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Expansion of the Shape of Numbers

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Abstract

This article extends the concept of the shape of numbers. Originally, a shape was defined as $[1, K_1, K_2, \cdots]$, $1 < K_1 < K_2 < \cdots$, $K_i \in N$. In this paper, the domain of a shape is extended from *N* to *Z*, the low bound is extended from 1 to *Z*, and $K_i < K_{i+1}$, $K_i = K_{i+1}$, $K_i > K_{i+1}$ are allowed, which prove that they can be calculated with the similar form $(T_0 + K_0)(T_1 + K_1)(T_2 + K_2)\cdots$. In this way, a lot of calculation formulas can be obtained. At the end, the form is obtained to calculate

$$K_1 \times \cdots \times K_M + (L + K_1) \times \cdots \times (L + K_M) + (2L + K_1) \times \cdots \times (2L + K_M) + (3L + K_1) \times \cdots \times (3L + K_M) + \cdots.$$

Subject Areas

Discrete Mathematics

Keywords

Shape of Numbers, Calculation Formula, Combinatorics, Congruence, Stirling Number

1. Introduction

Peng, J. has introduced Shape of numbers in [1] [2] [3]:

$(I_1, I_2, \cdots, I_M)$, $I_i \in N$, $I_1 < I_2 < \cdots < I_M$. There are $M - 1$ intervals between adjacent numbers. $I_{i+1} - I_i = 1$ means continuity, $I_{i+1} - I_i > 1$ means discontinuity.

Shape of numbers: collect $(I_1, I_2, \cdots, I_M)$ **with the same continuity and discontinuity at the same position into a catalog, call it a Shape.**

A shape has a min Item: $(1, K_1, K_2, \ldots)$ **that use the symbol** ***PS*** **= [min Item] to represent it.**

If $K_{i+1} - K_i = D > 1$**, only** $I_{i+1} - I_i \geq D$ **is allowed. If** $K_{i+1} - K_i = 1$**, only**

$I_{i+1} - I_i = 1$ **is allowed.**

The single $(I_1, I_2, \cdots, I_M)$ **is an item,** $I_1 \times I_2 \times \cdots \times I_M$ **is the product.** I_i **is a factor.**

Example:

$PS = [1,2] \to (1,2),(2,3),(3,4),(1000,1001) \in PS$

$PS = [1,3] \to (1,3),(1,4),(2,4),(1,5),(2,5),(3,5),(1000,2001) \in PS$

$PS = [1,4] \to (1,4),(1,5),(2,5),(1,6),(2,6),(3,6) \in PS$, $(3,5),(4,6) \notin PS$

$PS = [1,4,6] \to (1,4,7),(1,5,7),(2,5,7) \in PS$, $(3,5,7) \notin PS$

Define:

***SET*(*N*, *PS*) = set of items belonging to *PS* in [1, *N* − 1]**

***PM*(*PS*) = count of factors**

***PB*(*PS*) = count of discontinuities**

***MIN*(*PS*) = min product:** $MIN([1,2,3]) = 1 \times 2 \times 3$, $MIN([1,2,4]) = 1 \times 2 \times 4$

***IDX*(*PS*) = (max factor) + 1**

***PH*(*PS*) = *IDX*(*PS*) − *PB*(*PS*) − 2**

Basic Shape: intervals = 1 or 2

***BASE*(*PS*) = *BS*: if (1) *PB*(*BS*) = *PB*(*PS*), (2) *PM*(*BS*) = *PM*(*PS*), (3) *BS* is a Basic Shape, (4) *BS* has discontinuity intervals at the same positions of *PS*.**

Example:

$PS = [1,2] \to BASE(PS) = [1,2]$

$PS = [1,3],[1,4],[1,K>2] \to BASE(PS) = [1,3]$

$PS = [1,3,4],[1,4,5],[1,K>2,X=K+1] \to BASE(PS) = [1,3,4]$

$PS = [1,3,5],[1,4,9],[1,K>2,X>K+1] \to BASE(PS) = [1,3,5]$

***End*(*N*, *PS*) = set of items belonging to *PS* with the max factor = *N* − 1;**

|*SET*(*N*, *PS*)| = count of items in *SET*(*N*, *PS*);

***SUM*(*N*, *PS*) = sum of all products in *SET*(*N*, *PS*).**

Example:

$SUM(6,[1,2,4]) = 1 \times 2 \times 4 + 1 \times 2 \times 5 + 2 \times 3 \times 5$

$SUM(9,[1,4,7]) = 1 \times 4 \times 7 + 1 \times 4 \times 8 + 1 \times 5 \times 8 + 2 \times 5 \times 8$

[3] introduced the subset:

If *PB*(*PS*) = 0, *SET*(*N*, *PS*) is simple.

If *PB*(*PS*) > 0, then can fix some interval of discontinuities to get subsets.

***SET*(*N*, *PS*, *PT*) = subset of *SET*(*N*, *PS*), a valid**

$$PT = [1, T_1, \cdots, T_M] = \begin{cases} T_{i+1} - T_i = 1 : K_{i+1} - K_i = 1, & \text{means } I_{i+1} - I_i = 1 \\ T_{i+1} - T_i = 1 : K_{i+1} - K_i = D > 1, & \text{means } I_{i+1} - I_i = D \\ T_{i+1} - T_i = 2 : K_{i+1} - K_i = D > 1, & \text{means } I_{i+1} - I_i \ge D \end{cases} \quad (*)$$

PT only has the change at (*), when a change happens, make the interval fixed.

PCHG*(*PS*, *PT*) = count of change from *BASE*(*PS*) to *PT

Example:

$PCHG([1,3,5],[1,3,5]) = 0$

$PCHG([1,3,5],[1,2,4]) = PCHG([1,4,7],[1,2,4]) = 1$, changed at T_1

$PCHG([1,3,5],[1,3,4]) = PCHG([1,4,7],[1,3,4]) = 1$, changed at T_2

$PCHG([1,3,5],[1,2,3]) = PCHG([1,8,10],[1,2,3]) = 2$, changed at T_1, T_2

***SUM_SUBSET*(*N*, *PS*, *PT*)** is defined in [3] **= sum of all products in *SET*(*N*, *PS*, *PT*)**

Now, *SUM*() and *SUM_SUBSET*() are uniformly defined as *SUM*(*N*, *PS*, *PT*), *SUM*(*N*, *PS*, *BASE*(*PS*)) is abbreviated as *SUM*(*N*, *PS*)

Only valid PT is discussed below.

[1] [2] [3] came to the following conclusion:

(1.1) $\left|SET(N,PS,PT)\right| = \binom{N-PH(PS)-PCHG(PS,PT)-1}{PB(PT)+1}$

(1.2) $SUM(N,PS) = MIN(PS)\binom{N}{IDX(PS)}$, ***PS* is a Basic Shape**

The following uses count of $X \in K$ for count of

$$\{X_1, X_2, \cdots, X_M\} \in \{K_1, K_2, \cdots, K_M\}$$

(1.3) $PS = [1, K_1, \cdots, K_M]$, $PT = [1, T_1, T_2, \cdots, T_M]$

Use the form $(T_1+K_1)(T_2+K_2)\cdots(T_M+K_M) = \sum X_1 X_2 \cdots X_M$ **,** $X_i = T_i$ **or** K_i**.**

The expansion has 2^M **items, don't swap the factors of** $X_1 X_2 \cdots X_M$ **, then each** $X_1 X_2 \cdots X_M$ **corresponds to one expression =**

$$A_q \binom{N-PH(PS)-PCHG(PS,PT)}{IDX(PT)-q}$$

$$q = \text{count of } X \in K.$$

$$SUM(N,PS,PT) = \sum A_q \binom{N-PH(PS)-PCHG(PS,PT)}{IDX(PT)-q}.$$

$$A_q = \prod_{i=1}^{M}(X_i + D_i),\quad D_i = \begin{cases} -m: X_i = T_i, m = \text{count of } \{X_1,\cdots,X_{i-1}\} \in K \\ +m: X_i = K_i, m = \text{count of } \{X_1,\cdots,X_{i-1}\} \in T \end{cases}$$

Example:

$$PS = [1, K_1 \ge 3, K_2 \ge K_1 + 2, K_3 \ge K_2 + 2],$$

$$BS = BASE(PS) = [1,3,5,7],$$

$$IDX(BS) = 8$$

$$\begin{aligned}\text{The form} &= (3+K_1)(5+K_2)(7+K_3) \\ &= 3\times5\times7 + 3\times5\times K_3 + 3\times K_2\times7 + 3\times K_2\times K_3 \\ &\quad + K_1\times5\times7 + K_1\times5\times K_3 + K_1\times K_2\times7 + K_1\times K_2\times K_3\end{aligned}$$

$$\begin{aligned}P &= N - PH(PS) - PCHG(PS,PT) \\ &= N - \{IDX(PS) - PB(PS) - 2\} - 0 \\ &= N - \{K_3 + 1 - 3 - 2\} = N - K_3 + 4\end{aligned}$$

→

$$SUM(N,PS)=3\times5\times7\binom{P}{8}+3\times5\times(K_3+2)\binom{P}{7}$$
$$+3\times(K_2+1)\times(7-1)\binom{P}{7}+3\times(K_2+1)\times(K_3+1)\binom{P}{6}$$
$$+K_1\times(5-1)\times(7-1)\binom{P}{7}+K_1\times(5-1)\times(K_3+1)\binom{P}{6}$$
$$+K_1\times K_2\times(7-2)\binom{P}{6}+K_1\times K_2\times K_3\binom{P}{5}$$

An item $\in PS=\{\text{begin},K_1+E_1,\cdots,K_M+E_M\}$, K is fixed, E is variable.

A product $=\text{begin}\times(K_1+E_1)\cdots(K_M+E_M)=\text{begin}\times\sum F_1F_2\cdots F_M$, $F_i = E_i$ or K_i

That is, a product can be broken down into 2^M parts.

Define $SUM\_K(N,PS,PT,PF=F_1F_2\cdots F_M)$ **= Sum of one part in *SUM*(*N*, *PS*).**

***PF* indicates the part. $F_i = E_i$ or K_i**

Rewrite 1.3), add {braces}:

$$SUM(N,PS,PT)=\sum\text{product}=\sum\sum\text{begin}\times F_1\cdots F_M$$
$$=\sum\prod_{i=1}^{M}(X_i+D_i)\binom{A}{M_q}$$

$$X_i+D_i=\begin{cases}\{T_i-D_i\}:X_i=T_i,D_i=\text{count of }\{X_1,\cdots,X_{i-1}\}\in K\\\{K_i\}+\{D_i\}:X_i=K_i,D_i=\text{count of }\{X_1,\cdots,X_{i-1}\}\in T\end{cases}$$

Expand *SUM*(*N*, *PS*, *PT*) by {braces}:

(1.4) *SUM_K*(*N*, *PS*, *PT*, *PF*) = ΣExpansion of *SUM*() with same

$$\{K_i\}\in PF=\sum\prod_{i=1}^{M}Y_i\binom{A}{M_q},$$

$$Y_i=\begin{cases}0:F_i=K_i,X_i=T_i\\K_i:F_i=K_i,X_i=K_i\\T_i-D_i:F_i=E_i,X_i=T_i,D_i=\text{count of }\{X_1,\cdots,X_{i-1}\}\in K\\D_i:F_i=E_i,X_i=K_i,D_i=\text{count of }\{X_1,\cdots,X_{i-1}\}\in T\end{cases}$$

Example:

$$SUM(N,[1,K_1\ge3,K_2\ge K_1+2]),$$

$$\text{form}=(3+K_1)(5+K_2)\ \rightarrow$$

$$=15\binom{N-K_2+3}{6}+3(\{K_2\}+\{1\})\binom{N-K_2+3}{5}$$
$$+K_1(\{5-1\})\binom{N-K_2+3}{5}+K_1K_2\binom{N-K_2+3}{4}$$

Expand by the {braces}:

$$= \left\{15\binom{N-K_2+3}{6}+3\binom{N-K_2+3}{5}\right\}+3K_2\binom{N-K_2+3}{5}$$
$$+4K_1\binom{N-K_2+3}{5}+K_1K_2\binom{N-K_2+3}{4}$$
$$=\sum_{\text{begin}=1}^{N-K_2}\sum \text{begin}\times\left(K_1+E_{1,\text{begin}}\right)\left(K_2+E_{2,\text{begin}}\right)$$

→

$$SUM\_K\left(N,PS,BS,E_1E_2\right)$$
$$=\sum_{\text{all items}}\text{begin}*E_{1,i}E_{2,i}=15\binom{N-K_2+3}{6}+3\binom{N-K_2+3}{5}$$

$$SUM\_K\left(N,PS,BS,E_1K_2\right)=\sum_{\text{all items}}\text{begin}*E_{1,i}K_2=3K_2\binom{N-K_2+3}{5}$$

$$SUM\_K\left(N,PS,BS,K_1E_2\right)=\sum_{\text{all items}}\text{begin}*K_1E_{2,i}=4K_1\binom{N-K_2+3}{5}$$

$$SUM\_K\left(N,PS,BS,K_1K_2\right)=\sum_{\text{all items}}\text{begin}*K_1K_2=K_1K_2\binom{N-K_2+3}{4}$$

In this paper, we extend the definition of Shape of Numbers and generalize the corresponding results.

2. The Extension of Shape

Redefine:

$$PS=\left[\min \text{Item}\right]=\left[K_0,\cdots,K_M,\cdots\right],\quad \text{Item}=\left(I_0,\cdots,I_M,\cdots\right),$$
$$BASE\left(PS\right)=BS=\left[G_0=1,G_1,\cdots,G_M,\cdots\right]$$

1) change factor's domain of definition from *N* to *Z*, change K_0 from 1 to *Z*.

2) allow $K_0\le K_1\le\cdots\le K_M$ **, If** $K_{i+1}=K_i$ **, only** $I_{i+1}=I_i$ **is allowed.** $G_{i+1}-G_i=1$

3) allow $K_i>K_{i+1}$ **, only** $I_{i+1}=I_i$ **is allowed.** $G_{i+1}-G_i=1$.

Example:

$$PS=\left[3,5\right]\rightarrow BASE\left(PS\right)=\left[1,3\right]$$

$$SET\left(8,\left[3,5\right]\right)=\left\{\left(3,5\right),\left(3,6\right),\left(4,6\right),\left(3,7\right),\left(4,7\right),\left(5,7\right)\right\}$$
$$\ne SET\left(8,\left[1,3\right]\right)-SET\left(5,\left[1,3\right]\right)$$

$$PS=\left[3,5,4,6\right]\rightarrow BASE\left(PS\right)=\left[1,3,4,6\right]$$

$$SET\left(8,PS\right)=\left\{\left(3,5,4,6\right),\left(3,6,5,7\right),\left(4,6,5,7\right),\left(3,5,4,7\right)\right\}$$

Redefine:

Basic Shape: K_0 **= 1 and intervals = 1 or 2**

***SET*(*N*, *PS*) = set of items belonging to *PS* in [**K_0**, *N* − 1], Max Factor of item ≤ *N* − 1**

PB*(*PS*) = Count of discontinuities in *BS

***PH*(*PS*) = (Max Factor) − 1 − *PB*(*BS*)**

***IDX*(*PS*) = *IDX* of** $BS = \{\text{max factor of } BS\} + 1 = PM(BS) + PB(BS) + 1$

$D^1 f(n)$: if $f(n) = \sum A_i \binom{N-n_i}{m_i}$, **then** $D^1 f(n) = \sum A_i \binom{N-n_i-1}{m_i-1}$

2.1) $|SET(N,PS,PT)| = \binom{N-K_0-PH(PS)-PCHG(PS,PT)}{PB(PT)+1}$

2.2) Specify $\binom{N<M}{M} = 0$,

$$\sum_{n=0}^{N-1} n\binom{n-K}{M} = (M+1)\binom{N-K}{M+2} + (M+K)\binom{N-K}{M+1}$$

2.3) $PS = [K_0, K_1, \cdots, K_M]$, $PT = [1, T_1, T_2, \cdots, T_M]$, **can use the form** $(T_0+K_0)\cdots(T_M+K_M)$

$$SUM(N,PS,PT) = \sum A_q \binom{N-PH(PS)-PCHG(PS,PT)-1}{IDX(PT)-q},$$

$$A_q = \prod_{i=0}^{M}(X_i + D_i),$$

$$D_i = \begin{cases} -m : X_i = T_i, m = \text{count of } \{X_0,\cdots,X_{i-1}\} \in K \\ +m : X_i = K_i, m = \text{count of } \{X_0,\cdots,X_{i-1}\} \in T \end{cases}$$

$$q = \text{count of } X \in K$$

[Proof]

Here only prove *SUM*(*N*, *PS*), *SUM*(*N*, *PS*, *PT*) can use the same method.

$$BS = BASE(PS) = [1, G_1, G_2, \cdots, G_M]$$

Use the similar way of [2], by definition:

(1*) $SUM(N,PS) = \sum_{n=-\infty}^{N} \sum END(n,PS)$

(2*) $\sum END(N,PS) = D^1 SUM(N,PS)$

(3*) $SUM(N,[PS,K_{M+1} = 1+K_M]) = \sum_{n=-\infty}^{N-1} n \times \sum END(n,PS)$

(4*) $SUM(N,[PS,K_{M+1} = K_M]) = \sum_{n=-\infty}^{N-1} n \times \sum END(n+1,PS)$

(5*) $SUM(N,[PS,K_{M+1} > 1+K_M]) = \sum_{n=-\infty}^{N-1} n \times SUM(n-(K_{M+1}-K_M)+1,PS)$

Suppose $SUM(N,PS) = \sum X_0 X_1 \cdots X_M \binom{N-PH(PS)-1}{M_i}$, Max factor of *PS* $= K_M$

$$P = n - PH(PS) - 1 = n - [K_M - 1 - PB(BS)] - 1 = n - [K_M - PB(BS)]$$

$$Q = N - PH(PS) - 1$$

$$C = \text{Count of } \{X_0,\cdots,X_M\} \in K,$$

$$M_i = IDX(BS) - C$$

1) $PS1 = [PS, K_{M+1} = 1+K_M]$, $BS1 = BASE(PS1) = [BS, G_{M+1} = 1+G_M]$

$$SUM(N,PS1)=\sum_{n=-\infty}^{N-1} n\times\sum END(n,PS)=\sum_{n=-\infty}^{N-1} n\times D^1 SUM(n,PS)$$

$$=\sum_{n=-\infty}^{N-1} n\times\sum X_0\cdots X_M\binom{P-1}{M_i-1}$$

$$=\sum_{n=-\infty}^{N-1} n\times\sum X_0\cdots X_M\binom{n-[K_M-PB(BS)+1]}{M_i-1}\xrightarrow{(2.2)}$$

$$=\sum\left(X_0\cdots X_M M_i\binom{Q-1}{M_i+1}+X_0\cdots X_M(M_i-1+K_M-PB(BS)+1)\binom{Q-1}{M_i}\right)$$

$$M_i=IDX(BS)-C=1+G_M-C=G_{M+1}-C$$

$$\begin{aligned}&M_i-1+K_M-PB(BS)+1\\&=M_i+K_M-PB(BS)=IDX(BS)-C+K_M-PB(BS)\\&=(PM(BS)+PB(BS)+1)-C+K_M-PB(BS)\\&=K_M+1+PM(BS)-C=K_{M+1}+(M+1)-C\end{aligned}$$

→

$$\begin{aligned}&SUM(N,PS1)\\&=\sum X_0\cdots X_M(G_{M+1}-C)\binom{Q-1}{IDX(BS)-C+1}\\&\quad+\sum X_0\cdots X_M(K_{M+1}+M+1-C)\binom{Q-1}{IDX(BS)-C}\\&=\sum X_0\cdots X_M(G_{M+1}-C)\binom{N-PH(PS1)-1}{IDX(BS1)-C}\\&\quad+\sum X_0\cdots X_M(K_{M+1}+M+1-C)\binom{N-PH(PS1)-1}{IDX(BS1)-(C+1)}\end{aligned}$$

→ Match the form $(G_0+K_0)(G_1+K_1)\cdots(G_M+K_M)\{G_{M+1}+K_{M+1}\}$.

2) $PS1=[PS,K_{M+1}=K_M]$, $BS1=BASE(PS1)=[BS,G_{M+1}=1+G_M]$

$$\begin{aligned}&SUM(N,PS1)=\sum_{n=-\infty}^{N-1} n\times\sum END(n+1,PS)\\&=\sum_{n=-\infty}^{N-1} n\times D^1 SUM(n+1,PS)\\&=\sum_{n=-\infty}^{N-1} n\times\sum X_0\cdots X_M\binom{P}{M_i-1}\\&=\sum_{n=-\infty}^{N-1} n\times\sum X_0\cdots X_M\binom{n-[K_M-PB(BS)]}{M_i-1}\xrightarrow{(2.2)}\end{aligned}$$

$$=\sum\left(X_0\cdots X_M M_i\binom{Q}{M_i+1}+X_0\cdots X_M(M_i-1+K_M-PB(BS))\binom{Q}{M_i}\right)$$

$$\begin{aligned}&=\sum X_0\cdots X_M(G_{M+1}-C)\binom{N-PH(PS1)-1}{IDX(BS1)-C}\\&\quad+\sum X_0\cdots X_M(K_M+M+1-C)\binom{N-PH(PS1)-1}{IDX(BS1)-(C+1)}\end{aligned}$$

→ Match the form $(G_0+K_0)(G_1+K_1)\cdots(G_M+K_M)\{G_{M+1}+K_{M+1}\}$.

3) $PS1=[PS,\ K_{M+1}>K_M+1]$, $BS1=BASE(PS1)=[BS,2+G_M]$

$$SUM(N,PS1)=\sum_{n=-\infty}^{N-1} n\times SUM(n-(K_{M+1}-K_M-1),PS)$$
$$=\sum_{n=-\infty}^{N-1} n\times\sum X_0\cdots X_M\binom{n-(K_{M+1}-K_M-1)-PH(PS)-1}{M_i}$$
$$=\sum_{n=-\infty}^{N-1} n\times\sum X_0\cdots X_M\binom{n-[K_{M+1}-K_M+PH(PS)]}{M_i}$$

$$\begin{aligned}Q1&=N-[K_{M+1}-K_M+PH(PS)]\\&=N-[K_{M+1}-K_M+K_M-1-PB(BS)]\\&=N-[K_{M+1}-1-PB(BS)]\\&=N-[K_{M+1}-1-PB(BS1)]-1\\&=N-PH(PS1)-1\end{aligned}$$

$$SUM(N,PS1)\xrightarrow{(2.2)}$$
$$=\sum X_0\cdots X_M(M_i+1)\binom{Q1}{M_i+2}$$
$$+\sum X_0\cdots X_M(K_{M+1}-K_M+PH(PS)+M_i)\binom{Q1}{M_i+1}$$

$$M_i+1=IDX(BS)-C+1=1+G_M-C+1=G_{M+1}-C$$

$$K_{M+1}-K_M+PH(PS)+M_i=K_{M+1}-K_M+PH(PS)+IDX(BS)-C$$
$$=K_{M+1}-K_M+(K_M-1-PB(BS))+(PM(BS)+PB(BS)+1)-C$$
$$=K_{M+1}+PM(BS)-C=K_{M+1}+M+1-C$$

→

$$SUM(N,PS1)$$
$$=\sum X_0\cdots X_M(G_{M+1}-C)\binom{Q1}{M_i+2}$$
$$+\sum X_0\cdots X_M(K_{M+1}+M+1-C)\binom{Q1}{M_i+1}$$
$$=\sum X_0\cdots X_M(G_{M+1}-C)\binom{Q1}{IDX(BS)-C+2}$$
$$+\sum X_0\cdots X_M(K_{M+1}+M+1-C)\binom{Q1}{IDX(BS)-C+1}$$
$$=\sum X_0\cdots X_M(G_{M+1}-C)\binom{N-PH(PS1)-1}{IDX(BS1)-C}$$
$$+\sum X_0\cdots X_M(K_{M+1}+M+1-C)\binom{N-PH(PS1)-1}{IDX(BS1)-(C+1)}$$

→ Match the form $(G_0+K_0)(G_1+K_1)\cdots(G_M+K_M)\{G_{M+1}+K_{M+1}\}$.

4) $PS1=[PS,K_{M+1}<K_M]$, $BS1=BASE(PS1)=[BS,1+G_M]$

By definition:

$$SUM\left(N,\left[PS,K_{M+1}\right]\right)=\sum_{n=-\infty}^{N-1}\left(n+K_{M+1}-K_M\right)\sum END\left(n+1,PS\right)$$
$$=\sum_{n=-\infty}^{N-1}\left(n+K_{M+1}-K_M\right)\times D^1 SUM\left(n+1,PS\right)$$
$$=\sum_{n=-\infty}^{N-1}\left(n+K_{M+1}-K_M\right)\times\sum X_0\cdots X_M\binom{p}{M_i-1}$$
$$=\sum_{n=-\infty}^{N-1}n\times\sum X_0\cdots X_M\binom{P}{M_i-1}+\sum_{n=-\infty}^{N-1}\left(K_{M+1}-K_M\right)$$
$$\times\sum X_0\cdots X_M\binom{P}{M_i-1}$$

$$=\sum X_0\cdots X_M\left\{M_i\binom{Q}{M_i+1}+\left(M_i-1+K_M-PB\left(BS\right)\right)\binom{Q}{M_i}\right.$$
$$\left.+\left(K_{M+1}-K_M\right)\binom{Q}{M_i}\right\}$$

$$=\sum X_0\cdots X_M\left\{\left(IDX\left(BS\right)-C\right)\binom{Q}{M_i+1}+\left(M_i+K_{M+1}-PB\left(BS\right)-1\right)\binom{Q}{M_i}\right\}$$

$$M_i+K_{M+1}-PB\left(PS\right)-1=IDX\left(BS\right)-C+K_{M+1}-PB\left(BS\right)-1$$
$$=\left(PM\left(BS\right)+PB\left(BS\right)+1\right)-C+K_{M+1}-PB\left(BS\right)-1$$
$$=K_{M+1}+\left(M+1\right)-C$$

→

$$SUM\left(N,PS1\right)$$
$$=\sum X_0\cdots X_M\left(G_{M+1}-C\right)\binom{N-PH\left(PS1\right)-1}{IDX\left(BS1\right)-C}$$
$$+\sum X_0\cdots X_M\left(K_{M+1}+M+1-C\right)\binom{N-PH\left(PS1\right)-1}{IDX\left(BS1\right)-\left(C+1\right)}$$

→ Match the form $\left(G_0+K_0\right)\cdots\left(G_M+K_M\right)\left\{G_{M+1}+K_{M+1}\right\}$.

q.e.d.

Example:

$$N-PH\left(\left[-11,-7,-4\right]\right)-1=N-\left(-4-2-1\right)-1=N+6,$$

$$BASE\left(\left[-11,-7,-4\right]\right)=\left[1,3,5\right]$$

$SUM\left(N,\left[-11,-7,-4\right]\right)$ → form $=(1-11)(3-7)(5-4)$ →

$$=15\binom{N+6}{6}-118\binom{N+6}{5}+315\binom{N+6}{4}-308\binom{N+6}{3}$$

$$15=1\times3\times5;$$

$$-308=\left(-11\right)\times\left(-7\right)\times\left(-4\right)$$

$$-118=1\times3\times\left(-4+2\right)+1\times\left(-7+1\right)\ \times\left(5-1\right)+\left(-11\right)\times\left(3-1\right)\ \times\left(5-1\right)$$

$$315=1\times\left(-7+1\right)\times\left(-4+1\right)+\left(-11\right)\times\left(3-1\right)\times\left(-4+1\right)+\left(-11\right)\times\left(-7\right)\times\left(5-2\right)$$

$$
\begin{aligned}
& SUM(-2,[-11,-7,-4]) \\
& =(-11)\times(-7)\times(-4)+(-11)\times(-7)\times(-3) \\
& \quad +(-11)\times(-6)\times(-3)+(-10)\times(-6)\times(-3) \\
& =315-308\times 4=-917
\end{aligned}
$$

$$
\begin{aligned}
& SUM(-1,[-11,-7,-4]) \\
& =SUM(-2,[-11,-7,-4])+(-11)\times(-7)\times(-2)+(-11)\times(-6)\times(-2) \\
& \quad +(-11)\times(-5)\times(-2)+(-11)\times(-7)\times(-2) \\
& \quad +(-11)\times(-6)\times(-2)+(-11)\times(-5)\times(-2) \\
& =-118+315\times 5-308\times 10=-1623
\end{aligned}
$$

$$SUM(N,[4,7,11],[1,2,4]) \rightarrow \text{form}=(1+4)(2+7)(4+11) \rightarrow$$

$$=8\binom{N-10}{5}+62\binom{N-10}{4}+200\binom{N-10}{3}+308\binom{N-10}{2}$$

$$62=1\times 2\times(11+2)+1\times(7+1)\times(4-1)+4\times(2-1)\times(4-1)$$

$$200=1\times(7+1)\times(11+1)+4\times(2-1)\times(11+1)+4\times 7\times(4-2)$$

[1, 2, 4] means $I_1-I_0=K_1-K_0=7-4=3$, $I_2-I_1=K_2-K_1\geq 11-7=4$

$$
\begin{aligned}
& SUM(15,[4,7,11],[1,2,4]) \\
& =4\times 7\times 11+4\times 7\times 12+5\times 8\times 12+4\times 7\times 13+5\times 8\times 13 \\
& \quad +6\times 9\times 13+4\times 7\times 14+5\times 8\times 14+6\times 9\times 14+7\times 10\times 14 \\
& =8+62\times 5+200\times 10+308\times 10=5398
\end{aligned}
$$

$$N-PH([4,7,1,8])-1=N-(8-1-2)-1=N-6,\quad BASE([4,7,1,8])=[1,3,4,6]$$

$$SUM(N,[4,7,1,8]) \rightarrow \text{form}=(1+4)(3+7)(4+1)(6+8) \rightarrow$$

$$=72\binom{N-6}{7}+417\binom{N-6}{6}+922\binom{N-6}{5}+876\binom{N-6}{4}+224\binom{N-6}{3}$$

$$
\begin{aligned}
417 &= 1\times 3\times 4\times(8+3)+1\times 3\times(1+2)\times(6-1) \\
& \quad +1\times(7+1)\times(4-1)\times(6-1)+4\times(3-1)\times(4-1)\times(6-1)
\end{aligned}
$$

$$
\begin{aligned}
922 &= 1\times 3\times(1+2)\times(8+2)+1\times(7+1)\times(4-1)\times(8+2) \\
& \quad +4\times(3-1)\times(4-1)\times(8+2)+1\times(7+1)\times(1+1)\times(6-2) \\
& \quad +4\times(3-1)\times(1+1)\times(6-2)+4\times 7\times(4-2)\times(6-2)
\end{aligned}
$$

$$
\begin{aligned}
876 &= 4\times 7\times 1\times(6-3)+4\times 7\times(4-2)\times(8+1) \\
& \quad +4\times(3-1)\times(1+1)\times(8+1)+1\times(7+1)\times(1+1)\times(8+1)
\end{aligned}
$$

$$
\begin{aligned}
& SUM(13,[4,7,1,8]) \\
& =4\times 7\times 1\times 8+4\times 7\times 1\times 9+(4+5)\times 8\times 2\times 9+4\times 7\times 1\times 10 \\
& \quad +(4+5)\times 8\times 2\times 10+(4+5+6)\times 9\times 3\times 10+4\times 7\times 1\times 11 \\
& \quad +(4+5)\times 8\times 2\times 11+(4+5+6)\times 9\times 3\times 11+(4+5+6+7) \\
& \times 10\times 4\times 11+4\times 7\times 1\times 12+(4+5)\times 8\times 2\times 12+(4+5+6)\times 9\times 3\times 12 \\
& +(4+5+6+7)\times 10\times 4\times 12+(4+5+6+7+8)\times 11\times 5\times 12 \\
& =72+417\times 7+922\times 21+876\times 35+224\times 35=60853
\end{aligned}
$$

2.2. SUM_K(N, PS, PT, PF)

An item $\in PS = \{K_0 + E_0, K_1 + E_1, \cdots, K_M + E_M\}$, **$K$ is fixed, E is variable.**

A product $= (K_0 + E_0) \times (K_1 + E_1) \cdots (K_M + E_M) = \sum F_0 F_1 F_2 \cdots F_M$, $F_i = E_i$ or $F_i = K_i$

That is, a product can be broken down into 2^{M+1} parts.

Use the same method of [2]

2.4) *SUM_K*(*N*, *PS*, *PT*, *PF*) is similar to (1.4), except the form = $(T_0 + K_0)\cdots$

Example:

$$BASE([4,7,11]) = BS = [1,3,5]$$

$$\begin{aligned} &SUM(13,[4,7,11]) \\ &= 4\times7\times11 + 4\times7\times(11+1) + 4\times(7+1)\times(11+1) + (4+1)\times(7+1)\times(11+1) \\ &= \{4\times7\times11 + 4\times7\times11 + 4\times7\times11 + 4\times7\times11\} \\ &\quad + \{4\times7\times1 + 4\times7\times1 + 4\times7\times1\} + \{4\times1\times11 + 4\times1\times11\} + \{1\times7\times11\} \\ &\quad + \{4\times1\times1 + 4\times1\times1\} + \{1\times7\times1\} + \{1\times1\times11\} + \{1\times1\times1\} \end{aligned}$$

$$4\times7\times11 \to 308$$

$$\begin{aligned} &SUM\_K(13, PS, BS, K_0K_1K_2) \\ &= \{4\times7\times11 + 4\times7\times11 + 4\times7\times11 + 4\times7\times11\} = 308\binom{N-9}{3} \end{aligned}$$

$$4\times7\times(5-2) \to 4\times7\times3$$

$$\begin{aligned} &SUM\_K(13, PS, BS, K_0K_1E_2) \\ &= \{4\times7\times1 + 4\times7\times1 + 4\times7\times1\} = 4\times7\times3\binom{N-9}{4} \end{aligned}$$

$$4\times(3-1)\times(11+1) \to 4\times2\times11$$

$$SUM\_K(13, PS, BS, K_0E_1K_2) = \{4\times1\times11 + 4\times1\times11\} = 4\times2\times11\binom{N-9}{4}$$

$$1\times(7+1)\times(11+1) \to 1\times7\times11$$

$$SUM\_K(13, PS, BS, E_0K_1K_2) = \{1\times7\times11\} = 1\times7\times11\binom{N-9}{4}$$

$$4\times(3-1)\times(5-1) + 4\times(3-1)\times(11+1) \to 4\times3\times4 + 4\times2\times1$$

$$\begin{aligned} &SUM\_K(13, PS, BS, K_0E_1E_2) \\ &= \{4\times1\times1 + 4\times1\times1\} = 4\times3\times4\binom{N-9}{5} + 4\times2\times1\binom{N-9}{4} \end{aligned}$$

$$1\times(7+1)\times(5-1) + 1\times(7+1)\times(11+1) \to 1\times7\times4 + 1\times7\times1$$

$$\begin{aligned} &SUM\_K(13, PS, BS, E_0K_1E_2) \\ &= \{1\times7\times1\} = 1\times7\times4\binom{N-9}{5} + 1\times7\times1\binom{N-9}{4} \end{aligned}$$

$$1\times3\times(11+2)+1\times(7+1)\times(11+1)\to1\times3\times11+1\times1\times11$$

$$SUM\_K(13,PS,BS,E_0E_1K_2)$$

$$=\{1\times1\times11\}=1\times3\times11\binom{N-9}{5}+1\times1\times11\binom{N-9}{4}$$

$$1\times3\times5+[1\times3\times(11+2)+1\times(7+1)\times(5-1)+4\times(3-1)\times(5-1)]$$
$$+[1\times(7+1)\times(11+1)+4\times(3-1)\times(11+1)+4\times7\times(5-2)]$$
$$\to1\times3\times5+[1\times3\times2+1\times1\times4+0]+[1\times1\times1+0+0]=15+[10]+[1]$$

$$SUM\_K(13,PS,BS,E_0E_1E_2)$$

$$=\{1\times1\times1\}=15\binom{N-9}{6}+10\binom{N-9}{5}+\binom{N-9}{4}$$

3. Coefficient Analysis

$$K=[K_1,\cdots,K_M],$$

$$T=[T_1,\cdots,T_M]$$

Use the form $(T_1+K_1)\cdots(T_M+K_M)=\sum X_1X_2\cdots X_M$, $X_i = T_i$ or K_i

Define $H(K,T,N,S)=\sum B_N$, $0\le N\le M$, $N=$ count of $X\in T$

$$B_N=\prod_{i=1}^{M}(X_i+D_i),\quad D_i=\begin{cases}-mS: X_i=T_i, m=\text{count of }\{X_1,\cdots,X_{i-1}\}\in K\\ +mS: X_i=K_i, m=\text{count of }\{X_1,\cdots,X_{i-1}\}\in T\end{cases}$$

***H*(*K*, *T*, *N*, 1) is abbreviated as *H*(*K*, *T*, *N*)**

3.1) $H(K,T,M)=T_1\times T_2\times\cdots\times T_M$, $H(K,T,0)=K_1\times K_2\times\cdots\times K_M$

[3] has proved:

$$SUM(N+1,[1,1,\cdots,1],[1,2,\cdots,M])=\sum_{n=1}^{N}n^M=\sum_{K=1}^{M}K!S_2(M,K)\binom{N+1}{K+1}$$

$S_2(M, K)$ is Stirling number of the second kind. →

3.2) $H([1,1,\cdots,1],[2,3,\cdots,M],N)=(M-N)!\times S_2(M,M-N)$

$$SUM(N,[K_0=1,K_1,\cdots,K_M],[T_0=1,T_1,\cdots,T_M])$$

can use the form = $(T_1+K_1)\cdots(T_M+K_M)$ or $(T_0+K_0)(T_1+K_1)\cdots(T_M+K_M)$

For arbitrary K, T:

3.3) $H([P,K],[P,T],N,S)=P\times H(K,T,N,S)+P\times H(K,T,N-1,S)$

[Proof]

$$H([P,K,K_{M+1}],[P,T,T_{M+1}],N+1,S)$$
$$=(X_{M+1}=K_{M+1})+(X_{M+1}=T_{M+1})$$
$$=H([P,K],[P,T],N+1,S)(K_{M+1}+[N+1]\times S)$$
$$+H([P,K],[P,T],N,S)(T_{M+1}-[M+1-N]\times S)$$
$$=\{P\times H(K,T,N+1,S)+P\times H(K,T,N,S)\}(K_{M+1}+[N+1]\times S)$$
$$+\{P\times H(K,T,N,S)+P\times H(K,T,N-1,S)\}(T_{M+1}-[M+1-N]\times S)$$

$$= P\times\left\{H(K,T,N+1,S)\left(K_{M+1}+[N+1]\times S\right)\right.$$
$$\left.+H(K,T,N,S)\left(T_{M+1}-[M-N]\times S\right)\right\}$$
$$+P\times\left\{H(K,T,N,S)\left(K_{M+1}+N\times S\right)\right.$$
$$\left.+H(K,T,N-1,S)\left(T_{M+1}-[M-(N-1)]\times S\right)\right\}$$
$$= P\times H\left([K,K_{M+1}],[T,T_{M+1}],N+1,S\right)+P\times H\left([K,K_{M+1}],[T,T_{M+1}],N,S\right)$$

q.e.d.

this →

3.4) $SUM\left(N,[1,2,\cdots,n,K_1,\cdots,K_M],[1,2,\cdots,n,T_1,\cdots,T_M]\right)$

can use the form:

$$(T_1+K_1)\cdots(T_M+K_M)=n!\sum A_q\begin{pmatrix}N-PH(PS)-PCHG(PS,PT)+n-1\\ IDX(PT)-q\end{pmatrix}$$

[2] has proved:

3.5) $H(K,K,N,S)=\begin{pmatrix}M\\N\end{pmatrix}K_1\times K_2\times\cdots\times K_M$

1.3) can derive 1.2) from this.

3.6) if $K_i+S=K_{i+1}$, $T_i+S=T_{i+1}$, **then**

$$H(K,T,N,S)=\begin{pmatrix}M\\N\end{pmatrix}T_1\cdots T_N\times K_{N+1}\cdots K_M$$

[Proof]

Suppose $H(K,T,N,S)=\begin{pmatrix}M\\N\end{pmatrix}T_1\cdots T_N K_{N+1}K_{N+2}\cdots K_M$

$$H\left([K,K_{M+1}=S+K_M],[T,T_{M+1}=S+T_M],N+1,S\right)$$
$$=(X_{M+1}=K_{M+1})+(X_{M+1}=T_{M+1})$$
$$=H(K,T,N+1,S)\left(K_{M+1}+[N+1]\times S\right)$$
$$+H(K,T,N,S)\left(T_{M+1}-[M-N]\times S\right)$$
$$=\begin{pmatrix}M\\N+1\end{pmatrix}T_1\cdots T_{N+1}K_{N+2}\cdots K_M\left(K_{M+1}+[N+1]\times S\right)$$
$$+\begin{pmatrix}M\\N\end{pmatrix}T_1\cdots T_N K_{N+1}\cdots K_M\left(T_{M+1}-[M-N]\times S\right)$$
$$=T_1\cdots T_N K_{N+2}\cdots K_M\begin{pmatrix}M\\N+1\end{pmatrix}T_{N+1}\left(K_{M+1}+[N+1]\times S\right)$$
$$+T_1\cdots T_N K_{N+2}\cdots K_M\begin{pmatrix}M\\N\end{pmatrix}K_{N+1}\left(T_{M+1}-[M-N]\times S\right)$$
$$=T_1\cdots T_N K_{N+2}\cdots K_M\begin{pmatrix}M\\N+1\end{pmatrix}\left[T_{N+1}K_{M+1}+T_{N+1}(N+1)\times S\right]$$
$$+T_1\cdots T_N K_{N+2}\cdots K_M\begin{pmatrix}M\\N\end{pmatrix}\left[K_{M+1}-[M-N]\times S\right]$$
$$\times\left(\left[T_{N+1}+[M-N]\times S\right]-[M-N]\times S\right)$$

$$= T_1 \cdots T_N K_{N+2} \cdots K_M \left[\binom{M}{N+1} T_{N+1} K_{M+1} + \binom{M}{N} T_{N+1} K_{M+1} \right]$$

$$+ T_1 \cdots T_N K_{N+2} \cdots K_M \left[\binom{M}{N+1} T_{N+1}(N+1) - \binom{M}{N} T_{N+1}(M-N) \right] \times S$$

$$= T_1 \cdots T_N K_{N+2} \cdots K_M \left[\binom{M}{N+1} T_{N+1} K_{M+1} + \binom{M}{N} T_{N+1} K_{M+1} \right]$$

$$= \binom{M+1}{N+1} T_1 \cdots T_{N+1} K_{N+2} \cdots K_{M+1}$$

→ $H([K, K_{M+1}], [T, T_{M+1}], N+1, S)$ holds

q.e.d.

Define

$F_Q^K = \sum Q-$product with different factors $\in K$ **, the sum traverse all combinations.**

$E_Q^K = \sum Q-$product with factors $\in K$ **, the sum traverse all combinations.**

$F_Q^{\{1,2,\cdots,N\}}$ **is abbreviated as** F_Q^N **,** $E_Q^{\{1,2,\cdots,N\}}$ **is abbreviated as** E_Q^N **;**

$$F_0^K = E_0^K = 1; \quad F_{Q>|K|}^K = 0;$$

By definition:

$$E_Q^{N+1} = (N+1) E_{Q-1}^{N+1} + E_Q^N; \quad F_Q^{[K, K_{M+1}]} = K_{M+1} F_{Q-1}^K + F_Q^K.$$

3.7) if $T_i + 1 = T_{i+1}$**, then**

$$H(K, T, N) = T_1 \cdots T_N \left[F_{M-N}^K E_0^N + F_{M-N-1}^K E_1^N + \cdots + + F_0^K E_{M-N}^N \right]$$

[Proof]

Suppose $H(K, T, N)$ holds

$$H([K, K_{M+1}], [T, T_{M+1} = 1 + T_M], N+1)$$

$$= H(K, T, N+1)(K_{M+1} + N + 1) + H(K, T, N)(T_{M+1} - [M-N])$$

$$= T_1 \cdots T_{N+1} \left[F_{M-N-1}^K E_0^{N+1} + F_{M-N-2}^K E_1^{N+1} + \cdots + F_0^K E_{M-N-1}^{N+1} \right] (K_{M+1} + N + 1)$$

$$+ T_1 \cdots T_N \left[F_{M-N}^K E_0^N + F_{M-N-1}^K E_1^N + \cdots + F_0^K E_{M-N}^N \right] (T_{M+1} - [M-N])$$

$$= T_1 \cdots T_{N+1} \left[F_{M-N-1}^K E_0^{N+1} + F_{M-N-2}^K E_1^{N+1} + \cdots + F_0^K E_{M-N-1}^{N+1} \right] (K_{M+1} + N + 1)$$

$$+ T_1 \cdots T_{N+1} \left[F_{M-N}^K E_0^N + F_{M-N-1}^K E_1^N + \cdots + F_0^K E_{M-N}^N \right]$$

$$H([K, K_{M+1}], [T, T_{M+1} = 1 + T_M], N+1) / T_1 \cdots T_{N+1}$$

$$= \left[F_{M-N-1}^K E_0^{N+1} + F_{M-N-2}^K E_1^{N+1} + \cdots + F_0^K E_{M-N-1}^{N+1} \right] (K_{M+1} + N + 1)$$

$$+ \left[F_{M-N}^K E_0^N + F_{M-N-1}^K E_1^N + \cdots + F_0^K E_{M-N}^N \right]$$

Sum of all items with count of factors $\in K = M - N - Q$

$$= F_{M-[N+1]-Q}^K E_Q^{N+1} K_{M+1} + F_{M-[N+1]-(Q-1)}^K E_{Q-1}^{N+1}(N+1) + F_{M-N-Q}^K E_Q^N$$

$$= F_{M-[N+1]-Q}^K E_Q^{N+1} K_{M+1} + F_{M-N-Q}^K \left[(N+1) E_{Q-1}^{N+1} + E_Q^N \right]$$

$$= F_{M-[N+1]-Q}^K E_Q^{N+1} K_{M+1} + F_{M-N-Q}^K E_Q^{N+1}$$

$$= F_{M-N-Q}^{[K, K_{M+1}]} E_Q^{N+1} = F_{[M+1]-[N+1]-Q}^{[K, K_{M+1}]} E_Q^{N+1}$$

→ $H([K, K_{M+1}], [T, T_{M+1}], N+1)$ holds

q.e.d.

Example:

$$\begin{aligned}&H([A,B,C,D],[1,2,3,4],2)\\&=1\times 2\times(C+2)(D+2)+1\times(B+1)\times(3-1)(D+2)\\&\quad+A\times(2-1)\times(3-1)(D+2)+1\times(B+1)(C+1)(4-2)\\&\quad+A\times(2-1)\times(C+1)(4-2)+AB(3-1)(4-2)\\&=1\times 2[(C+2)(D+2)+(B+1)(D+2)+A(D+2)\\&\quad+(B+1)(C+1)+A(C+1)+AB]\\&=1\times 2[(CD+BD+AD+BC+AC+AB)\\&\quad+(C+D+B+A)(1+2)+(1\times 2+1\times 1+2\times 2)]\end{aligned}$$

3.8) In $SUM(N,[K_1,\cdots,K_M],[1,2,\cdots,M])$, K_i can switch the order.

3.9) if $T_i+S=T_{i+1}$, then

$$\begin{aligned}&H(K,T,N,S)\\&=T_1\cdots T_N\left[F_{M-N}^{K}E_0^{\{S,2S,\cdots,NS\}}+F_{M-N-1}^{K}E_1^{\{S,2S,\cdots,NS\}}+\cdots+F_0^{K}E_{M-N}^{N\{S,2S,\cdots,NS\}}\right]\end{aligned}$$

$F_{M-N}^{M}=S_1(M+1,N+1)$, **S_1 is the first kind of unsigned Stirling number.**

From 3.5) and 3.7)→

$$\begin{aligned}&H([1,\cdots,M-1],[1,\cdots,M-1],N)=(M-1)!\binom{M-1}{N}\\&=N!\left[F_{M-1-N}^{M-1}E_0^N+F_{M-1-N-1}^{M-1}E_1^N+\cdots+F_0^{M-1}E_{M-1-N}^N\right] \rightarrow\\&=N!\left[S_1(M,N+1)E_0^N+S_1(M,N+2)E_1^N+\cdots+S_1(M,M)E_{M-1-N}^N\right]\end{aligned}$$

3.10)

$$S_1(M,N+1)E_0^N+S_1(M,N+2)E_1^N+\cdots+S_1(M,M)E_{M-1-N}^N=\binom{M-1}{N}\frac{(M-1)!}{N!}$$

4. $K_1\times\cdots\times K_M+(L+K_1)\times\cdots\times(L+K_M)+(2L+K_1)\times\cdots\times(2L+K_M)+\cdots$

$K=[K_1,\cdots,K_M]$, $T=[T_1,\cdots,T_M]$, Max Factor = K_M

$$\begin{aligned}SUM(N,PS,PT)&=\sum_{n=-\infty}^{N}\sum END(n,PS,PT)\\&=\sum_{n=K_M+1}^{N}\sum END(n,PS,PT)\\&=\sum A_q\binom{N-PH(PS)-PCHG(PS,PT)-1}{IDX(PT)-q}\end{aligned}$$

When $N=K_M+1$

$$\begin{aligned}&N-PH(PS)-PCHG(PS,PT)-1\\&=N-(K_M-1-PB(BS))-PCHG(PS,PT)-1\\&=PB(BS)-PCHG(PS,PT)+1\\&=PB(PT)+1=IDX(PT)-M\end{aligned}$$

$$\rightarrow\ SUM(N, PS, PT) = A_M$$

$$\text{A product} = (K_1 + E_1)\cdots(K_M + E_M) = \sum F_1 F_2 \cdots F_M\ ,\quad F_i = E_i\ \text{ or }\ F_i = K_i$$

SUM(*N*, *PS*, *PT*) can be broken down into 2^M parts.

SUM_K(*N*, *PS*, *PT*, *PF*) can explain why *SUM*(*N*, *PS*, *PT*) has that strange form:

We can calculate every part of *SUM*() by some way without the form. There may be complex relationships between the parts, but their sum just match a simple form.

SUM_K(*N*, *PS*, *PT*, *PF*) use the form =

$$(T_1 + K_1)\cdots(T_M + K_M) = \sum \prod_{i=1}^{M} Y_i \binom{A}{M_q}$$

$$Y_i = \begin{cases} 0 : F_i = K_i, X_i = T_i \\ K_i : F_i = K_i, X_i = K_i \\ T_i - D_i : F_i = E_i, X_i = T_i, D_i = \text{count of } \{X_1, \cdots, X_{i-1}\} \in K \\ D_i : F_i = E_i, X_i = K_i, D_i = \text{count of } \{X_1, \cdots, X_{i-1}\} \in T \end{cases}$$

When T_i and D_i all increase *L* times. If $PT = [1, 2, \cdots, M]$, when *N* increase, $|End(N, PS, PT)| = 1$, match the corresponding *SUM_K*().

Define

$$SUML(N, PS, 1) = SUM(N + K_M, PS, [1, 2, \cdots, M])$$

$$SUML(N, PS, L) = K_1 \times \cdots \times K_M + (L + K_1) \times \cdots \times (L + K_M) + \cdots + ((N-1)L + K_1) \times \cdots \times ((N-1)L + K_M)$$

***SUML_K*(*N*, *PS*, *PF*, *L*) = corresponding part of *SUML*(*N*, *PS*, *L*)**

Above →

4.1) *SUML*(*N*, *PS*, *L*), *SUML_K*(*N*, *PS*, *PF*, *L*),

$PT = [T_1, \cdots, T_M] = [1 \times L, 2 \times L, \cdots, M \times L]$, **can use the from** $(T_1 + K_1)\cdots(T_M + K_M)$

$SUML(N, PS, L) = \sum A_q \binom{N}{M + 1 - q}$, $q = \text{count of } X \in K$, 2^M **items in total.**

$$A_q = \prod_{i=1}^{M} (X_i + D_i),\quad D_i = \begin{cases} -mL : X_i = T_i, m = \text{count of } \{X_1, \cdots, X_{i-1}\} \in K \\ +mL : X_i = K_i, m = \text{count of } \{X_1, \cdots, X_{i-1}\} \in T \end{cases}$$

Example:

$$SUML(N, [3, 5, 8], 4)\ \rightarrow$$

$$\text{form} = (1 \times 4 + 3)(2 \times 4 + 5)(3 \times 4 + 8) = (4 + 3)(8 + 5)(12 + 8)\ \rightarrow$$

$$= 384\binom{N}{4} + 896\binom{N}{3} + 636\binom{N}{2} + 120\binom{N}{1}$$

$$384 = 4 \times 8 \times 12;\quad 120 = 3 \times 5 \times 8$$

$$896 = 4 \times 8 \times (8 + 2 \times 4) + 4 \times (5 + 1 \times 4) \times (12 - 1 \times 4) + 3 \times (8 - 1 \times 4) \times (12 - 1 \times 4)$$

$$636 = 3\times5\times(12-2\times4)+3\times(8-1\times4)\times(8+1\times4)$$
$$+4\times(5+1\times4)\times(8+1\times4)$$

$$SUML(6,[3,5,8],4)$$
$$=3\times5\times8+7\times9\times12+11\times13\times16+15\times17\times20$$
$$+19\times21\times24+23\times25\times28$$
$$=384\times15+896\times20+636\times15+120\times6=33940$$

$$SUML(N,[1],2)=1+3+\cdots+(2N-1)=2\binom{N}{2}+\binom{N}{1}=N^2$$

4.2) P is a prime number, For arbitrary $K_1, K_2, \cdots, K_M$ **:**

If $M < P-1$**, then**

$$K_1\times K_2\times\cdots\times K_M+(L+K_1)\times\cdots\times(L+K_M)+\cdots$$
$$+((P-1)L+K_1)\times\cdots\times((P-1)L+K_M)\equiv 0\ MOD\ P$$

If $M = P-1$ **and** $(L,P)=1$ **then**

$$K_1\times K_2\times\cdots\times K_M+(L+K_1)\times\cdots\times(L+K_M)+\cdots$$
$$+((P-1)L+K_1)\times\cdots\times((P-1)L+K_M)\equiv -1\ MOD\ P$$

[Proof]

$$\text{The expression} = SUML(P,PS,L)=\sum A_q\binom{P}{M+1-q}$$

If $M < P-1$, then $M+1 < P$

If $M = P-1$ and $(L,P)=1$, then

$$SUML(N,PS,L)=L^{P-1}(P-1)!\binom{P}{P}+\sum A_q\binom{P}{M+1-q}, q>0$$
$$\equiv L^{P-1}(P-1)!\equiv -1\ MOD\ P$$

q.e.d.

5. Conclusions

The whole process of [1-3] is reviewed and this paper:

[1] tries to calculate all products of k distinct integers in [1, $N-1$], introduces the concept of Shape of numbers. The idea divides all products of k distinct integers in [1, $N-1$] into 2^{K-1} catalogs and derives the calculation formula of every catalog, that is 1.2).

[1] only introduces the basic shape. The conclusion is obtained through the derivation process.

[2] introduces the shape $PS=[1,K_1,\cdots,K_M]$, tries to calculate $SUM(N, PS)$, the form $(G_1+K_1)(G_2+K_2)\cdots(G_M+K_M)$ is guessed by observation and proved by induction.

At the same time, *SUM_K*() is introduced.

[3] introduces the subset, and shows the way to calculate $1^M+2^M+3^M+\cdots+N^M$.

In this paper, the Shape and the form are further extended. So a lot of numbers's series can be calculated.

Some new congruences are also obtained in [1] [2] [3] and this article.

The whole foundation is just $\binom{N}{M}+\binom{N}{M+1}=\binom{N+1}{M+1}$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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