Retraction Notice

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The Asymptotic Expansions of the Largest Eigenvalues in the Presence of a Finite Number of Inclusions

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Abstract

In this paper, we provide a rigorous derivation of asymptotic formula for the largest eigenvalues using the convergence estimation of the eigenvalues of a sequence of self-adjoint compact operators based on the polarisation tensors of perturbations resulting from the presence of small inhomogeneities reported with numerical tests for the Laplace operator.

Subject Areas

Mathematical Analysis

Keywords

Largest Eigenvalues, Asymptotic Expansion, Convergence Estimation, Small Inhomogeneities, Laplace Operator

1. Introduction

In the past decades, the Laplacian spectrum has attracted wide attention and become an area of great interest. It had been widely applied to solve problems in fields, such as: randomized algorithms, combinational optimization problem and machine learning.

One of the fields that paid particular attention to Laplacian spectrum is the so-called the largest eigenvalue [1] [2]. It plays an important role in many techniques of multivariate statistics including the Principal Component Analysis (PCA). Furthermore, considering the study of sample covariance matrices in fundamental multivariate analysis as an example, it possibly works as a test statistic used in statistical hypothesis testing [3]. But still little is known about the distribution of the largest eigenvalue [4] [5] [6], and Principal Component Analysis (PCA) is a known technique of multivariate data analysis. The origins of PCA lie in multivariate data analysis; however, it has a wide range of other applications, one of eigenvalues of the covariance matrix. It is a linear dimensionality reduction procedure, which can also be thought of as a model selection technique. Inspired by this notion, we consider recovering as much of the total variance in the data as possible while reducing the dimensionality of the problem from p to k. In genetic studies, for example, it is not uncommon to have p (the number of genes in this context) of the order of 1000 and n (the number of patients) of order 100.

In other parts, eigenvalues and eigenvectors of several graph matrices appear in numerous papers on various subjects relevant to information and communication technologies. Efficient computation of eigenvectors and eigenvalues (especially for corresponding the largest or smallest eigenvalues) of matrix is an important problem in engineering. Recently, many researches tackle this problem by using neural networks mostly by focusing on computing the eigenvectors of positive definite symmetric matrices corresponding to the largest or smallest eigenvalues. A more general case will be studied in [7], which proposes a neural network approach to compute the eigenvectors corresponding to the largest or the smallest eigenvalues of any real symmetric matrix. [7] [8] [9] [10] proposed an iterative method for computing the largest eigenvalue of an irreducible non-negative tensor. Depending on the polynomial optimization techniques, this method aims to extend the non-negative tensors and examine the maximum eigenvalue of an essentially non-negative tensor. They demonstrated that finding the maximum eigenvalue of an essentially non-negative tensor is equivalent to solving a sum of squares (SOS) polynomial optimization problem, which, in turn, can be equivalently rewritten as a semi-definite programming problem.

The eigenvalues of the higher order have become an important subject of study in a new branch of applied mathematics and digital multilinear algebra, with many practical applications. The present paper differs from the research about the largest eigenvalues in [2] [7] [8] [9] [10]. We have presented the Laplace-Neumann eigenvalue problem in domain contains a finite number of inclusion using the convergence estimation of eigenvalues of a sequence of self-adjoint compact operators of perturbation resulting from the presence of small inhomogeneities with a theorem developed by Osborn applied from compact operators.

The novelty of this work, it that to give the asymptotic expansion of the largest eigenvalues using the tensor polarisation tensors [11] which are symmetric definite positive matrix with order $n \times n$. Our method is different from the work of [2] [7] [8] [9] [10].

Let Ω be a bounded domain in $\mathbb{R}^2$, with Lipchitz boundary $\partial\Omega$. Let ν denote the out unit normal to $\partial\Omega$ and assume it has a smooth background conductivity 1. We suppose that Ω contains a finite number of small inhomogeneities each of the form $z_l + \varepsilon \mathcal{B}_l$, where $\mathcal{B}_l \subset \mathbb{R}^2$ is a bounded smooth

(C^∞) domain containing the origin. The total collection of imperfections thus takes the form $\mathcal{D}_\varepsilon = \bigcup_{l=1}^{N} \mathcal{D}_\varepsilon^l$, where $\mathcal{D}_\varepsilon^l = z_l + \varepsilon \mathcal{B}_l$. The points $z_l \in \Omega$, $l = 1, 2, \cdots, N$, that determines the locations of the inhomogeneities (see Figure 1).

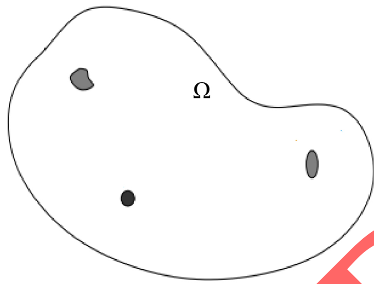

Figure 1. Examples of domain contain three imperfections.

Let λ_i be the ith eigenvalue of multiplicity m for the Laplacian in the absence of any inhomogeneities. Then there exist m nonzero solutions $\{u^{ij}\}_{j=1}^{m}$ to

$$\begin{cases} -\Delta u^{ij} = \lambda_i u^{ij} \quad \text{in } \Omega, \\ \left.\dfrac{\partial u^{ij}}{\partial \nu}\right|_{\partial\Omega} = 0. \end{cases} \tag{1}$$

The eigenvalues problem in the presence of imperfections consists of finding $\{\lambda_\varepsilon^{ij}\}_{j=1}^{m}$ such that there exists a nontrivial eigenfunction $\{u_\varepsilon^{ij}\}_{j=1}^{m}$ that is solution to

$$\begin{cases} -\nabla \cdot \left(1 + \sum_{l=1}^{N} \left((k_l - 1)\chi\left(D_\varepsilon^l\right)\right)\right) \nabla u_\varepsilon^{ij} = \tilde{\lambda}_\varepsilon u_\varepsilon^{ij} \quad \text{in } \Omega, \\ \left.\dfrac{\partial u_\varepsilon^{ij}}{\partial \nu}\right|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

It is well known that all eigenvalues of (1) are real, of finite multiplicity, have no finite accumulation points and there corresponding eigenfunctions which make up an orthonormal basis of $L^2(\Omega)$.

This paper is organized as follows. In Section 2, we introduce the eigenvalue Laplace-Neumann problem (2) and (1) and the geometry properties of the domain Ω and we give some preliminary results. In Section 3, we derive the asymptotic expansion for the Largest eigenvalue using the convergence estimate of the eigenvalues of a sequence of self-adjoint compact operators and applying the Osborn's formula such that Theorem 3.7 the main result of this paper. In Section 4, we present some numerical tests given by Freefem++.

2. Some Preliminaries Results

To derive the asymptotic formula for the eigenvalues we will use a convergence estimate of the eigenvalues of a sequence of self-adjoint compact operators. Let

$\mathcal{X}$ be a (real) Hilbert space and suppose we have a compact, self-adjoint linear operator $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ along with a sequence of compact, self-adjoint linear operators $\mathcal{T}_\varepsilon:\mathcal{X}\to\mathcal{X}$ such that $\mathcal{T}_\varepsilon\to\mathcal{T}$ pointwise as $\varepsilon\to 0$ and the sequence $\{\mathcal{T}_\varepsilon\}$ is collectively compact. Let μ be a nonzero eigenvalue of $\mathcal{T}$ of multiplicity *m*. Then we know that for small ε, each $\mathcal{T}_\varepsilon$ has a set of eigenvalues counted according to multiplicity, $\{\mu_\varepsilon^1,\cdots,\mu_\varepsilon^m\}$ such that for each *j*, $\mu_\varepsilon^j\to\mu$ as $\varepsilon\to 0$. Define the average

$$\bar{\mu}_\varepsilon=\frac{1}{m}\sum_{j=1}^{m}\frac{1}{\mu_\varepsilon^j}. \tag{3}$$

If $\phi^1,\cdots,\phi^m$ is an orthonormal basis of eigenfunctions associated with the eigenvalue μ, then there exists a constant $\mathcal{C}$ such that for $j=1,\cdots,m$ the following Osborn's formula holds

$$\left|\mu-\bar{\mu}_\varepsilon-\frac{1}{m}\sum_{j=1}^{m}\left\langle\left(\mathcal{T}-\mathcal{T}_\varepsilon\right)\phi^j,\phi^j\right\rangle\right|\le\mathcal{C}\left\|\left(\mathcal{T}-\mathcal{T}_\varepsilon\right)\big|_{span\{\phi^j\}_{1\le j\le m}}\right\|^2, \tag{4}$$

where $\left(\mathcal{T}-\mathcal{T}_\varepsilon\right)\big|_{span\{\phi^j\}_{1\le j\le m}}$ denotes the restriction of $\left(\mathcal{T}-\mathcal{T}_\varepsilon\right)$ to the *m*-dimensional vector space spanned by $\phi^j_{1\le j\le m}$.

In our case, let $\mathcal{X}$ be $L^2(\Omega)$ with the standard inner product. For any $g\in L^2(\Omega)$, we define $\mathcal{T}_\varepsilon g=u_\varepsilon$ and $\mathcal{T}g=u$, where u_ε is the solution to

$$\begin{cases}-\nabla\cdot\left(1+\sum_{l=1}^{N}\left(\left(k_l-1\right)\chi\left(D_\varepsilon^l\right)\right)\right)\nabla u_\varepsilon=g \quad \text{in } \Omega,\\ \left.\dfrac{\partial u_\varepsilon}{\partial\nu}\right|_{\partial\Omega}=0,\end{cases} \tag{5}$$

and u is the solution of

$$\begin{cases}\Delta u=g \quad \text{in } \Omega,\\ \left.\dfrac{\partial u}{\partial\nu}\right|_{\partial\Omega}=0.\end{cases} \tag{6}$$

The function $g\mapsto(-\Delta)^{-1}g$ is continuous from $L^2(\Omega)$ to $H_0^1(\Omega)$. Clearly $\mathcal{T}_\varepsilon$ and $\mathcal{T}$ are compact operators from $L^2(\Omega)$ to $L^2(\Omega)$. From the standard H^1 estimates we get the following lemma:

Lemma 2.1 *$\mathcal{T}_\varepsilon$ and $\mathcal{T}$ are compact, self-adjoint operators from $L^2(\Omega)$ to $L^2(\Omega)$. Moreover, the family of operators $\{\mathcal{T}_\varepsilon\}$ is collectively compact.*

Let (μ^i,u^i), and $(\mu_\varepsilon^i,u_\varepsilon^i)$ be the ith normalised eigenpairs of $\mathcal{T}$ and $\mathcal{T}_\varepsilon$ respectively. Then if $\lambda_i=\frac{1}{\mu^i}$ and $\lambda_{i\varepsilon}=\frac{1}{\mu_\varepsilon^i}$, then u_ε^i and u^i are the solution of (2) and (1) respectively. From the spectral theory, if λ_i has a multiplicity *m* with a correspond set of orthonormal eigenfunctions $\{u^{ij}\}_{j=1}^m$ then there exist *m* eigenvalues $\tilde{\lambda}_\varepsilon$ that satisfy the following lemma.

Lemma 2.2 *Let Ω be a bounded domain in $I\!R^2$, and $\varepsilon>0$ a small number. Then the eigenvalues of Laplacian-Neumann operator satisfy the following* *expansion*

$$\lambda_i - \tilde{\lambda}_\varepsilon = \theta(1), \text{ as tends to } 0, \tag{7}$$

in the other word,

$$\lambda_i \tilde{\lambda}_\varepsilon = \lambda_i^2 + \lambda_i \theta(1), \text{ as tends to } 0, \tag{8}$$

for any $j = 1, 2, \cdots, m$, *such that* $\theta(1)$ *independent of i.*

Proof. From [12], the proof is based on the following asymptotic expansion

$$\omega_\varepsilon - \omega_0 = \frac{1}{2\sqrt{\pi}} \sum_{p=1}^{\infty} \frac{1}{p} \sum_{n=p}^{+\infty} \varepsilon^n tr \int_{\partial V_{\delta_0}} B_{n,p}(\omega) \mathrm{d}\omega \tag{9}$$

$$+ \frac{1}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(2\pi)^n}{n} \int_{\partial V_{\delta_0}} \left[\frac{1}{\ln(\eta\omega\varepsilon)} \times (D_\Omega^\varepsilon)\left[N_\Omega^\varepsilon(.,z)\right](x) - \frac{\ln cap(\partial B)}{2\pi} \right]^n \mathrm{d}\omega, \tag{10}$$

where

$$B_{n,p}(\omega) = (-1)^p \sum_{n_i} \left(A_0(\omega) + (\ln(\omega\varepsilon)) B_0(\omega)\right)^{-1} \left(A_{n_1} + (\ln(\omega\varepsilon)) B_0(\omega)\right) \tag{11}$$

$$\times \cdots \left(A_0(\omega)(\ln(\omega\varepsilon)) B_0(\omega)\right)^{-1} \left(A_{n_1} + (\ln(\omega\varepsilon)) B_{n,p}(\omega)\right) \omega^n, \tag{12}$$

then ω_ε is the characteristic eigenvalue and $\mathcal{D}_\Omega$ is the double layer potential and $cap(\partial B)$ called the capacity of ∂B. We obtain from this asymptotic expansion the following leading-order term of $\lambda_i - \lambda_\varepsilon^{ij}$ in two dimensions as follows

$$\lambda_i - \tilde{\lambda}_\varepsilon = \frac{-2\pi}{\ln\left(\varepsilon\sqrt{\lambda_i}\right)} \left|u^{ij}(z)\right|^2 + \theta\left(\frac{1}{\left|\ln(\varepsilon)\right|}\right), \tag{13}$$

this formula is exactly the one derived by Ozawa in [13], see also Besson [14], so as ε tends to 0 we get our desired result.

3. Asymptotic Expansion of the Largest Eigenvalues

In this part, we assume that the domains $\mathcal{D}_\varepsilon^l$, $l = 1, 2, \cdots, m$, satisfy

$$0 < d_0 \le |z_l - z_k| \quad \forall l \ne k, \quad dist(z_l, \partial\Omega) \ge d_0 \quad \forall l, \tag{14}$$

which the following theorem holds

Theorem 3.1 *Suppose that* Ω *contains an inclusion as the form* $\mathcal{D} = z + \varepsilon B$ *which is far from the boundary. Then the solutions* $\{u^{ij}\}_{j=1}^m$ *to* (1) *satisfy*

- $\left\|u^{ij}\right\|_{L^\infty(D)} \le \mathcal{C}$, *where* $\mathcal{C}$ *is independent of* ε *and* λ_i.
- $\left\|\frac{\nabla u^{ij}}{\sqrt{\lambda_i}}\right\|_{L^\infty(D)} \le \mathcal{C}$, *where* $\mathcal{C}$ *is independent of* ε *and* λ_i.
- $\left\|\frac{\nabla^2 u^{ij}}{\lambda_i}\right\|_{L^\infty(D)} \le \mathcal{C}$, *where* $\mathcal{C}$ *is independent of* ε *and* λ_i.

To prove the Theorem 3.1, we need this following lemma

Lemma 3.2 *For any* $i, s = 0, 1, 2, \cdots$, *we have*

$$u_s^{ij}(r,\phi)=\frac{J_s\left(\beta_{si}\frac{r}{R}\right)}{R\sqrt{\pi\left\{1-\frac{s^2}{\beta_{si}^2}\right\}J_s^2(\beta_{si})}}\mathrm{e}^{\pm I s\phi}, \tag{15}$$

where $\mathcal{I}^2=-1$, $\mathcal{J}_s(r)$ *is the Bessel function of integer order s*, β_{si} *denotes the ith zero of* $\mathcal{J}_s'(r)$ *and* $\lambda_{si}=\left(\frac{\beta_{si}}{R}\right)^2$ *is the associated eigenvalue.*

Proof. For a round disk of radius *R*, the Helmholtz equation $(\Delta+\lambda_{si})u^{ij}=0$ can be expressed in the polar coordinate system in the following form (we can see [15])

$$\frac{\partial^2 u^{ij}}{\partial r^2}+\frac{1}{r}\frac{\partial u^{ij}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u^{ij}}{\partial\theta^2}+\lambda_{si}u^{ij}=0,$$

with $0\le r\le R$, $0\le\theta\le 2\pi$.

One can seek a solution of the last equation as Fourrier expansion over $0\le\theta\le 2\pi$, that

$$u^{ij}=\sum_{-\infty}^{\infty}U_{si}(r)\mathrm{e}^{is\theta}.$$

Then, for each $U_{si}(r)$, due to linear independence of trigonometric functions, we arrive at to ordinary differential equation

$$U_{si}''(r)+\frac{1}{r}U_{si}'(r)+\left(\lambda_{si}-\frac{s^2}{r^2}\right)U_{si}(r)=0,$$

whose only solution regular inside the disk is the Bessel function of the first kind and order *m*. Therefore a complete system of linearly independent solutions for our equation, can be chosen as $\left\{J_s\times\mathrm{e}^{(\pm I s\phi)}\right\}$, with the boundary Neumann condition $J_s'=0$.

By a separation of the variables we can write $u_s^{ij}(r,\phi)=c_{si}U_{si}(r)\mathrm{e}^{\pm I s\phi}$, where c_{si} is a constant and $U_{si}(r)$ satisfies

$$\begin{cases}U_{si}''(r)+\frac{1}{r}U_{si}'(r)+\left(\lambda_{si}-\frac{s^2}{r^2}\right)U_{si}(r)=0, & 0\le r<R,\\ U_{si}'(R)=0.\end{cases}$$

Using the definition of the Bessel function [16] we can deduce that,

$$U_{si}(r,\phi)=\mathcal{J}_s\left(\beta_{si}\frac{r}{R}\right),$$

we get,

$$u_s^{ij}(r,\phi)=\frac{\mathcal{J}_s\left(\beta_{si}\frac{r}{R}\right)}{R\sqrt{\pi\left\{1-\frac{s^2}{\beta_{si}^2}\right\}\mathcal{J}_s^2(\beta_{si})}}\mathrm{e}^{\pm I s\phi}.$$

Now, we give an important result about the eigenvalue λ_i, which will be described by this lemma

Lemma 3.3 *As i tends to* ∞, *we have*:

$$\lambda_i \approx i. \tag{16}$$

Proof. We proof our lemma firstly in the case of a disk and after we study the general case.

For the case of a disk, our proof are based on this relation $\lambda_s^i = \left(\frac{\beta_{si}}{R}\right)^2$. From, [17], if we fix s thus β_{si} have the following asymptotic expansion

$$\beta_{si} = \beta'_{si} + \theta\left(\frac{1}{\beta'_{si}}\right),$$

where $\beta'_{si} = \left(i + \frac{s}{2} - \frac{3}{4}\right)\pi$.

Then, we have for the general case, provided Ω is bounded and the boundary $\partial\Omega$ is sufficently regular, the Neumann Laplacian has a discrete spectrum of infinitely many positive eigenvalues with no infinite accumulation point

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots.$$

Here we use the Weyl's Asymptotic Formula in [18] [19] [20] to get the asymptotic expansion for the eigenvalues. Let X be a Riemannian manifold and Δ be the Laplace operator on X. It is well-known that if X is compact then the spectrum of $-\Delta$ is discrete and consists of an increasing sequence $\{\lambda_i\}_{i=1}^{\infty}$ of the eigenvalues (counted according to their multiplicities) where $\lambda^1 = 0$ and $\lambda_i \to \infty$ as $i \to \infty$. Moreover, if $n = dimX$ then Weyl's [19] asymptotic formula says that, for large values of i

$$\lambda_i \approx \frac{4\pi^2 i^{\frac{2}{n}}}{\left(c_n \mu(X)\right)^{\frac{2}{n}}}, \tag{17}$$

where μ is the Reimannian measure on $\mathcal{X}$ and $c_n > 0$ is a constant depending only on n. According to the last theory in our case when, i is large, $\lambda^1 = 0$ corresponding to $u^1 = \frac{1}{\sqrt{|\Omega|}}$, $X = \Omega$ and $\mu(\Omega) = |\Omega|$ we can prove the above lemma. □

Now, we prove Theorem 3.1.

Proof of Theorem 3.1. We derive this theorem firstly in the simple case when Ω is a disk and after we study the general case; For the case of a disk, the eigenvalue $\{\lambda_{is}\}_{i,s=0,1,2,\cdots}$ of $-\Delta$ in a disk Ω of radius R in $I\!R^2$ have two multiplicity and they are the solutions of the following system:

$$\begin{cases} (\Delta + \lambda_{is}) u_s^{ij}(r,\phi) \text{ in } \Omega, \\ \left.\frac{\partial u_s^{ij}(r,\phi)}{\partial r}\right|_{r=R} = 0. \end{cases} \tag{18}$$

From the Lemma 3.1, we have the following result

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$$u_s^{ij}(r,\phi) = \frac{\mathcal{J}_s\left(\beta_{si}\frac{r}{R}\right)}{R\sqrt{\pi\left\{1-\frac{s^2}{\beta_{si}^2}\right\}\mathcal{J}_s^2(\beta_{si})}} \mathrm{e}^{\pm Is\phi}. \tag{19}$$

Then, we study the eigenfunction

1) As $r \to \infty$ we have $\mathcal{J}_s(r) \simeq \frac{s!}{2^s} r^s$.

2) As $\beta \to \infty$ we have $\mathcal{J}_s(\beta) \simeq \sqrt{\frac{2}{\pi\beta}} \cos\left(\beta - \frac{2S+1}{2}\frac{\pi}{4}\right) + \theta\left(\frac{1}{\beta^{\frac{3}{2}}}\right)$.

3) If we fix and we choose $i = E\left(\frac{1}{\varepsilon^\alpha}\right)$, we find

$$\beta_{si} \approx \frac{1}{\varepsilon^\alpha} + \theta\left(\varepsilon^{-\alpha}\right).$$

Using, this three assertions we can deduce that $\left|\frac{\mathcal{J}_s\left(\beta_{si}\frac{r}{R}\right)}{\mathcal{J}_s(\beta_{si})}\right|$ is uniform bound for s and $r \in [0, R]$, so $\|u^{ij}\|_{L^\infty(\mathcal{D})} \le \mathcal{C}$, where $\mathcal{C}$ is independent of ε and λ_i, where satisfied for the case of a disk.

Now, let's turn to the general case. From [21], u^{ij} can be represented as

$$u^{ij} = S_{\mathcal{D}}^{\sqrt{\lambda_i}} \varphi(x), \quad x \in \mathcal{D},$$

where $S_{\mathcal{D}}^{\sqrt{\lambda_i}} \varphi(x)$ the single layer potential of the density function φ on $\partial\mathcal{D}$, which can be defined as

$$S_{\mathcal{D}}^{\sqrt{\lambda_i}} \varphi(x) = \int_{\partial\mathcal{D}} \Gamma(x-y)\varphi(y)\mathrm{d}\sigma(y), \quad x \in I\!R^2.$$

On the other hand, we have the following result, let $\varphi \in L^2(\partial\mathcal{D})$, $\tilde{\varphi}(x) = \varepsilon\varphi(\varepsilon x + z), x \in \partial\mathcal{B}$. Then, for $x \in \partial\mathcal{B}$, we have

$$S_{\mathcal{D}}^{\sqrt{\lambda_i}} \varphi(\varepsilon x + z) = \frac{1}{2\pi}\sum_{n=0}^{+\infty}(-1)^n \frac{\left(\sqrt{\lambda_i}\varepsilon\right)^{2n}}{2^{2n}(n!)^2} \times \int_{\partial B}|x-y|^{2n}\left(\ln\left(\sqrt{\lambda_i}\varepsilon|x-y|\right) + \ln\gamma - \sum_{j=1}^{n}\frac{1}{j}\right)\tilde{\varphi}(y)\mathrm{d}\sigma(y).$$

Then, using the Lemma 3.2 we can get the assertion (1) deduced easily such that,

$$\|u^{ij}\|_{L^\infty(\mathcal{D})} = \left\{\sup\left|u^{ij}\right|, x \in \mathcal{D}\right\}.$$

Finally, the last two assertions may be deduced easily after we calculate $\frac{\nabla u^{ij}}{\sqrt{\lambda_i}}$ and $\frac{\nabla^2 u^{ij}}{\lambda_i}$. □

3.1. Estimation Energy

In this section, when the inclusions are not degenerate (*i.e.* their conductivity $k_l > 0$, $k_l \neq 1$) the first term the expansion of u_ε solution to (5) is the background potential u solution to (6). In fact, u_ε converges strongly in $H^1(\Omega)$. This is the consequence of the following estimate of the $H^1(\Omega)$ norm of $u_\varepsilon - u$.

Lemma 3.4 *Let* u_ε *be the solution to* (5) *and u solution to* (6) *for a given* $g \in L^2(\Omega)$. *Then there exists a constant* C, *independent of* ε, *u and the set of points* $(z_l)_{l=1}^N$ *such that the following estimate holds*:

$$\|u_\varepsilon - u\|_{H^1(\Omega)} \le C\left(\|\nabla_x u\|_{L^\infty(\mathcal{D}_\varepsilon)}\varepsilon^{\frac{3}{2}} + \|\nabla_x^2 u\|_{L^\infty(\mathcal{D}_\varepsilon)}\varepsilon^2 + \|g\|_{L^\infty(\mathcal{D}_\varepsilon)}\varepsilon^2\right). \tag{20}$$

Proof. The proof of the above lemma is based on the following estimate

$$\begin{aligned}&\left\|\nabla_y\left(u_\varepsilon(\varepsilon y) - u(\varepsilon y) - \varepsilon\upsilon(y)\right)\right\|_{L^2(\tilde{\Omega})} \\ &\le C\left(\|\nabla_x u\|_{L^\infty(\mathcal{D}_\varepsilon)}\varepsilon^{\frac{3}{2}} + \|\nabla_x^2 u\|_{L^\infty(\mathcal{D}_\varepsilon)}\varepsilon^2 + \|g\|_{L^\infty(\mathcal{D}_\varepsilon)}\varepsilon^2\right).\end{aligned} \tag{21}$$

where υ is the unique solution of the following transmission problem:

$$\begin{cases}\Delta\upsilon = 0 & \text{in } I\!R^2 \setminus \left(\overline{\bigcup_{l=1}^m B_l}\right)\\ \Delta\upsilon = 0 & \text{in } B_l,\ \forall l = 1,2,3,\cdots,m.\\ \upsilon|_- = \upsilon|_+ & \text{on } \partial B_l\\ \left.\dfrac{\partial\upsilon}{\partial\nu}\right|_+ - k\left.\dfrac{\partial\upsilon}{\partial\nu}\right|_- = (k-1)\nabla_x u(z_l)\cdot\nu_l, & \text{on } \partial B_l,\\ \lim_{|y|\to\infty}\upsilon(y) = 0, & \end{cases} \tag{22}$$

and $\tilde{\Omega} = \dfrac{1}{\varepsilon}\Omega$. Then, from [22] we put

$$\omega(\varepsilon) = u_\varepsilon(\varepsilon y)\varepsilon - u(\varepsilon y) - \varepsilon\upsilon(y).$$

Using the unperturbed problem and setting $\lambda = \lambda_j(\varepsilon)$, we see that $\omega(\varepsilon)$ solves:

$$-\Delta\omega(\varepsilon) = \lambda\omega + (\lambda - \lambda_0)\left(u(\varepsilon y) + \varepsilon\upsilon(y)\right).$$

For $z \in I\!R$, we define the function θ by

$$\theta(z) = \lambda z + (\lambda - \lambda_0)\|u(\varepsilon y) - \varepsilon\upsilon(y)\|_{L^\infty(D_\varepsilon)},$$

then trivially remark,

$$|\theta(z)| \le |\theta(0)| + \lambda|z|, \quad \forall z \in I\!R,$$

and consequently,

$$|\theta(\omega(z))| \le |\theta(0)| + \lambda|z|. \tag{23}$$

Now, it turns out from the definition of ω that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$ and so by $\|u(\varepsilon y) - \varepsilon\upsilon(y)\|_{L^\infty(\mathcal{D}_\varepsilon)} > 0$, we get,

$$\left|\omega(\varepsilon)(x)\right| \leq 2\left\|u(\varepsilon y)+\varepsilon \upsilon(y)\right\|_{L^{\infty}(\mathcal{D}_{\varepsilon})}, \text{ for } x \in \mathcal{D}_{\varepsilon}.$$

Moreover, we recall that $\lambda(\varepsilon) \to \lambda_0$. Now it is useful to introduce the following function

$$\tilde{\theta}(\omega(\varepsilon)) = \lambda\omega + (\lambda - \lambda_0)\big(u(\varepsilon y) + \varepsilon\upsilon(y)\big),$$

and the second term is bounded by

$$\lambda\left|\omega(\varepsilon)\right| \leq 2\lambda_0 \left\|u(\varepsilon y)+\varepsilon \upsilon(y)\right\|_{L^{\infty}(\mathcal{D}_{\varepsilon})}.$$

These estimate give

$$\left\|\tilde{\theta}(\omega(\varepsilon))\right\|_{L^{\infty}(\Omega)} \leq (1+2\lambda)\left\|u(\varepsilon y)+\varepsilon \upsilon(y)\right\|_{L^{\infty}(\mathcal{D}_{\varepsilon})}. \tag{24}$$

Next, we can write in Ω,

$$-\Delta\omega(\varepsilon) = \tilde{\theta}(\omega(\varepsilon)).$$

By integrating by parts, we find that the function $\omega(\varepsilon)$ is a solution to the following problem

$$\forall v \in H_0^1(\Omega), \quad \int_{\Omega} \nabla\omega(\varepsilon)\nabla v \mathrm{d}x = \int_{\Omega} v(\omega(\varepsilon)) v \mathrm{d}x.$$

If we take $v = \omega(\varepsilon)$ we can deduce that:

$$\left\|\nabla\omega\right\|_{L^2(\Omega)}^2 = \int_{\Omega} \Omega(\omega)\bar{\omega}\mathrm{d}x,$$

Then, by Poincare's inequality, there exist some positive constant $\mathcal{C}(\tilde{\Omega})$ such that

$$\left\|\omega\right\|_{L^2(\Omega)} \leq \mathcal{C}(\Omega)\left\|\nabla\omega\right\|_{L^2(\Omega)}. \tag{25}$$

Since ω and $\nabla\omega$ are uniformly bounded in Ω. There exist some constant independent of ε such that $\mathcal{C}(\Omega) \leq \mathcal{C}_0$, which give

$$\left\|\nabla\omega\right\|_{L^2(\Omega)} \leq \mathcal{C}\left\|u(\varepsilon y)+\varepsilon \upsilon(y)\right\|_{L^{\infty}(\mathcal{D}_{\varepsilon})}, \tag{26}$$

which concludes the proof.

Now, we have this remark

Remark 3.5 *The function v is connected to polarisation tensors* $\mathcal{M}^l$ *for any* $l = 1,\cdots,N$, *which are given by*

$$\mathcal{M}_{pq}^l = (1-k_l)|B_l|\delta_{pq} + (1-k_l)^2 \int_{\partial B_l} y_p \left.\frac{\partial\phi_p^{(l)}}{\partial\nu}\right|_{-} \mathrm{d}\sigma_y, \tag{27}$$

where for $p = 1,2$, $\phi_p^{(l)}$ *is the unique function which satisfies*

$$\begin{cases} \Delta\phi_p^{(l)} = 0 & \text{in } IR^2 \setminus \overline{B_l} \\ \Delta\phi_p^{(l)} = 0 & \text{in } B_l, \\ \left.\dfrac{\partial\upsilon}{\partial\upsilon}\right|_{+} - k\left.\dfrac{\partial\phi_p^{(l)}}{\partial\upsilon}\right|_{-} = \upsilon_l, & \text{on } \partial B_l, \end{cases} \tag{28}$$

with $\phi_p^{(l)}$ *continuous across* ∂B_l *and* $\lim_{|y|\to\infty}\phi_p^{(l)} = 0$.

3.2. Derivation of the Asymptotic Expansion for the Largest Eigenvalues

In this section, we restrict to the case of a single inhomogeneity ($N = 1$), by iteration, we can get the more general case. So, we suppose that this inhomogeneity is centred at the origin, so it is of the form $\mathcal{D} = \varepsilon \mathbf{B}$, with conductivity k. The general case may be verified by fairly direct iteration of the argument we present here, adding one inhomogeneity at a time. By according Osborn's (see [23]) formula in (4), we obtain

$$\left|\frac{1}{\lambda_i} - \frac{1}{m}\sum_{j=1}^{m}\frac{1}{\tilde{\lambda}_\varepsilon} - \frac{1}{m}\sum_{j=1}^{m}\left\langle \frac{1}{\lambda_i}u^{ij} - \upsilon_\varepsilon^{ij}, u^{ij}\right\rangle\right| \le C\left\|\frac{1}{\lambda_i}u^{ij} - \upsilon_\varepsilon^{ij}\right\|_{L^2(\Omega)}^2, \tag{29}$$

where $\upsilon_\varepsilon^{ij}$ satisfies

$$\begin{cases} \nabla\cdot\left(1+\sum_{l=1}^{m}(k_l-1)\chi\left(\mathcal{D}_\varepsilon^l\right)\right)\nabla \upsilon_\varepsilon^{ij} = u^{ij} & \text{in } \Omega, \\ \left.\dfrac{\partial \upsilon_\varepsilon^{ij}}{\partial \nu}\right|_{\partial\Omega} = 0. \end{cases}$$

If we take $i = E\left(\frac{1}{\varepsilon^\alpha}\right)$ with $0 \le \alpha \le 1$, we deduce from Lemma 3.5,

$$\lambda_i = \theta\left(\varepsilon^{-\alpha}\right), \text{ as } \varepsilon \text{ tends to } 0. \tag{30}$$

Then according to Theorem 3.1 and Lemma 3.7 with $u = \frac{u^{ij}}{\lambda_i}$, $u_\varepsilon = \upsilon_\varepsilon^{ij}$ and $g = u^{ij}$ we obtain

$$\left\|\frac{u^{ij}}{\lambda_i} - \upsilon_\varepsilon^{ij}\right\|_{L^2(\Omega)} \le C\frac{\varepsilon^{\frac{3}{2}}}{\sqrt{\lambda_i}}.$$

This gives

$$\frac{1}{\lambda_i} - \frac{1}{m}\sum_{j=1}^{m}\frac{1}{\tilde{\lambda}_\varepsilon} = \frac{1}{m}\sum_{j=1}^{m}\left\langle \frac{1}{\lambda_i}u^{ij} - \upsilon_\varepsilon^{ij}, u^{ij}\right\rangle + \theta\left(\frac{\varepsilon^3}{\lambda_i}\right). \tag{31}$$

Let $\langle .,. \rangle$ be the L^2-inner product, by integration by parts and by using the transmission conditions satisfies by $\upsilon_\varepsilon^{ij}$ across $\partial\mathcal{D}_\varepsilon$, we get

$$\begin{aligned}\left\langle \frac{1}{\lambda_i}u^{ij} - \upsilon_\varepsilon^{ij}, u^{ij}\right\rangle &= \int_\Omega\left(\frac{1}{\lambda_i}u^{ij} - \upsilon_\varepsilon^{ij}\right)u^{ij}\mathrm{d}y \\ &= \frac{k-1}{\lambda_i k}\int_D\left|u^{ij}\right|^2\mathrm{d}y + \frac{1-k}{\lambda_i}\int_{\partial D}\left.\frac{\partial \upsilon_\varepsilon^{ij}}{\partial \nu}\right|_{-}u^{ij}\mathrm{d}\sigma_x.\end{aligned}$$

Suppose

$$r_\varepsilon^{ij}(x) = \upsilon_\varepsilon^{ij}(x) - \frac{1}{\lambda_i}u^{ij}(x) - \varepsilon\upsilon\left(\frac{x}{\varepsilon}\right),$$

where υ is defined in (22) (with $\frac{1}{\lambda_i}u^{ij}$ in place of u), inserting this into the above formula we get

$$\left\langle \frac{1}{\lambda_i} u^{ij} - \upsilon_\varepsilon^{ij}, u^{ij} \right\rangle = \frac{k-1}{\lambda_i k} \int_D \left| u^{ij} \right|^2 \mathrm{d}x + \frac{1-k}{\lambda_i} \int_{\partial\mathcal{D}} \frac{\partial r_\varepsilon^{ij}}{\partial \nu_x}\bigg|_- u^{ij} \mathrm{d}\sigma_x$$
$$+ \frac{1-k}{\lambda_i} \left(\frac{1}{\lambda_i} \frac{\partial u^{ij}}{\partial \nu_x}(x) + \frac{\varepsilon}{\lambda_i} \int_{\partial\mathcal{D}} \frac{\partial \upsilon}{\partial \nu_x}\bigg|_- \left(\frac{x}{\varepsilon}\right) \right) u^{ij} \mathrm{d}\sigma_x. \tag{32}$$

Note that

$$\Delta_x r_\varepsilon^{ij}(x) = \frac{1}{k} u^{ij}(x) - \frac{1}{\lambda_i} \Delta_x u^{ij}(x).$$

From (17) we also have

$$\left\| \nabla_\xi r_\varepsilon^{ij}(\varepsilon\xi_1, \varepsilon\xi_2) \right\|_{L^2(\tilde{\Omega})} \le C(k,B) \frac{\varepsilon^{\frac{3}{2}}}{\sqrt{\lambda_i}},$$

this gives that

$$\frac{1-k}{\lambda_i} \int_{\partial\mathcal{D}} \frac{\partial r_\varepsilon^{ij}}{\partial \upsilon_x}\bigg|_- u^{ij} \mathrm{d}\sigma_x = (1-k)\varepsilon \int_{\mathcal{B}} \nabla_\xi r_\varepsilon^{ij}(\varepsilon\xi) \cdot \frac{\nabla_x u^{ij}(\varepsilon\xi)}{\lambda_i} \mathrm{d}\xi$$
$$+ (1-k)\varepsilon^2 \int_{\mathcal{B}} \left(\frac{1}{k} u^{ij}(\varepsilon\xi) - \frac{1}{\lambda_i} \Delta_x u^{ij}(\varepsilon\xi) \right) \frac{u^{ij}}{\lambda_i}(\varepsilon\xi) \mathrm{d}\xi.$$

Using the Lemmas 3.5 and Theorem 3.1, we deduce that

$$\left| (1-k)\varepsilon \int_{\mathcal{B}} \nabla_\xi r_\varepsilon^{ij}(\varepsilon\xi) \cdot \frac{\nabla_x u^{ij}(\varepsilon\xi)}{\lambda_i} \mathrm{d}\xi \right|$$
$$\le C(k,B)\varepsilon \left\| \nabla_\xi r_\varepsilon^{ij}(\varepsilon\xi) \right\|_{L^2(\tilde{\Omega})} \left\| \frac{\nabla_x u^{ij}}{\lambda_i} \right\|_{L^\infty(D)} \le C(k,B) \frac{\varepsilon^{\frac{5}{2}}}{\sqrt{\lambda_i}},$$

where $C(k,B)$ independent of *i*.

We conclude that

$$\frac{1-k}{\lambda_i} \int_{\partial\mathcal{D}} \frac{\partial r_\varepsilon^{ij}}{\partial \upsilon_x}\bigg|_- u^{ij} \mathrm{d}\sigma_x$$
$$= \frac{1-k}{k\lambda_i} \varepsilon^2 \int_{\mathcal{C}} \left(u^{ij}\right)^1 (\varepsilon\xi) \mathrm{d}\xi - \frac{1-k}{\lambda^{i2}} \varepsilon^2 \int_{\mathcal{B}} \Delta_x u_\varepsilon^{ij}(\varepsilon\xi) u^{ij}(\varepsilon\xi) \mathrm{d}\xi + \theta\left(\frac{\varepsilon^{\frac{5}{2}}}{\sqrt{\lambda_i}} \right). \tag{33}$$

In the same time we have

$$(1-k)\int_{\partial\mathcal{D}} \frac{1}{\lambda_i} \frac{\partial u^{ij}}{\partial \nu_x}(x) + \frac{\partial \upsilon}{\partial \nu_\xi}\bigg|_- \left(\frac{x}{\varepsilon}\right) \frac{u^{ij}}{\lambda_i} \mathrm{d}\sigma_x$$
$$= (1-k)\int_{\partial\mathcal{D}} \frac{1}{\lambda_i} \frac{\partial u^{ij}}{\partial \nu} \frac{u^{ij}}{\lambda_i} \mathrm{d}\sigma_x + (1-k)\varepsilon \int_{\partial\mathcal{D}} \frac{\partial \upsilon}{\partial \nu_\xi}\bigg|_- \left(\frac{x}{\varepsilon}\right) \frac{u^{ij}}{\lambda_i}(x) \mathrm{d}\sigma_x.$$

From Theorem 3.1, the Taylor expansion of u^{ij} is

$$\frac{u^{ij}}{\lambda_i}(x) = \frac{u^{ij}}{\lambda_i}(0) + \frac{\nabla_x u^{ij}}{\lambda_i}(0) \cdot x + \theta\left(x^2\right), \tag{34}$$

where $\theta\left(x^2\right)$ independent of *i*. This gives that:

$$\begin{aligned}&\frac{1-k}{\lambda_i^2}\int_{\partial\mathcal{D}}\frac{\partial u^{ij}}{\partial\nu_x}u^{ij}(x)\mathrm{d}x\\&=\frac{1-k}{\lambda_i^2}\int_{\mathcal{D}}\Delta_x u^{ij}(x)u^{ij}(x)\mathrm{d}x+(1-k)\int_{\partial\mathcal{D}}\frac{\left|\nabla_x u^{ij}\right|^2}{\lambda_i^2}(x)\mathrm{d}x\\&=\varepsilon^2\frac{1-k}{\lambda_i^2}\int_{\mathcal{B}}\Delta_x u^{ij}(\varepsilon\xi)u^{ij}(\varepsilon\xi)\mathrm{d}x+\varepsilon^2\frac{1-k}{\lambda_i^2}|\mathcal{B}|\left|\nabla_x u^{ij}(0)\right|^2+\theta\left(\frac{\varepsilon^3}{\sqrt{\lambda_i}}\right),\end{aligned}\tag{35}$$

where, $\theta(\varepsilon^3)$ independent of *i*.

At the same time, by a Taylor expansion of u^{ij} in (38) about $x=0$, we obtain

$$\begin{aligned}&(1-k)\int_{\partial\mathcal{D}}\frac{\partial\upsilon}{\partial\nu_\xi}\bigg|_-\left(\frac{x}{\varepsilon}\right)\frac{u^{ij}}{\lambda_i}(x)\mathrm{d}\sigma_x\\&=(1-k)\int_{\partial\mathcal{D}}\frac{\partial\upsilon}{\partial\nu_\xi}\bigg|_-\left(\frac{x}{\varepsilon}\right)\left\{\frac{u^{ij}}{\lambda_i}(0)+\frac{\nabla_x u^{ij}}{\lambda_i}(0)\cdot x+\theta\left(x^2\right)\right\}\mathrm{d}\sigma_x\\&=\varepsilon^2(1-k)\int_{\partial\mathcal{B}}\frac{\partial\upsilon}{\partial\nu_\xi}\bigg|_-(\xi)\frac{\nabla_x u^{ij}}{\lambda_i}(0)\cdot\xi\mathrm{d}\sigma_\xi+\theta\left(\frac{\varepsilon^3}{\sqrt{\lambda_i}}\right),\end{aligned}\tag{36}$$

where $\theta(\varepsilon^3)$ independent of *i*.

We choose $0\le\alpha\le 1$ and inserting the above identity, (34), (33) in (31), we get

$$\begin{aligned}&\left\langle\frac{1}{\lambda_i}u^{ij}-\upsilon_\varepsilon^{ij},u^{ij}\right\rangle\\&=\varepsilon^2\frac{1-k}{\lambda_i^2}\left(|\mathcal{B}|\left|\nabla_x u^{ij}(0)\right|^2+\lambda_i\int_{\partial\mathcal{B}}\frac{\partial\upsilon}{\partial\nu_\xi}\bigg|_-(\xi)\frac{\nabla_x u^{ij}}{\lambda_i}(0)\cdot\xi\mathrm{d}\sigma_\xi\right)+\theta\left(\frac{\varepsilon^{\frac{5}{2}}}{\sqrt{\lambda_i}}\right).\end{aligned}$$

We now use the fact

$$\upsilon(\xi)=\frac{1-k}{\lambda_i}\sum_{p=1}^{2}\frac{\partial u^{ij}}{\partial x_p}(0)\phi_p(\xi),\tag{37}$$

where ϕ_p is defined in (28). Putting $v^{ij}=\frac{u^{ij}}{\sqrt{\lambda_i}}$ we obtain the following expansion

$$\frac{1}{\lambda_i}-\frac{1}{m}\sum_{j=1}^{m}\frac{1}{\tilde{\lambda}_\varepsilon}=\frac{\varepsilon^2}{m\lambda_i}\sum_{j=1}^{m}\nabla_x\upsilon^{ij}(0)\cdot\mathcal{M}\nabla_x\upsilon^{ij}(0)+\theta\left(\frac{\varepsilon^{\frac{5}{2}}}{\sqrt{\lambda_i}}\right),$$

where $\mathcal{M}$ is the polarisation tensor which is defined in (27) and $\theta\left(\varepsilon^{\frac{5}{2}}\right)$ independent of *i*.

Our main result in this section is the following:

Theorem 3.6 *For any* $i=E\left(\frac{1}{\varepsilon^\alpha}\right)$. *If* λ_i *is an eigenvalue of* (1) *of multiplicity* *m then there is m eigenvalue of* (2) *which converges to* λ_i, *such that*

$$\left|\overline{\lambda_{i\varepsilon}} - \lambda_i\right| = \theta\left(\varepsilon^{2-\alpha}\right), \text{ for any } 0 \le \alpha \le 1,$$

for any small ε.

Proof. We easily see that far all $j = 1, \cdots, m$ (see [22] for more details)

$$\left|\mu_\varepsilon - \mu\right| = \left|\frac{\lambda_{i\varepsilon} - \lambda_i}{\lambda_{i\varepsilon}\lambda_i}\right|,$$

which we can write,

$$\left|\lambda_{i\varepsilon} - \lambda_i\right| = \left|\mu_\varepsilon - \mu\right|\left|\lambda_{i\varepsilon}\lambda_i\right|.$$

Then according to Lemma 2.2, we have

$$\lambda_{i\varepsilon} - \lambda_i = \varepsilon^2\left(\lambda_i + \theta(1)\right)\nabla_x \upsilon^{ij}(0) \cdot \mathcal{M}\nabla_x \upsilon^{ij}(0) + \theta\left(\varepsilon^{\frac{5}{2}}\left(\lambda_i + \theta\left(\varepsilon^{2-\alpha}\right)\right)\right). \quad (38)$$

The fact that

$$\left|\nabla_x \upsilon^{ij}(0) \cdot \mathcal{M}\nabla_x \upsilon^{ij}(0)\right| \le \mathcal{C},$$

where $\mathcal{C}$ is independent of λ_i and ε which complete the proof.

So, the following theorem holds.

Theorem 3.7 *Suppose* $\lambda^{E\left(\frac{1}{\varepsilon^\alpha}\right)}$ *is an eigenvalue of multiplicity m of* (1), *with an* L^2 *orthonormal basis of eigenfunctions* $\left\{u^{E\left(\frac{1}{\varepsilon^\alpha}\right)j}\right\}_{j=1}^{m}$. *Suppose* $\lambda_\varepsilon^{E\left(\frac{1}{\varepsilon^\alpha}\right)j}$ *are eigenvalues of* (2) *which converge to* $\lambda^{E\left(\frac{1}{\varepsilon^\alpha}\right)}$. *For any* $0 \le \alpha \le 1$, *the following asymptotic expansion holds*:

$$\overline{\lambda_\varepsilon^{E\left(\frac{1}{\varepsilon^\alpha}\right)}} - \lambda^{E\left(\frac{1}{\varepsilon^\alpha}\right)} = \varepsilon^2 \sum_{j=1}^{m}\sum_{l=1}^{N} \nabla_x u^{\left(\frac{1}{\varepsilon^\alpha}\right)j}(z_l) \cdot \mathcal{M}^l \nabla_x u^{\left(\frac{1}{\varepsilon^\alpha}\right)j}(z_l) + \Theta\left(\varepsilon^{\left(\frac{5}{2}-\alpha\right)}\right), \quad (39)$$

where $\mathcal{M}^l$ *is the polarisation tensor associated to* $\mathcal{B}^l$ *and* $\overline{\lambda_\varepsilon^{E\left(\frac{1}{\varepsilon^\alpha}\right)}}$ *is the harmonic average of the* $\lambda_\varepsilon^{E\left(\frac{1}{\varepsilon^\alpha}\right)j}$.

Proof. We have

$$\frac{1}{\lambda_i} - \frac{1}{m}\sum_{j=1}^{m}\frac{1}{\tilde{\lambda}_\varepsilon} = \frac{1}{m}\sum_{j=1}^{m}\left\langle \frac{1}{\lambda_i}u^{ij} - \upsilon_\varepsilon^{ij}, u^{ij}\right\rangle$$

$$= \varepsilon^2 \frac{1-k}{\lambda_i^2}\left(\left\|\mathcal{B}\right|\nabla_x u^{ij}(0)\right|^2 + \lambda_i \int_{\partial\mathcal{B}} \left.\frac{\partial \upsilon}{\partial \nu_\xi}\right|_{-}(\xi)\frac{\nabla_x u^{ij}}{\lambda_i}(0)\cdot\xi \,\mathrm{d}\sigma_\xi\right) + \theta\left(\frac{\varepsilon^{\frac{5}{2}}}{\sqrt{\lambda_i}}\right),$$

and using the assertion (37), we get

$$\frac{1}{\lambda_i} - \frac{1}{m}\sum_{j=1}^{m}\frac{1}{\tilde{\lambda}_\varepsilon} = \frac{\varepsilon^2}{m\lambda_i}\sum_{j=1}^{m}\nabla_x \upsilon^{ij}(0)\cdot\mathcal{M}^l\nabla_x \upsilon^{ij}(0) + \theta\left(\frac{\varepsilon^{\frac{5}{2}}}{\sqrt{\lambda_i}}\right).$$

Then, inserting this in Theorem 3.6 in connection with $v^{ij} = \frac{u^{ij}}{\sqrt{\lambda_i}}$, we prove that

$$\overline{\lambda_{i\varepsilon}} - \lambda_i = \varepsilon^2 \sum_{j=1}^{m} \sum_{l=1}^{N} \nabla_x u^{E\left(\frac{1}{\varepsilon^\alpha}\right)j}(z_l) \cdot \mathcal{M}^l \nabla_x u^{E\left(\frac{1}{\varepsilon^\alpha}\right)j}(z_l) + \Theta\left(\varepsilon^{\left(\frac{5}{2}-\alpha\right)}\right),$$

where $\mathcal{M}^l$ is the polarisation tensor associated to $\mathcal{B}^l$. Finally, we take $i = E\left(\frac{1}{\varepsilon^\alpha}\right)$, for $0 \le \alpha \le 1$, we get our main result. □

4. Numerical Experiments

In this section, we want to give some numerical examples using Freefem++ such that $\mathcal{D} = \varepsilon\mathcal{B}$, the inclusion centred in the origin, for some different values of ε and α we give the largest eigenvalues in any case. Now, we give some arbitrary values of ε and α then we get ε^α (by replacing ε and α each time by its values) as shown in **Table 1**.

Then, according to the values of ε^α we give the different values of the largest eigenvalues as shown in **Table 2**.

Then, we present this following graft how is plotted using Microsoft Office Excel as presented by **Figure 2**.

From this figure we show that the variation of the largest eigenvalues and i are dependent. One more important result presented by this paper that

$$\left|\overline{\lambda_{i\varepsilon}} - \lambda_i\right| = \theta\left(\varepsilon^{2-\alpha}\right), \text{ for any } 0 \le \alpha \le 1,$$

for any small ε.

We remember that $0 \le \alpha \le 1$, so we take some arbitrary values of α and ε to test $\left|\overline{\lambda_{i\varepsilon}} - \lambda_i\right|$. By numerical examples, for the case of $\alpha = \frac{1}{4}$, $\alpha = \frac{1}{2}$ and

Table 1. The values of ε^α.

	$\alpha_1 = \frac{1}{4}$	$\alpha_2 = \frac{1}{2}$	$\alpha_3 = \frac{1}{3}$
$\varepsilon_1 = 2\times10^{-3}$	0.2114	0.044	0.128
$\varepsilon_2 = 1\times10^{-5}$	0.056	0.0031	0.022
$\varepsilon_3 = 4\times10^{-4}$	0.141	0.2	0.075

Table 2. The different values of the largest eigenvalues.

	$\alpha_1 = \frac{1}{4}$	$\alpha_2 = \frac{1}{2}$	$\alpha_3 = \frac{1}{3}$
$\varepsilon_1 = 2\times10^{-3}$	1589.86	2989.19	1610.88
$\varepsilon_2 = 1\times10^{-5}$	8416.03	602192	11956.8
$\varepsilon_3 = 4\times10^{-4}$	291.085	1018.37	22.116

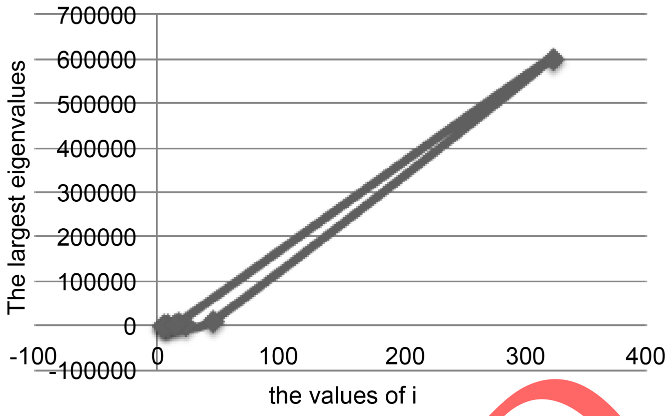

Figure 2. The variation of the largest eigenvalues depending of *i*.

$\varepsilon = 2\times10^{-3}$, we have:

$$\varepsilon^{2-\alpha} = 4.11148321e, \quad \text{for } \alpha = \frac{1}{4},$$

and

$$\varepsilon^{2-\alpha} = 3.994148321e, \quad \text{for } \alpha = \frac{1}{2}.$$

In other part, if we calculate numerically $\left|\overline{\lambda_{i\varepsilon}} - \lambda_i\right|$, we get:

$$\left|\overline{\lambda_{i\varepsilon}} - \lambda_i\right| = 4.1688e - 012, \quad \text{for } \alpha = \frac{1}{4} \text{ and } \varepsilon = 2\times10^{-3},$$

and

$$\left|\overline{\lambda_{i\varepsilon}} - \lambda_i\right| = 3.8467e - 011, \quad \text{for } \alpha = \frac{1}{2} \text{ and } \varepsilon = 2\times10^{-3}.$$

5. Conclusion

In this paper, a rigorous derivation of asymptotic formula was provided for the largest eigenvalues using the convergence estimation of the eigenvalues of a sequence of self-adjoint compact operators satisfying the following property:

$$\left|\overline{\lambda_{i\varepsilon}} - \lambda_i\right| = \theta\left(\varepsilon^{2-\alpha}\right), \quad \text{for any } 0 \le \alpha \le 1,$$

which are tested by numerical experiments.

Conflicts of Interest

The authors declare no conflicts of interest.

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