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Stability & Chaos Tests of 2D Discrete Time Dynamical System with Hidden Attractors

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Abstract

In this research two-dimensional dynamical system was taken. The system was analyzed through its fixed points, stability analysis, chaos diagnoses and adaptive control technique. It was found that the system had a hidden attractor and unstable fixed points. The largest value of Lyapunov exponent equals 1.853981 and from binary test we get: k = 0.9305, which indicates that the system is chaotic. Adaptive control technique was performed, and it was found that the system is stable and regular after control.

Subject Areas

Dynamical System

Keywords

Hidden Attractors, Chaos, Lyapunov Exponent, Lyapunov Dimension, Binary Test, Adaptive Control

1. Introduction

In recent years, we have seen that there is great interest in the subject of hidden and self-Excited attractors, the attractors in chaotic dynamical systems are classified into two categories: dynamical systems with self-Excited attractors and dynamical systems with hidden attractors, as it is said for the systems whose basin of attraction intersects with the neighborhood. The Lorenz system [1] is one of the chaotic systems that have self-Excited attractors, which was proposed in 1963, while the systems in which the basin of attraction does not intersect with the open neighborhood of equilibrium and which does not have fixed points or equilibrium or it has only one stable fixed point or nodal points, so it is called a dynamical system with hidden attractors. Jerk-like system [2] is one of the systems that have hidden attractors. There are many researchers who have provided a number of models in dynamical systems that have these types of attractors, Jafari and Sprott [3] managed to create a list of 17 chaotic models with nonlinear quadratic boundaries that have no equilibrium points, which were classified by Leonov and Kuznetsov [4] [5] as systems with hidden attractors, and Chan and Wang introduced an anarchic system with only one stable point equilibrium [6] through some modifications to Sprott system [7], and Wei [8] designed a four-dimensional system, which turned out to be a hyper-chaotic system without equilibrium points.

Models in dynamical systems have hidden attractors for many applications, especially in the field of engineering designs, because they do not allow for unforeseen accidents [7] such as potential disturbances in the structure of bridges and aircraft wings [9] and others. This paper was arranged as follows: system description, system analysis and finding fixed points, stability analysis of fixed points using stability tests: characteristic roots test, jury test [10] [11], study of the dynamical behavior of the system, finding phase space and bifurcation diagrams [12] [13] and Chaos diagnosing using chaos tests: the Lyapunov exponent test and Lyapunov dimension [14] [15], binary (0 - 1) test [16] [17]. The problem of chaos control attract attention of mathematician, researchers and engineers and there are many practical reasons for controlling chaos one of them chaos causes irregular behavior in nonlinear dynamical systems, therefor chaos control should be eliminated as much as possible or totally suppressed, adaptive control technique [18] [19] were used to control.

2. System Description

Two-dimensional discrete time dynamical system, suggested by Panahi [20] [21] and the define is as follows:-

$$F=\begin{cases} x_{t+1}=x_t+y_t \\ y_{t+1}=0.1x_t^2+0.1+y_t\left(-x_t-by_t\right)+y_t \end{cases} \tag{1}$$

$$b=2 \tag{2}$$

where x_t, y_t represent variables system and b represent parameter system.

3. System analysis

In this section analysis the system was done by means of its fixed points. Let us assume that:-

$$f_1\left(x_t, y_t\right)=x_t$$
$$f_2\left(x_t, y_t\right)=y_t$$

To find the fixed points of system (1), suppose that:

$$y_t=0 \tag{3}$$

$$0.1x_t^2+0.1+y_t\left(-x_t-by_t\right)+y_t=y_t \tag{4}$$

By solving Equations (3) and (4), we obtain the following points:-

$$p_0 = (i, 0), \quad p_1 = (-i, 0)$$

Since the obtained fixed points are nodal points, we can say that the dynamical system (1) have hidden attractors.

The Jacobian matrix of system (1) obtained as:

$$J_{(x_t, y_t)} = \begin{bmatrix} 1 & 1 \\ 0.2x_t - y_t & -x_t - 2by_t + 1 \end{bmatrix} \quad (5)$$

Proposition (1):- Let

$$L(\lambda) = a_2\lambda^2 + a_1\lambda + a_0 = 0 \quad (6)$$

A characteristic Equation of (5) and $|\lambda_i|, i = 1, 2$ are values of the roots of Equation (6), then the following cases are true:

1) If $|\lambda_i| < 1$, then the fixed point of system (1) is locally asymptotically stable and is called the sink.

2) If $|\lambda_i| > 1$, then the fixed point of system (1) is unstable and is usually called source, but if at least one of the values of the roots of Equation (6) is greater than one, then the fixed point is called Saddle.

3) If $|\lambda_i| = 1$, then the fixed point of system (1) is called Non-Hyperbolic point, but if $|\lambda_i| \neq 1$, then the fixed point is called Hyperbolic point.

Proposition (2):-

From the Equation (6), then the jury Table 1 is obtain as:

Such that

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, k = 0, 1, \ n = 2, \quad c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}, k = 0, n = 2$$

Then the fixed point of system (1) is called stable if the satisfies following conditions:

$$L(1) > 0, \quad (-1)^n L(-1) > 0, \quad |a_0| < a_n, \quad |b_0| > |b_{n-1}|, \quad |c_0| > |c_{n-2}|,$$

where L is the characteristic equation. Otherwise, the fixed point is called unstable.

4. Stability Analysis

In this section the stability analysis of the fixed points of the system (1) is analyzed.

Table 1. Jury table.

λ^0	λ^1	λ^2
a_0	a_1	a_2
a_2	a_1	a_0
b_0	b_1	
b_1	b_0	
c_0		

Characteristic Equation Roots Test

We will test the stability of the fixed points of system (1) by using the test of roots characteristic equation. To test the stability of the fixed point p_0, we substitute the point $p_0 = (i,0)$ in Equation (5), we obtain:-

$$J_{(i,0)} = \begin{bmatrix} 1 & 1 \\ 0.2i & -i+1 \end{bmatrix}$$

Finding the determinant ($\text{Det}(\lambda I - J) = 0$), we get:-

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0.2i & -i+1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \lambda^2 + (-2+i)\lambda + (1-1.2i) = 0 \tag{7}$$

By analyzing Equation (7), we get:-

$$\lambda_1 = 1, \quad \lambda_2 = 1$$

From the Proposition (1) we obtain the following values:-

$$|\lambda_1| = 1, \quad |\lambda_2| = 1$$

Then the fixed point $p_0 = (i,0)$ is a non-hyperbolic point, and in the same way point $p_1 = (-i,0)$ was tested, and its results are shown in Table 2, which turned out to be non-hyperbolic as well.

Note (1)

In this system, we did not use Jury test to study the stability of fixed points of the system, since Jury test depends on the coefficients Equation (7) which represent complex values.

5. Dynamical Behavior and Numerical Results of System (1)

After studying the theoretical side of the system, we will discuss the practical side of the system, and at the beginning we will use Newton Raphson's numerical method, Newton Raphson's numerical method was used to generate the best values for the system (1) with the least possible error, as the system parameter was fixed at the value $b = 2$, the following values were obtained: $(x, y) = (1.8, -0.39)$ and an error of (0.0001). Time behavior of system (1) was studied by generating a time series of system states with time ($t = 100$), shown in Figure 1, which shows the unstable behavior of system states x_t, y_t with time. The phase space was found for the variables system (1), shown in Figure 2, which shows us the paths for variables system (1) with hidden attractors. The bifurcation diagrams

Table 2. Results of the points test p_0, p_1 for system (1) using the characteristic equation roots test.

Fixed points	Results of the roots of the characteristic equations of points	Status
$p_0 = (i,0)$	$\lambda_1 = 1, \lambda_2 = 1$	Non-hyperbolic
$p_1 = (-i,0)$	$\lambda_1 = 1, \lambda_2 = 1$	Non-hyperbolic

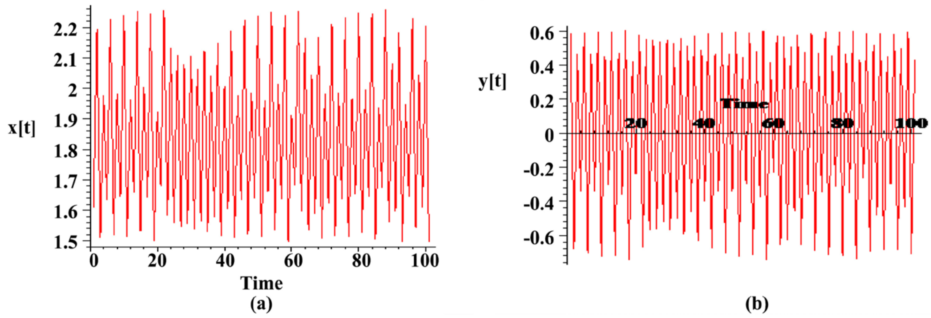


Figure 1. Time behavior of system (1). (a): x_t versus Time; (b): y_t versus Time.

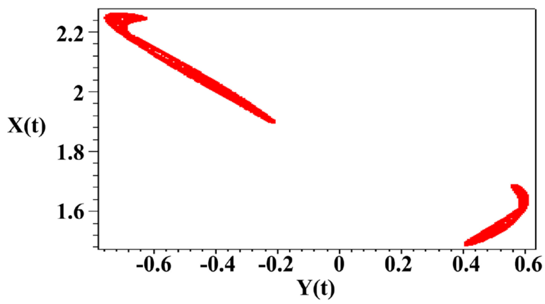


Figure 2. The phase space of variables system (1).

of the bifurcation parameter of the system (1) were found, the parameter b was fixed at the interval ranging from 1.975 to 2, shown in Figure 3, which shows the bifurcation of parameter b with variable x_t, y_t indicating chaotic behavior of the system (1).

6. Lyapunov Exponent and Lyapunov Dimension

To study the chaotic behavior of system (1) we will use Lyapunov exponent test, and after applying and calculating Lyapunov exponent of system (1) the following values were obtained:-

$$L_1 = -0.250000, \quad L_2 = 1.853981$$

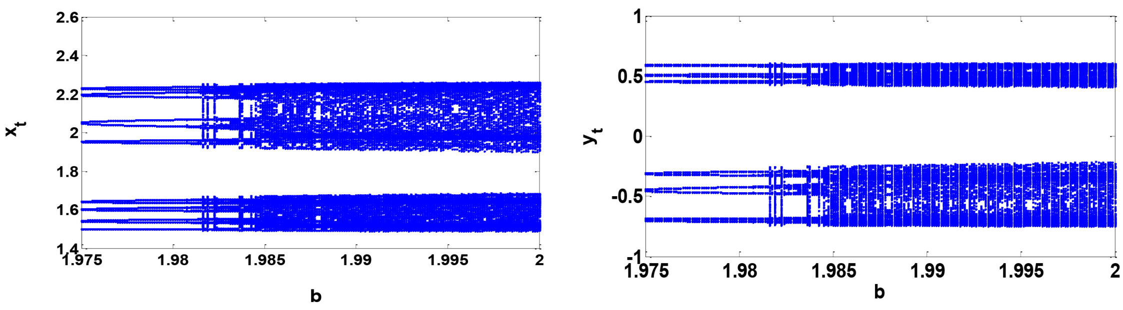
And since one of the values of Lyapunov exponent is a positive value, this indicates that system (1) is a chaotic system, and Figure 4 shows us the chaotic behavior of system (1).

To find Lyapunov dimension, we use the following formula:

$$p + \frac{L_1}{|L_2|} = 1 + \frac{-0.250000}{|1.853981|} = 1 - 0.134844 = 0.865156 = D_L$$

7. Binary Test

To study and define the mess more broadly, binary test (0 - 1) was used to determine the chaos of system (1), and by using a mathematical program in MATLAB, the system binary test was applied as a time series of system states and (1000) were generated from the iterations with a fixed value Parameter $b = 2$ and $(x, y) = (1.8, -0.39)$, and $q_c(t)$ was calculated against $p_c(t)$ and shown



(a) (b)

Figure 3. Bifurcation for parameter b with variables system (1). (a) x_t versus b; (b) y_t versus b.

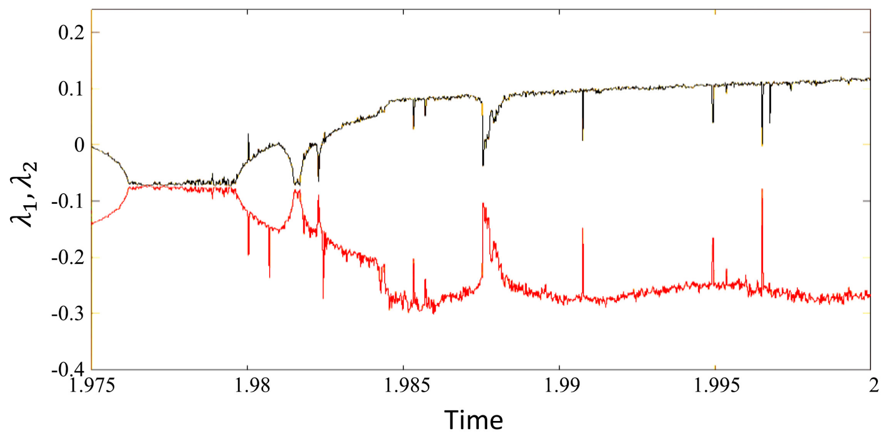


Figure 4. Lyapunov exponent of system (1).

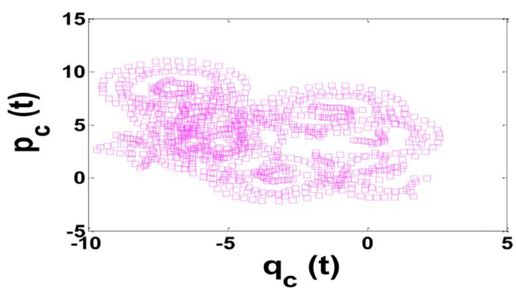


Figure 5. $p_c(t)$ versus $q_c(t)$ for system (1).

in **Figure 5**, which shows us a behavior similar to Brownian motion, and the mean square of the displacement $M_c(t)$ was found with time (t = 100) and shown in **Figure 6**, which shows the linear growth of the average square of the displacement with time, and the mean (k) was found for the growth aligned with K_c with (c) where $c \in (0, \pi)$, as it was Getting $k = 0.9305 \cong 1$, which indicates chaotic of the system (1), shown in **Figure 7**.

8. Adaptive Control Technique

To address the chaos of system (1) we will use the adaptive control technique and design an adaptive control law with the unknown parameter b of the system.

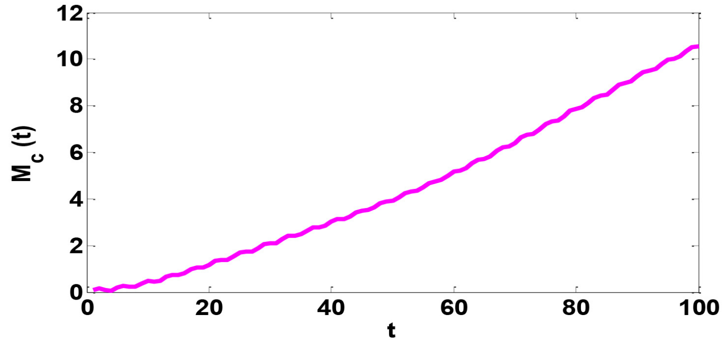


Figure 6. $M_c(t)$ versus time t for system (1).

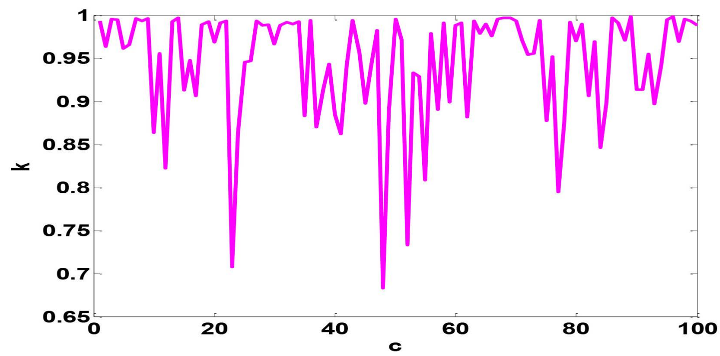


Figure 7. k versus c for system (1).

By adding the control units to the system, we get:-

$$\begin{cases} x_{t+1} = x_t + y_t + u_1 \\ y_{t+1} = 0.1x_t^2 + 0.1 + y_t\left(-x_t - by_t\right) + y_t + u_2 \end{cases} \tag{8}$$

where u_1, u_2 are the controllers for the adaptive feeding, defined as follows:-

$$\begin{cases} u_1 = -x_t - y_t - M_1 x_t \\ u_2 = -0.1x_t^2 - 0.1 + x_t y_t + \beta y_t^2 - y_t - M_1 y_t \end{cases} \tag{9}$$

where β is an approximate parameter of b and M_1, M_2 are positive constants by substituting (9) in (8) we get:

$$\begin{cases} x_{t+1} = -M_1 x_t \\ y_{t+1} = \left(-b + \beta\right) y_t^2 - M_2 y_t \end{cases} \tag{10}$$

Let the error for the discretionary parameter as follows:

$$e_b = -b + \beta \tag{11}$$

By substituting (11) into (10):

$$\begin{cases} x_{t+1} = -M_1 x_t \\ y_{t+1} = e_b y_t^2 - M_2 y_t \end{cases} \tag{12}$$

8.1. Numerical Results of System (10)

In this section we will stability test of fixed points p_0, p_1 for system (1) in system (10), where $M_1 = 0.7$, $M_2 = 0.5$ and parameter β is an estimated parameter of b which is estimated at = 2.1.

8.1.1. Characteristic Equation Roots Test

To study the stability of fixed points in System (1), using the characteristic equation root test, we will find the Jacobian Matrix of System (10) as follows:

$$J_{(x_t, y_t)} = \begin{bmatrix} -M_1 & 0 \\ 0 & 2(-b+\beta)y_t - M_2 \end{bmatrix} \tag{13}$$

Substituting fixed point $p_0 = (i, 0)$ and the values of M_1, M_2, into Equation (13) we get:-

$$J_{(0,0)} = \begin{bmatrix} -0.7 & 0 \\ 0 & -0.5 \end{bmatrix}$$

Finding the determinant ($\text{Det}(\lambda I - J) = 0$) we obtain the following equation:-

$$\lambda^2 + 1.2\lambda + 0.35 = 0 \tag{14}$$

By analyzing the quadratic Equation (14), and Proposition (1) we get:-

$$|\lambda_1| = 0.5, |\lambda_2| = 0.7$$

Therefore, the point p_0 is stable. Since point p_1 is symmetric to point p_0, it has the same results, that is, it is also a stable point, which leads to the system (9) being a stable system.

8.1.2. Jury Test

After the adaptive control technique of system was carried out and the characteristic equation roots were tested for the points in the controlled system (10), distinct equations were obtained with the values of their real coefficients, and thus it became possible for us to use a jury test to study the stability of fixed points in system (10). The coefficients of characteristic Equation (14) are:

$$a_0 = 0.35, \quad a_1 = 1.2, \quad a_2 = 1$$

Accordingly, we form the Jury test Table 3 for point p_0 as follows:

Since all of the conditions for Jury test for point p_0 are satisfies Proposition (2), then point p_0 is stable, and point p_1 is stable too, which leads to the system (10) is stable.

8.2. Lyapunov Exponent of System (10)

In this section we will use Lyapunov exponent test to study the chaotic behavior of

Table 3. Table of Jury test for point p_0 in the system (10).

λ^0	λ^1	λ^2
0.35	1.2	1
1	1.2	0.35
−0.8775	−0.78	
−0.78	−0.8775	
0.1616		

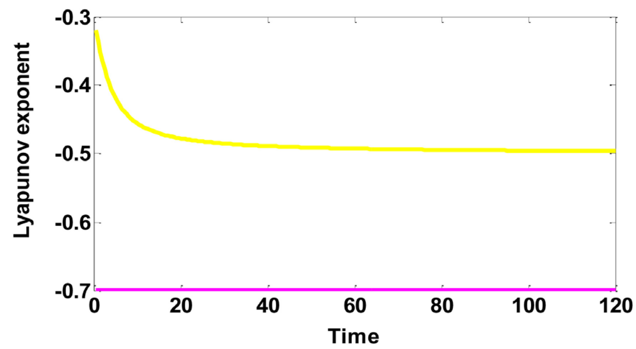


Figure 8. Lyapunov exponent of system (10).

system (10), and by using a written program in MATLAB, we obtained:-

$$L_1 = -0.700000, \quad L_2 = -0.418868$$

And since all the obtained values are negative values, this means that system (10) is a regular system as shown in Figure 8.

9. Conclusion

In this paper, the stability and chaos of a two-dimensional discrete time dynamical system with hidden attractors was studied, and the fixed points were found: $p_0 = (i, 0)$, $p_1 = (-i, 0)$ which shows us that the points are complex. The stability analysis of fixed points was analyzed by using the characteristic equation roots test, which shows us that the points are non-hyperbolic points, so we could not use jury test for the stability, the dynamic behavior of system was analyzed and a time series was generated for each state of system and phase space and bifurcation diagrams of the system were found. Chaos diagnosis in the system by using Lyapunov exponent test and it was obtained: $L_1 = -0.250000$, $L_2 = 1.853981$. Lyapunov dimension: $D_L = 0.865156$, and from the binary test (0 - 1) we get the index value k = 0.9305, which is close to one, all show that the system (1) is chaotic. The adaptive control technique of the chaotic system was carried out. Stability tests: roots of the characteristic equation and jury test for points in the controlling system (10), and Lyapunov exponent test obtaining the values: $L_1 = -0.700000$, $L_2 = -0.41886$, which all show that the system after control is stable and regular.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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