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# Elastic Layer on the Elastic Half-Space: The Solution in Matrixes

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## Abstract

If to apply bidimensional Fourier's transform to homogeneous system of equations of the theory of elasticity, then we will receive system of ordinary differential equations. The general solution of this system contains 6 arbitrary constants and allows to solve problems for the layer and the multilayer environment. It is shown that it is convenient to do statement and the solution of such tasks in the matrix form. The task for the layer on the elastic half-space is solved. Ways of inverse of Fourier's transformation are considered.

## Subject Areas

Theory of Elasticity, Matrixes

## Keywords

Bidimensional Fourier's Transformation

## 1. Introduction

Tens of monographs devoted to the solution of various tasks of the classical theory of elasticity are published. Authors apply different methods of modern mathematics. However, modern mathematical programs for computers give the real chance to effectively apply one more method: statement and the solution of tasks in the matrix form after bidimensional Fourier's transform. This work is devoted to it. We will consider the task for the layer on the elastic half-space which has the particular interest for sciences of the Earth.

## 2. Bidimensional Fourier's Transformation

Bidimensional Fourier's transform will be used in the look

$$\tilde{f}(\xi,\eta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{e}^{(\xi x+\eta y)i}\mathrm{d}\xi\mathrm{d}\eta=\int_{0}^{\infty}\int_{-\pi}^{\pi} rf(x,y)\mathrm{e}^{ir\rho\sin\tau}\mathrm{d}\tau\mathrm{d}r \tag{2.1}$$

where $i=\sqrt{-1}$, $r=\sqrt{x^2+y^2}$, $\rho=\sqrt{\xi^2+\eta^2}$ and in the second integral $x=\frac{r}{\rho}(\xi\sin\tau+\eta\cos\tau)$, $y=\frac{r}{\rho}(-\xi\cos\tau+\eta\sin\tau)$.

Then inverse transformation receives the form

$$f(x,y)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}(\xi,\eta)\mathrm{e}^{-(\xi x+\eta y)i}\mathrm{d}\xi\mathrm{d}\eta=\frac{1}{4\pi^2}\int_{0}^{\infty}\int_{-\pi}^{\pi}\rho\tilde{f}(\xi,\eta)\mathrm{e}^{ir\rho\sin\tau}\mathrm{d}\tau\mathrm{d}\rho \quad (2.2)$$

wherein the second integral $\xi=-\frac{\rho}{r}(x\sin\tau+y\cos\tau)$, $\eta=\frac{\rho}{r}(x\cos\tau-y\sin\tau)$.

Transformations of derivatives are defined by formulas

$$Fo\left(\frac{\partial^n f(x,y)}{\partial x^n}\right)=(-i\xi)^n\frac{\partial\tilde{f}}{\partial\xi},\quad Fo\left(\frac{\partial^n f(x,y)}{\partial y^n}\right)=(-i\eta)^n\frac{\partial\tilde{f}}{\partial\eta} \quad (2.3)$$

The following representations of Bessel functions are useful to inverse transformation of Fourier

$$\int_{-\pi}^{\pi}\mathrm{e}^{iz\sin\tau}\mathrm{d}\tau=2\pi J_0(z),\quad \int_{-\pi}^{\pi}i\sin(\tau)\mathrm{e}^{iz\sin\tau}\mathrm{d}\tau=-2\pi J_1(z),\quad \int_{-\pi}^{\pi}\cos(\tau)\mathrm{e}^{iz\sin\tau}\mathrm{d}\tau=0 \quad (2.4)$$

## 3. Displacements and Stresses

Homogeneous equations of balance of homogeneous isotropic elastic environment in displacements have the form

$$\mathrm{drad}\,\mathrm{div}\,\boldsymbol{u}+(1-2\nu)\nabla^2\boldsymbol{u}=0 \quad (3.1)$$

where $\boldsymbol{u}=[u,v,w]$ is displacement vector, $\nabla$ is the nabla-operator, $\nu$ is Poisson's coefficient.

If to (3.1) to apply bidimensional Fourier's transform on *x*, *y*, then the general solution of such system receives the form

$$\begin{aligned}\tilde{u}&=(\chi A_1+\xi zG)\mathrm{e}^{-\rho z}+(\chi B_1+\xi zQ)\mathrm{e}^{\rho z}\\ \tilde{v}&=(\chi A_2+\eta zG)\mathrm{e}^{-\rho z}+(\chi B_2+\eta zQ)\mathrm{e}^{\rho z}\\ \tilde{w}&=(\chi A_3-\rho zGi)\mathrm{e}^{-\rho z}+(\chi B_3+\rho zQi)\mathrm{e}^{\rho z}\end{aligned} \quad (3.2)$$

where $A_i$ and $B_i$ are arbitrary constants, $G=-\frac{\xi A_1+\eta A_2-\rho A_3 i}{\rho}$, $Q=\frac{\xi B_1+\eta B_2+\rho B_3 i}{\rho}$, $\chi=3-4\nu$.

For the first time such solution was received in work [1]. In the matrix type of the transform of movements and stresses take the form

$$\tilde{\boldsymbol{u}}(z)=\frac{\mathrm{e}^{-\rho z}}{\rho}\cdot U\times A+\frac{\mathrm{e}^{\rho z}}{\rho}\cdot V\times B,\quad \tilde{\boldsymbol{s}}(z)=\frac{\mu\mathrm{e}^{-\rho z}}{\rho}\cdot S\times A+\frac{\mu\mathrm{e}^{\rho z}}{\rho}\cdot P\times B \quad (3.3)$$

where $\mu$ is the shear modulus, $\times$ is multiplication of matrixes,

$$\tilde{\boldsymbol{u}}(z)=\begin{Vmatrix}\tilde{u}\\ \tilde{v}\\ \tilde{w}\end{Vmatrix},\quad \tilde{\boldsymbol{s}}(z)=\begin{Vmatrix}\tilde{\sigma}_{zx}\\ \tilde{\sigma}_{zy}\\ \tilde{\sigma}_{zz}\end{Vmatrix},\quad A=\begin{Vmatrix}A_1\\ A_2\\ A_3\end{Vmatrix},\quad B=\begin{Vmatrix}B_1\\ B_2\\ B_3\end{Vmatrix},\quad \alpha=1-2\nu,\quad \beta=1-\nu$$

$$U=\begin{Vmatrix} \chi\rho-\xi^2 z & -\xi\eta z & \xi\rho zi \\ -\xi\eta z & \chi\rho-\eta^2 z & \eta\rho zi \\ \xi\rho zi & \eta\rho zi & \rho(\chi+\rho z) \end{Vmatrix},\quad V=\begin{Vmatrix} \chi\rho+\xi^2 z & \xi\eta z & \xi\rho zi \\ \xi\eta z & \chi\rho+\eta^2 z & \eta\rho zi \\ \xi\rho zi & \eta\rho zi & \rho(\chi-\rho z) \end{Vmatrix}$$

$$S=\begin{Vmatrix} 2\xi^2 z\rho-\chi\rho^2-\xi^2 & \xi\eta(2\rho z-1) & -2\xi\rho(\rho z+\alpha)i \\ \xi\eta(2\rho z-1) & 2\eta^2 z\rho-\chi\rho^2-\eta^2 & -2\eta\rho(\rho z+\alpha)i \\ -2\xi\rho(\rho z-\alpha)i & -2\eta\rho(\rho z-\alpha)i & 2\rho^2(\rho z+2\beta) \end{Vmatrix}$$

$$P=\begin{Vmatrix} 2\xi^2 z\rho+\chi\rho^2+\xi^2 & \xi\eta(2\rho z+1) & 2\xi\rho(\rho z-\alpha)i \\ \xi\eta(2\rho z+1) & 2\eta^2 z\rho+\chi\rho^2+\eta^2 & 2\eta\rho(\rho z-\alpha)i \\ 2\xi\rho(\rho z+\alpha)i & 2\eta\rho(\rho z+\alpha)i & -2\rho^2(\rho z-2\beta) \end{Vmatrix}$$

## 4. Statement and the Solution of the Problem.

We consider it expedient to remind that the algebra of matrixes differs from algebra of numbers. The main difference consists that matrix multiplication is noncommutativity.

Let's consider the problem about the elastic layer on the surface of the elastic half-space. To simplify calculations, we will consider Poisson's coefficient identical in the layer and the half-space and $\nu=1/4$. At calculations it is necessary to carry out the main simplification: $\xi^2+\eta^2=\rho^2$.

The layer $-h\le z\le 0$ is defined by formulas

$$\tilde{\boldsymbol{u}}(z)=\frac{e^{-\rho z}}{\rho}\cdot U\times A+\frac{e^{\rho z}}{\rho}\cdot V\times B,\quad \tilde{\boldsymbol{s}}(z)=\frac{\mu e^{-\rho z}}{\rho}\cdot S\times A+\frac{\mu e^{\rho z}}{\rho}\cdot P\times B,\quad -h\le z\le 0 \tag{4.1}$$

and the half-space $z\ge 0$

$$\tilde{\boldsymbol{w}}(x)=\frac{e^{-\rho z}}{\rho}\cdot U\times C,\quad \tilde{\boldsymbol{g}}(z)=\frac{m\mu e^{-\rho z}}{\rho}\cdot S\times C \tag{4.2}$$

where $\tilde{\boldsymbol{w}}(x)$ is matrix of displacements, $\tilde{\boldsymbol{g}}(z)$ is matrix of stresses, $m\mu$ is the shear modulus in half-space, $m\ne 1$ and $C$ is matrix of arbitrary constants.

On a layer surface at $z=-h$ the following single forces are possible:

$\sigma_{zz}(x,y,-h)=-\delta(x)\delta(y)$, $\sigma_{zx}(x,y,-h)=\sigma_{zy}(x,y,-h)=0$, matrix is $G$;

$\sigma_{zx}(x,y,-h)=-\delta(x)\delta(y)$, $\sigma_{zz}(x,y,-h)=\sigma_{zy}(x,y,-h)=0$, matrix is $Gx$;

$\sigma_{zy}(x,y,-h)=-\delta(x)\delta(y)$, $\sigma_{zz}(x,y,-h)=\sigma_{zx}(x,y,-h)=0$, matrix is $Gy$.

where $\delta$ is delta-function.

The mentioned matrixes have the form

$$G=\begin{Vmatrix} 0 \\ 0 \\ -1 \end{Vmatrix},\quad Gx=\begin{Vmatrix} -1 \\ 0 \\ 0 \end{Vmatrix},\quad Gy=\begin{Vmatrix} 0 \\ -1 \\ 0 \end{Vmatrix} \tag{4.3}$$

Thus, we have two boundary conditions: on demarcation of at $z=0$ the continuity of movements and stresses is observed and on the surface of the layer at $z=-h$ single force is applied. As a result, system of equations for definition of matrixes of $A$, $B$ and $C$ receive the look

$$\tilde{\boldsymbol{u}}(0)=\tilde{\boldsymbol{w}}(0),\quad \tilde{\boldsymbol{s}}(0)=\tilde{\boldsymbol{g}}(0),\quad \tilde{\boldsymbol{s}}(-h)=G \tag{4.4}$$

In the right-hand member of the third Equation (4.4) it is possible to substitute matrixes of $Gx$ or $Gy$ depending on the objective.

The first two equations of system (4.4) in expanded form have the form

$$A+B=C\,,\quad So\times A+Po\times B=m\cdot So\times C \tag{4.5}$$

where $So$ and $Po$ are matrixes $S$ и $P$ at $z=0$.

The system (4.4) has the solution

$$B=(m-1)\cdot Nb\times C\,,\quad A=Na\times C \tag{4.6}$$

where $Nb=\left(So^{-1}\times Po-E\right)^{-1}$, $Na=E-(m-1)\cdot Nb$.

Then the equations for the layer receive the form

$$\begin{aligned}\tilde{\boldsymbol{u}}(z)&=\frac{\mathrm{e}^{-\rho z}}{\rho}\cdot U\times Na\times C+\frac{\mathrm{e}^{\rho z}(m-1)}{\rho}\cdot V\times Nb\times C\\ \tilde{\boldsymbol{s}}(z)&=\frac{\mu\mathrm{e}^{-\rho z}}{\rho}\cdot S\times Na\times C+\frac{\mu\mathrm{e}^{\rho z}(m-1)}{\rho}\cdot P\times Nb\times C\end{aligned} \tag{4.7}$$

and the third equation of system (4.4) receives the form

$$\frac{\mu\mathrm{e}^{\rho h}}{\rho}\cdot Sh\times A+\frac{\mu\mathrm{e}^{-\rho h}}{\rho}\cdot Ph\times B=G \tag{4.8}$$

where $Sh$ and $Ph$ are matrixes $S$ и $P$ at $z=-h$.

Solution of (4.8) is

$$C=\frac{\rho}{\mu}\cdot\left(\mathrm{e}^{\rho h}\cdot Sh\times Na+\mathrm{e}^{-\rho h}\cdot(m-1)\cdot Ph\times Nb\right)^{-1}\times G \tag{4.9}$$

or

$$\begin{aligned}C&=\frac{\rho}{\mu}\cdot\mathrm{e}^{-\rho h}\cdot Cn\\ &=\frac{\rho}{\mu}\cdot\mathrm{e}^{-\rho h}\cdot\left\|\begin{matrix}\dfrac{3i\xi\left(4\rho h(m-1)\mathrm{e}^{-2\rho h}+2\rho hm-3m+4\rho h\right)}{8\rho^3F}\\ \dfrac{3i\eta\left(4\rho h(m-1)\mathrm{e}^{-2\rho h}+2\rho hm-3m+4\rho h\right)}{8\rho^3F}\\ -\dfrac{3\left(4(m-1)(\rho h-1)\mathrm{e}^{-2\rho h}-4-4\rho h-5m-2\rho hm\right)}{8\rho^2F}\end{matrix}\right\|\end{aligned} \tag{4.10}$$

where $\begin{aligned}F&=2(m-1)^2\mathrm{e}^{-4\rho h}+(m-1)\left(4\rho^2h^2m+5m+4+8\rho^2h^2\right)\mathrm{e}^{-2\rho h}\\ &\quad+(m+2)(2m+1)\end{aligned}$.

Now solutions for a layer and a half-space take a form

$$\begin{aligned}\tilde{\boldsymbol{u}}(z)&=\frac{\mathrm{e}^{-\rho(z+h)}}{\mu}\cdot U\times Na\times Cn+\frac{\mathrm{e}^{-\rho(h-z)}(m-1)}{\mu}\cdot V\times Nb\times Cn\\ \tilde{\boldsymbol{s}}(z)&=\mathrm{e}^{-\rho(z+h)}\cdot S\times Na\times Cn+\mathrm{e}^{-\rho(h-z)}(m-1)\cdot P\times Nb\times Cn\\ \tilde{\boldsymbol{w}}(z)&=\frac{\mathrm{e}^{-\rho(z+h)}}{\mu}\cdot U\times Cn\,,\quad \tilde{\boldsymbol{g}}(z)=m\mathrm{e}^{-\rho(z+h)}\cdot S\times Cn\end{aligned} \tag{4.11}$$

It is reasonable to carry out calculations on formulas (4.11) at the numerical values *m*. In the final solution it makes sense to make substitution $z \to z-h$, and then the layer and the half-space will occupy areas $0 \le z \le h$ and $z \ge h$.

## 5. The Result

We will not provide all solutions which can be received on formulas (4.11). Let's give only the formula for displacements in the half-space. Calculations were made at $m=5$ and substitution $z \to z-h$ was made. Displacements have the form

$$\tilde{\boldsymbol{w}}(z) = \frac{e^{-\rho z}}{2\mu} \left\| \begin{matrix} \dfrac{3i\xi\left(f_1\rho z + f_2\right)}{2\rho^2 f_0} \\ \dfrac{3i\eta\left(f_1\rho z + f_2\right)}{2\rho^2 f_0} \\ \dfrac{3\left(f_1\rho z + f_3\right)}{2\rho f_0} \end{matrix} \right\| \tag{5.1}$$

where $z \ge h$, $f_0 = 32e^{-4\rho h} + \left(112\rho^2h^2 + 116\right)e^{-2\rho h} + 77$,
$f_1 = \left(8 - 16\rho h\right)e^{-2\rho h} + 22$, $f_2 = \left(8\rho h + 16\rho^2h^2\right)e^{-2\rho h} - 8\rho h - 15$,
$f_3 = \left(16\rho^2h^2 - 24\rho h + 16\right)e^{-2\rho h} - 8\rho h + 29$.

Inverse transformations are carried out by means of section 2 formulas. Let's give two examples for arbitrary $H(\rho)$ function

$$Fo^{-1}\left(H(\rho)\right) = \frac{1}{4\pi^2}\int_0^\infty \rho H(\rho) \int_{-\pi}^{\pi} e^{ir\rho\sin(\tau)} d\tau d\rho = \frac{1}{2\pi}\int_0^\infty \rho H(\rho) J_0(r\rho) d\rho \tag{5.2}$$

and

$$\begin{aligned} Fo^{-1}\left(\frac{i\xi}{\rho}H(\rho)\right) &= \frac{1}{4\pi^2}\int_0^\infty \rho H(\rho) \int_{-\pi}^{\pi} \frac{i\xi}{\rho} e^{ir\rho\sin(\tau)} d\tau d\rho \\ &= -\frac{1}{4\pi^2 r}\int_0^\infty \rho H(\rho) \int_{-\pi}^{\pi} i\left(x\sin\tau + y\cos\tau\right) e^{ir\rho\sin(\tau)} d\tau d\rho \\ &= -\frac{x}{4\pi^2 r}\int_0^\infty \rho H(\rho) \int_{-\pi}^{\pi} i\sin\tau e^{ir\rho\sin(\tau)} d\tau d\rho \\ &= \frac{x}{2\pi r}\int_0^\infty \rho H(\rho) J_1(r\rho) d\rho \end{aligned} \tag{5.3}$$

Calculation of the received integrals takes 1 - 2 seconds of machine time and allows to investigate the decision in detail.

## 6. Conclusion

By the offered method, it is possible to solve several other problems for the layer. The solution in all cases turns out rather simple and is uniform. It is sometimes simpler to solve again the known problem, than to look for its solution in literature.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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