^{M-k}S_2(M,k)k!,M>0$

[Proof]

$$n^M=\nabla SUM(n,[1,\cdots,1][1,\cdots,M])$$

$$\xrightarrow{\text{Form}_2}\sum_{K=0}^{M}(-1)^{M-K}K!S_2(M,K)\binom{n+K-1}{K}$$

$$\sum_{n=0}^{N-1}q^n n^M=\sum_{n=0}^{N-1}q^n\sum_{K=0}^{M}(-1)^{M-K}K!S_2(M,K)\binom{n+K-1}{K}$$

$$=\sum_{K=0}^{M}(-1)^{M-K}K!S_2(M,K)\left\{q^N\sum_{g=0}^{K}(-1)^g\frac{\binom{N+K-2-g}{K-g}}{(q-1)^{g+1}}+\frac{(-1)^{K+1}q}{(q-1)^{K+1}}\right\}$$

$$\sum_{K=0}^{M}(-1)^{M-K}K!S_2(M,K)\sum_{g=0}^{K}(-1)^g\frac{q^N\binom{N+K-2-g}{K-g}}{(q-1)^{g+1}}\xrightarrow{\text{Same way as 9.1)}}$$

$$=\frac{q^N}{(q-1)^{M+1}}\sum_{g=0}^{M}(q-1)^{M-g}(-1)^g\nabla^g(N-1)^M$$

$$\text{Compare with 9.1)}\ \to\sum_{K=0}^{M}(-1)^{M-K}K!S_2(M,K)\frac{(-1)^{K+1}q}{(q-1)^{K+1}}=\frac{(-1)^{M+1}A_q^M}{(q-1)^{M+1}}$$

q.e.d.

$$n^M=\nabla SUM(n,[1,\cdots,1][1,\cdots,M])\xrightarrow{\text{Form}_1}\sum_{K=0}^{M}k!S_2(M+1,k+1)\binom{n-1}{K}\to$$

9.3) $A_q^M=\sum_{k=0}^{M}(1-q)^{M-k}q^{K+1}S_2(M+1,k+1)k!,\ M>0$

9.4) $\nabla^g(N-1)^M=\sum_{k=0}^{M-g}(-1)^{M-g-k}\binom{M}{k}N^K S_2(M-K+1,g+1)g!$

[Proof]

$$S_2(M,g)=\frac{1}{g!}\sum_{k=0}^{g}(-1)^k\binom{g}{k}(g-k)^M$$

$$\xrightarrow{j=g-k}=\frac{1}{g!}\sum_{j=0}^{g}(-1)^{g-j}\binom{g}{j}j^M=\frac{1}{(g-1)!}\sum_{j=1}^{g}(-1)^{g-j}\binom{g-1}{j-1}j^{M-1}$$

$$=\frac{1}{(g-1)!}\sum_{j=0}^{g-1}(-1)^{g-1-j}\binom{g-1}{j}(j+1)^{M-1}$$

$$\nabla^g (N-1)^M \xrightarrow{\text{By definition}} \sum_{j=0}^{g} (-1)^j \binom{g}{j} (N-j-1)^M$$

$$= \sum_{j=0}^{g} (-1)^j \binom{g}{j} \sum_{K=0}^{M} \binom{M}{k} N^k (j+1)^{M-k} (-1)^{M-K}$$

$$= \sum_{k=0}^{M} (-1)^{M-k} \binom{M}{k} N^k \left(\sum_{j=0}^{g} (-1)^j \binom{g}{j} (j+1)^{M-K} \right)$$

$$= \sum_{k=0}^{M} (-1)^{M-k-g} \binom{M}{k} N^k \left(\sum_{j=0}^{g} (-1)^{g+j} \binom{g}{j} (j+1)^{M-K} \right)$$

$$= \sum_{k=0}^{M} (-1)^{M-k-g} \binom{M}{k} N^K S_2 (M-K+1, g+1) g!$$

q.e.d.

9.5) ① $A_q^M = \sum_{k=0}^{M} (q-1)^{M-k} S_2 (M+1, k+1) k!$

② $\sum_{n=0}^{N-1} q^n n^M = \frac{q^N}{(q-1)^{M+1}} \sum_{g=0}^{M} (-1)^{M-g} A_q^{M-g} \binom{M}{g} (q-1)^g N^g + \frac{(-1)^{M+1} A_q^M}{(q-1)^{M+1}}$

[Proof]

$$f(N) = \sum_{n=0}^{N-1} q^n n^M = \frac{q^N}{(q-1)^{M+1}} \{\ldots\} + \frac{(-1)^{M+1} A_q^M}{(q-1)^{M+1}}$$

$$= \frac{q^N}{(q-1)^{M+1}} \left\{ \sum_{g=0}^{M} (q-1)^{M-g} (-1)^g \nabla^g (N-1)^M \right\} + \frac{(-1)^{M+1} A_q^M}{(q-1)^{M+1}}$$

$\{\ldots\} \xrightarrow{9.4)}$

$$= \sum_{g=0}^{M} (q-1)^{M-g} (-1)^g \sum_{k=0}^{M-g} (-1)^{M-g-k} \binom{M}{k} N^K S_2 (M-K+1, g+1) g!$$

$$= \sum_{g=0}^{M} (q-1)^{M-g} \sum_{k=0}^{M-g} (-1)^{M-k} \binom{M}{k} N^K S_2 (M-K+1, g+1) g!$$

$$= (q-1)^M \left\{ (-1)^M \binom{M}{0} N^0 S_2 (M+1, 1) 0! + (-1)^{M-1} \binom{M}{1} N^1 S_2 (M, 1) 0! + \cdots \right\}$$

$$+ (q-1)^{M-1} \left\{ (-1)^M \binom{M}{0} N^0 S_2 (M+1, 2) 1! + (-1)^{M-1} \binom{M}{1} N^1 S_2 (M, 2) 1! + \cdots \right\}$$

$$+ \cdots + (q-1)^0 \left\{ (-1)^M \binom{M}{0} N^0 S_2 (M+1, M+1) M! \right\}$$

Arrange by N^g

$$= \sum_{g=0}^{M} (q-1)^g N^g \binom{M}{g} (-1)^{M-g} \sum_{k=0}^{M-g} (q-1)^{M-g-k} S_2 (M-g+1, k+1) k!$$

take $g = 0$, $\{\ldots\} = (-1)^M \sum_{k=0}^{M} (q-1)^{M-k} S_2 (M+1, k+1) k!$

$$f(0) = 0 \to ①$$

$$① \to \sum_{k=0}^{M-g} (q-1)^{M-g-k} S_2 (M-g+1, k+1) k! = A_q^{M-g} \to ②$$

q.e.d.

Example 9.1

$$\sum_{n=0}^{N-1} q^n i = \frac{q^N\left(\left(q-1\right)N-q\right)+q}{\left(q-1\right)^2}$$

$$\sum_{i=0}^{N-1} 2^i = 2^N - 1$$

$$\sum_{i=0}^{N-1} 2^i i = 2^N\left(N-2\right)+2$$

$$\sum_{i=0}^{N-1} 2^i i^2 = 2^N\left(N^2-4N+6\right)-6$$

$$\sum_{i=0}^{N-1} 2^i i^3 = 2^N\left(N^3-6N^2+18N-26\right)+26$$

$$\sum_{i=0}^{N-1} 2^i i^4 = 2^N\left(N^4-8N^3+36N^2-104N+150\right)-150$$

$$\sum_{i=0}^{N-1} 2^i i^5 = 2^N\left(N^5-10N^4+60N^3-260N^2+750N-1082\right)+1082$$

$$\sum_{i=0}^{N-1} 3^i = \frac{3^N-1}{2}$$

$$\sum_{i=0}^{N-1} 3^i i = \frac{3^N\left(2N-3\right)+3}{4}$$

$$\sum_{i=0}^{N-1} 3^i i^2 = \frac{3^N\left(4N^2-2\times3\times2N+12\right)-12}{8}, 2\times3\times2=\left(3-1\right)A_3^1\binom{2}{1}$$

$$\sum_{i=0}^{N-1} 3^i i^3 = \frac{3^N\left(8N^3-4\times3\times3N^2+2\times12\times3N-66\right)+66}{16},$$

$$4\times3\times3=\left(3-1\right)^2 A_3^1\binom{3}{2}$$

$$\sum_{i=0}^{N-1} 3^i i^4 = \frac{3^N\left(16N^4-8\times3\times4N^3+4\times12\times6N^2-2\times66\times4N+480\right)-480}{32}$$

$$\sum_{i=0}^{N-1} 4^i = \frac{4^N-1}{3}$$

$$\sum_{i=0}^{N-1} 4^i i = \frac{4^N\left(3N-4\right)+4}{9}$$

$$\sum_{i=0}^{N-1} 4^i i^2 = \frac{4^N\left(9N^2-3\times4\times2N+20\right)-20}{27}$$

$$\sum_{i=0}^{N-1} 4^i i^3 = \frac{4^N\left(27N^3-9\times4\times3N^2+3\times20\times3N-132\right)+132}{81}$$

It is hard to imagine that the four expressions of A_q^M are equal, but they can stand verification.

$$\begin{aligned} A_q^M &= \sum_{k=0}^{M}\left(1-q\right)^{M-k} q^K S_2\left(M,k\right)k! \\ &= \sum_{k=0}^{M}\left(1-q\right)^{M-k} q^{K+1} S_2\left(M+1,k+1\right)k!, M>0 \\ &= \sum_{k=0}^{M}\left(q-1\right)^{M-k} S_2\left(M+1,k+1\right)k! \\ &= q\sum_{k=0}^{M}\left(q-1\right)^{M-k} S_2\left(M,k\right)k!, M>0 \end{aligned}$$

9.6) $M > 0, A_2^M = 2\sum_{k=0}^{M} S_2(M,k)k! = \sum_{k=0}^{M} S_2(M+1,k+1)k!$

$= \sum_{k=0}^{M}(-1)^{M-k} S_2(M,k)k!2^K = \sum_{k=0}^{M}(-1)^{M-k} S_2(M+1,k+1)k!2^{K+1}$

$\xrightarrow{2.3)} PS = [1,1,\cdots], PT = [1,2,\cdots,M]\sum_{q=0}^{M} H_1(q) = \sum_{q=0}^{M} H_2(q)2^q \to$

$\sum_{k=0}^{M} k!S_2(M+1,k+1) = \sum_{k=0}^{M}(-1)^{M-k} k!S_2(M,k)2^K \to$ match 9.6).

Conflicts of Interest

The author declares no conflicts of interest.

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