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Subset of the Shape of Numbers

Ji Peng

Department of Electronic Information, Nanjing University, Nanjing, China
Email: mcfroo@sina.com

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Abstract

This article is based on the concept of Shape of numbers, introduces subset of the Shape and obtains its calculation formula. This article also makes some analysis and draws new conclusions, especially the calculation method of $1^M + 2^M + 3^M + \cdots + N^M$. The Shape's concept becomes clearer and richness.

Subject Areas

Discrete Mathematics

Keywords

Shape of Numbers, Calculation Formula, Sum of Powers of Integers, Combinatorics, Congruence, Stirling Number

1. Introduction

Peng, J. has introduced Shape of numbers in [1] [2]:

$$(I_1, I_2, \cdots, I_M), I_i \in N, I_1 < I_2 < \cdots < I_M$$

There are *M*-1 intervals between adjacent numbers.

$I_{i+1} - I_i = 1$ **means continuity,** $I_{i+1} - I_i > 1$ **means discontinuity.**

Shape of numbers: collect $(I_1, I_2, \cdots, I_M)$ **with the same continuity and discontinuity at the same position into a catalog, call it a Shape. A Shape has a min Item:**

$(1, K_1, K_2, \cdots)$, **Use the symbol *PS* = [min Item] to represent it. If** $K_{i+1} - K_i = D > 1$ **then only** $I_{i+1} - I_i \geq D$ **is allowed.**

The single $(I_1, I_2, \cdots, I_M)$ **is an item,** $I_1 I_2 \cdots I_M$ **is the product.** I_i **is a factor.**

Example:

$$PS = [1,2] \to (1,2),(2,3),(3,4),(1000,1001) \in PS$$

$$PS=[1,3]\to(1,3),(1,4),(2,4),(1,5),(2,5),(3,5),(1000,2001)\in PS$$

$$PS=[1,4]\to(1,4),(1,5),(2,5),(1,6),(2,6),(3,6)\in PS,(3,5),(4,6)\notin PS$$

$$PS=[1,4,6]\to(1,4,7),(1,5,7),(2,5,7)\in PS,(3,5,7)\notin PS$$

Define:

***SET*(*N*, *PS*) = set of items belonging to PS in** $[1,N-1]$

***PM*(*PS*) = count of factors**

***PB*(*PS*) = count of discontinuities**

***MIN*(*PS*) = min product:** $MIN([1,2,3])=1\times2\times3$, $MIN([1,2,4])=1\times2\times4$

***IDX*(*PS*) = (max factor)+1**

***PH*(*PS*) = *IDX*(*PS*) − *PB*(*PS*) − 2**

Basic Shape: intervals = 1 or 2

***BASE*(*PS*) = *BS*: if 1) *PB*(*BS*) = *PB*(*PS*), 2) *PM*(*BS*) = *PM*(*PS*), 3) *BS* is a Basic Shape 4) *BS* has discontinuity intervals at the same positions of *PS*.**

Example:

$$PS=[1,2]\to BASE(PS)=[1,2]$$

$$PS=[1,3],[1,4],[1,K>2]\to BASE(PS)=[1,3]$$

$$PS=[1,3,4],[1,4,5],[1,K>2,X=K+1]\to BASE(PS)=[1,3,4]$$

$$PS=[1,3,5],[1,4,9],[1,K>2,X>K+1]\to BASE(PS)=[1,3,5]$$

|*SET*(*N*, *PS*)| = Count of items in *SET*(*N*, *PS*)

***SUM*(*N*, *PS*) = Sum of all products in *SET*(*N*, *PS*)**

Example:

$$SUM(6,[1,2,4])=1\times2\times4+1\times2\times5+2\times3\times5$$

$$SUM(9,[1,4,7])=1\times4\times7+1\times4\times8+1\times5\times8+2\times5\times8$$

[1] [2] came to the following conclusion:

1.1) $|SET(N,PS)|=\binom{N-PH(PS)-1}{PB(PS)+1}$

1.2) $SUM(N,PS)=MIN(PS)\binom{N}{IDX(PS)}$**, *PS* is a Basic Shape**

The following uses count of $X\in K$ for count of $\{X_1,X_2,\cdots,X_M\}\in\{K_1,K_2,\cdots,K_M\}$

1.3) $PS=[1,K_1,\cdots,K_M]$, $BS=BASE(PS)=[1,G_1,\cdots,G_M]$

Use the form $(G_1+K_1)(G_2+K_2)\cdots(G_M+K_M)=\sum X_1X_2\cdots X_M$ **,** $X_i=G_i$ **or** K_i**.**

The expansion has 2^M **items, don't swap the factors of** $X_1X_2\cdots X_M$**, then each** $X_1X_2\cdots X_M$ **corresponds to one expression =** $A_q\binom{N-PH(PS)}{IDX(BS)-q}$**,** q **= count of** $X\in K$

$SUM(N,PS)=\sum A_q\binom{N-PH(PS)}{IDX(BS)-q}$**,** 2^M **items in total.**

$$A_q = \prod_{i=1}^{M}(X_i + D_i),\quad D_i = \begin{cases} -m : X_i = G_i, m = \text{count of } \{X_1, \cdots, X_{i-1}\} \in K \\ +m : X_i = K_i, m = \text{count of } \{X_1, \cdots, X_{i-1}\} \in G \end{cases}$$

Example:

$PS = [1, K_1 \geq 3, K_2 \geq K_1 + 2, K_3 \geq K_2 + 2]$, $BS = BASE(PS) = [1,3,5,7]$

$$\begin{aligned}\text{The form} &= (3+K_1)(5+K_2)(7+K_3) \\ &= 3\times5\times7 + 3\times5\times K_3 + 3\times K_2\times7 + 3\times K_2\times K_3 \\ &\quad + K_1\times5\times7 + K_1\times5\times K_3 + K_1\times K_2\times7 + K_1\times K_2\times K_3\end{aligned}$$

$$\begin{aligned}P &= N - PH(PS) = N - \{IDX(PS) - PB(PS) - 2\} \\ &= N - \{K_3 + 1 - 3 - 2\} = N - K_3 + 4\end{aligned}$$

$IDX(BS) = 8$

$\rightarrow$

$$\begin{aligned}SUM(N,PS) &= 3\times5\times7\binom{P}{8} + 3\times5\times(K_3+2)\binom{P}{7} \\ &\quad + 3\times(K_2+1)\times(7-1)\binom{P}{7} + 3\times(K_2+1)\times(K_3+1)\binom{P}{6} \\ &\quad + K_1\times(5-1)\times(7-1)\binom{P}{7} + K_1\times(5-1)\times(K_3+1)\binom{P}{6} \\ &\quad + K_1\times K_2\times(7-2)\binom{P}{6} + K_1\times K_2\times K_3\binom{P}{5}\end{aligned}$$

An item $\in PS = \{\text{begin}, K_1 + E_1, \cdots, K_M + E_M\}$, K is fixed, E is variable.

A product $= \text{begin}\times(K_1+E_1)\cdots(K_M+E_M) = \text{begin}\times\sum F_1F_2\cdots F_M$, $F_i = E_i$ or $F_i = K_i$

That is, a product can be broken down into 2^M parts.

Define

$SUM\_K(SET(N,PS), PF = F_1F_2\cdots F_M)$ **= Sum of one part of** $SUM(N,PS)$

PF **indicates the part.** $PF = F_1F_2\cdots F_M$ **,** $F_i = E_i$ **or** $F_i = K_i$

Rewrite 1.3) and add {braces}:

$$\begin{aligned}SUM(N,PS) &= \sum\text{product} = \sum\sum\text{begin}\times F_1\cdots F_M \\ &= \sum\prod_{i=1}^{M}(X_i + D_i)\binom{A}{M_q}\end{aligned}$$

$$X_i + D_i = \begin{cases} \{G_i - D_i\} : X_i = G_i, D_i = \text{count of } \{X_1, \cdots, X_{i-1}\} \in K \\ \{K_i\} + \{D_i\} : X_i = K_i, D_i = \text{count of } \{X_1, \cdots, X_{i-1}\} \in G \end{cases}$$

Let $SUM1(N,PS) = SUM(N,PS)$ expand by the {braces}:

1.4) $SUM\_K(SET(N,PS), PF) = \sum\text{Expansion of } SUM1(N,PS)$ **with same** $\{K_i\} \in PF = \sum\prod_{i=1}^{M} Y_i\binom{A}{M_q}$,

$$Y_i = \begin{cases} 0 : F_i = K_i, X_i = G_i \\ K_i : F_i = K_i, X_i = K_i \\ G_i - D_i : F_i = E_i, X_i = G_i, D_i = \text{count of } \{X_1, \cdots, X_{i-1}\} \in K \\ D_i : F_i = E_i, X_i = K_i, D_i = \text{count of } \{X_1, \cdots, X_{i-1}\} \in G \end{cases}$$

Example:

$$SUM\left(N,[1,K_1 \ge 3,K_2 \ge K_1+2]\right),\ \text{form}=(3+K_1)(5+K_2)\ \to$$

$$=15\binom{N-K_2+3}{6}+3\left(\{K_2\}+\{1\}\right)\binom{N-K_2+3}{5}$$

$$+K_1\left(\{5-1\}\right)\binom{N-K_2+3}{5}+K_1K_2\binom{N-K_2+3}{4}$$

Expand by the {braces}:

$$=\left\{15\binom{N-K_2+3}{6}+3\binom{N-K_2+3}{5}\right\}+3K_2\binom{N-K_2+3}{5}$$

$$+4K_1\binom{N-K_2+3}{5}+K_1K_2\binom{N-K_2+3}{4}$$

$$=\sum_{\text{begin}=1}^{N-K_2}\sum \text{begin}\times\left(K_1+E_{1,\text{begin}}\right)\left(K_2+E_{2,\text{begin}}\right)$$

$\to$

$$SUM\_K\left(SET(N,PS),E_1E_2\right)$$

$$=\sum_{\text{all items}}\text{begin}*E_{1,i}E_{2,i}=15\binom{N-K_2+3}{6}+3\binom{N-K_2+3}{5}$$

$$SUM\_K\left(SET(N,PS),E_1K_2\right)=\sum_{\text{all items}}\text{begin}*E_{1,i}K_2=3K_2\binom{N-K_2+3}{5}$$

$$SUM\_K\left(SET(N,PS),K_1E_2\right)=\sum_{\text{all items}}\text{begin}*K_1E_{2,i}=4K_1\binom{N-K_2+3}{5}$$

$$SUM\_K\left(SET(N,PS),K_1K_2\right)=\sum_{\text{all items}}\text{begin}*K_1K_2=K_1K_2\binom{N-K_2+3}{4}$$

This can explain why 1.3) has that strange form:

We can calculate every part of 1.3) by some way without 1.3). There may be complex relationships between the parts, but their sum just match a simple form.

1.5) Use the symbol of 1.3), when $G_i=K_i$, $N_1=$ Count of $X\in K$, $N_1+N_2=M$

$$H(N_1,N_2,K)=\sum A_{q=N_1}=\sum\prod_{i=1}^{M}(X_i+D_i)=K_1K_2\cdots K_M\binom{M}{N_1,N_2}$$

Sum traverses all (N_1, N_2)-Choice of K

This $\to$ 1.3) is compatible with 1.2)

1.6) P is a prime number, {$PS1$, $PS2$, …} are all of the Basic Shapes,

$PM(PS1)=PM(PS2)=\cdots$, $PB(PS1)=PB(PS2)=\cdots>0$, $IDX(PS1)=IDX(PS2)=\cdots=P$,

That is, them are Basic shapes, have same count of factors and same count of discontinuities > 0, and max factor = $P-1$, then

$MIN(PS1)+MIN(PS2)+\cdots\equiv 0\ MOD\ P$

Example:

$$1\times2\times4\times6+1\times3\times4\times6+1\times3\times5\times6$$
$$\equiv1\times2\times3\times4\times6+1\times2\times3\times5\times6+1\times2\times4\times5\times6+1\times3\times4\times5\times6$$
$$\equiv 0\ MOD\ 7$$

2. Subset of *SET*(*N*, *PS*)

$PS = [1, K_1, K_2, \cdots, K_M]$, $PT = [1, T_1, T_2, \cdots, T_M]$,

Item $= \{I_0, I_1, I_2, \cdots, I_M\} \in SET(N, PS)$

If $PB(PS) = 0$, items $\in SET(N, PS)$ is very simple.

If $PB(PS) > 0$, some changes appear in $SET(N, PS)$.

We can fix some discontinuities of the Shape to get subsets.

Define *SET*(*N*, *PS*, *PT*) =Subset of *SET*(*N*, *PS*), a valid

$$PT = [1, T_1, \cdots, T_M]$$

$$= \begin{cases} T_{i+1} - T_i = 1 : K_{i+1} - K_i = 1, \text{ means } I_{i+1} - I_i = 1 \\ T_{i+1} - T_i = 1 : K_{i+1} - K_i = D > 1, \text{ means } I_{i+1} - I_i = D \\ T_{i+1} - T_i = 2 : K_{i+1} - K_i = D > 1, \text{ means } I_{i+1} - I_i \ge D \end{cases} \quad (*)$$

others are invalid.

Example:

$SET(N, [1,3,5], [1,3,5]) = SET(N, [1,3,5])$

$SET(N, [1,3,5], [1,4,5])$, $SET(N, [1,3,5], [1,3,6])$, $SET(N, [1,2,9], [1,3,4])$ is invalid.

$SET(N, [1,3,5], [1,2,4])$

$= \{(1,3,5), (1,3,6), (2,4,6), (1,3,7), (2,4,7), (3,5,7), \cdots\}$

$I_2 - I_1 = (3-1) = 2$, $I_3 - I_2 \ge (5-3) = 2$

$SET(N, [1,3,5], [1,3,4])$

$= \{(1,3,5), (1,4,6), (2,4,6), (1,5,7), (2,5,7), (3,5,7), \cdots\}$

$I_3 - I_2 = (5-3) = 2$, $I_2 - I_1 \ge (3-1) = 2$

$SET(N, [1,3,5], [1,2,3]) = \{(1,3,5), (2,4,6), (3,5,7), \cdots\}$

$SET(N, [1,4,8], [1,2,4])$

$= \{(1,4,8), (1,4,9), (2,5,9), (1,4,10), (2,5,10), (3,6,10), \cdots\}$

$I_2 - I_1 = (4-1) = 3$, $I_3 - I_2 \ge (8-4) = 4$

PT only has the change at (*). When a change happens, make the interval fixed.

The more changes, the fewer items:

Define *PCHG*(*PS*, *PT*) = count of change from *BASE*(*PS*) to *PT*

Example:

$PCHG([1,3,5], [1,2,4]) = PCHG([1,4,7], [1,2,4]) = 1$, changed at T_1

$PCHG([1,3,5], [1,3,4]) = PCHG([1,4,7], [1,3,4]) = 1$, changed at T_2

$PCHG([1,3,5], [1,2,3]) = PCHG([1,8,10], [1,2,3]) = 2$, changed at T_1, T_2

2.1) $SET(N, PS) = SET(N, PS, BASE(PS))$

2.2) $|SET(N, PS, PT)| = \binom{N - PH(PS) - 1 - PCHG(PS, PT)}{PB(PT) + 1}$

If PT1 only change T_i of PT, Obvious: $PCHG(PS,PT1)=PCHG(PS,PT)+1$

2.3) If $PT1$ only change T_i of PT,

$PT1=[1,T_1,\cdots,T_{i-1},T_i-1,T_{i+1}-1,\cdots,T_M-1]$.

Let $PS1=[1,K_1,\cdots,K_{i-1},K_i+1,K_{i+1}+1,\cdots,K_M+1]$**, then**

$$SET(N,PS,PT)=SET(N,PS,PT1)\cup SET(N,PS1,PT)$$

$$SET(N,PS,PT1)=SET(N,PS,PT)-SET(N,PS1,PT)$$

In particular: $PCHG(PS,PT)=1$ →

$SET(N,PS,PT)=SET(N,PS)-SET(N,PS1)$

[Proof]

PT1 change T_i of PT→ $T_i-T_{i-1}=2$ → $K_i-K_{i-1}>1$ →

$PCHG(PS,PT)=PCHG(PS1,PT)$

$$\begin{aligned}
&|SET(N,PS,PT1)|+|SET(N,PS1,PT)|\\
&=\binom{N-PH(PS)-1-PCHG(PS,PT1)}{PB(PT1)+1}\\
&\quad+\binom{N-PH(PS1)-1-PCHG(PS1,PT)}{PB(PT)+1}\\
&=\binom{N-PH(PS)-2-PCHG(PS,PT)}{PB(PT)}\\
&\quad+\binom{N-PH(PS)-2-PCHG(PS1,PT)}{PB(PT)+1}\\
&=\binom{N-PH(PS)-2-PCHG(PS,PT)}{PB(PT)}\\
&\quad+\binom{N-PH(PS)-2-PCHG(PS,PT)}{PB(PT)+1}\\
&=\binom{N-PH(PS)-1-PCHG(PS,PT)}{PB(PT)+1}\\
&=|SET(N,PS,PT)|
\end{aligned}$$

Count of the Items is equal.

Every item in $SET(N, PS1, PT)$ is in $SET(N, PS, PT)$, and not in $SET(N, PS, PT1)$.

q.e.d.

if $PCHG(PS,PT)=2$, PT changes at T_i and T_j, then

$$PT=[1,G_1,\cdots,G_{i-1},G_i-1,\cdots,G_{j-1}-1,G_j-2,G_{j+1}-2,\cdots,G_M-2]$$

Let

$PTA=[1,G_1,\cdots,G_{i-1},G_i-1,\cdots,G_M-1]$,

$PTB=[1,G_1,\cdots,G_{j-1},G_j-1,\cdots,G_M-1]$

$PSA=[1,K_1,\cdots,K_{i-1},K_i+1,\cdots,K_M+1]$,

$$PSB = \left[1, K_1, \cdots, K_{j-1}, K_j + 1, \cdots, K_M + 1\right]$$

$$PS2 = \left[1, K_1, \cdots, K_{i-1}, K_i + 1, \cdots, K_{j-1} + 1, K_j + 2, K_{j+1} + 2, \cdots, K_M + 2\right]$$

$\rightarrow$

$$\begin{aligned}
&SET(N, PS, PT) = SET(N, PS, PTA) - SET(N, PSA, PTA) \\
&= \left\{SET(N, PS, BS) - SET(N, PSB, BS)\right\} \\
&\quad - \left\{SET(N, PSA, BS) - SET(N, PS2, BS)\right\} \\
&= SET(N, PS, BS) - \left[SET(N, PSA, BS) \cup SET(N, PSB, BS)\right] \\
&\quad + SET(N, PS2, BS) \\
&= SET(N, PS) - [SET(N, PSA) \cup SET(N, PSB)] + SET(N, PS2)
\end{aligned}$$

General:

2.4) The relationship between *SET*(*N*, *PS*, *PT*) and *SET*(*N*, *PSX*) is similar to the Inclusion Exclusion Principle.

3. Calculation formula of *SET*(*N*, *PS*, *PT*)

Define:

***SUM_SUBSET*(*N*, *PS*, *PT*) = Sum of all products in *SET*(*N*, *PS*, *PT*)**

When *PT* is invalid, *SUM_SUBSET*(*N*, *PS*, *PT*) = 0

Only valid *PT* is discussed below.

$$PS = [1, K_1, K_2, \cdots, K_M], \quad BS = BASE(PS) = [1, G_1, G_2, \cdots, G_M],$$
$$PT = [1, T_1, T_2, \cdots, T_M]$$

3.1) Use the form $(T_1 + K_1)(T_2 + K_2)\cdots(T_M + K_M) = \sum X_1 X_2 \cdots X_M$ **, then**

$$SUM\_SUBSET(N, PS, PT) = \sum A_q \binom{N - PH(PS) - PCHG(PS, PT)}{IDX(PT) - q}$$

$$A_q = \prod_{i=1}^{M} (X_i + D_i), \quad D_i = \begin{cases} -m : X_i = T_i, m = \text{count of } \{X_1, \cdots, X_{i-1}\} \in K \\ +m : X_i = K_i, m = \text{count of } \{X_1, \cdots, X_{i-1}\} \in T \end{cases}$$

$q = \text{count of } X \in K$

[Proof]

1) If $PT = BS$, then $SUM\_SUBSET(N, PS, BS) = SUM(N, PS)$ $\rightarrow$ the formula holds.

2) If $M = 1$ and *PT* has 1 change, then $PS = [1, K_1 > 2]$, $PT = [1, T_1] = [1, 2]$, $BS = [1, G_1] = [1, 3]$,

Let $PS1 = [1, K_1 + 1]$, 2.3) $\rightarrow$

$$\begin{aligned}
&SUM_SUBSET(N, PS, PT) \\
&= SUM(N, PS) - SUM(N, PS1) \\
&= \left\{G_1 \binom{N - PH(PS)}{IDX(BS)} + K_1 \binom{N - PH(PS)}{IDX(BS) - 1}\right\} \\
&\quad - \left\{G_1 \binom{N - PH(PS1)}{IDX(BS)} + (K_1 + 1) \binom{N - PH(PS1)}{IDX(BS) - 1}\right\}
\end{aligned}$$

$$=\left\{G_1\binom{N-PH(PS1)}{IDX(BS)}+K_1\binom{N-PH(PS1)}{IDX(BS)-1}\right.$$
$$\left.+G_1\binom{N-PH(PS1)}{IDX(BS)-1}+K_1\binom{N-PH(PS1)}{IDX(BS)-2}\right\}$$
$$-\left\{G_1\binom{N-PH(PS1)}{IDX(BS)}+(K_1+1)\binom{N-PH(PS1)}{IDX(BS)-1}\right\}$$
$$=(G_1-1)\binom{N-PH(PS1)}{IDX(BS)-1}+K_1\binom{N-PH(PS1)}{IDX(BS)-2}$$
$$=T_1\binom{N-PH(PS1)}{IDX(BS)-1}+K_1\binom{N-PH(PS1)}{IDX(BS)-2}$$
$$=T_1\binom{N-PH(PS)-1}{IDX(PT)}+K_1\binom{N-PH(PS)-1}{IDX(PT)-1}$$
$$=T_1\binom{N-PH(PS)-PCHG(PS,PT)}{IDX(PT)}$$
$$+K_1\binom{N-PH(PS)-PCHG(PS,PT)}{IDX(PT)-1}$$

The form = $(T_1+K_1)\rightarrow$ The formula holds.

3) If $M>1$ and PT only has 1 change at T_M, then $PT=[1,G_1,\cdots,G_{M-1},G_M-1]$

Let $PS1=[1,K_1,\cdots,K_{M-1},K_M+1]$, 2.3) $\rightarrow PH(PS1)=PH(PS)+1$:

$$SUM(N,PS,PT)=SUM(N,PS)-SUM(N,PS1)$$
$$=\sum_{i=IDX(BS)-M}^{IDX(BS)}\sum A_i\binom{N-PH(PS)}{i}-\sum_{i=IDX(BS)-M}^{IDX(BS)}\sum B_i\binom{N-PH(PS1)}{i}$$
$$=\sum_{i=IDX(BS)-M}^{IDX(BS)}\sum A_i\left\{\binom{N-PH(PS1)}{i}+\binom{N-PH(PS1)}{i-1}\right\}$$
$$-\sum_{i=IDX(BS)-M}^{IDX(BS)}\sum B_i\binom{N-PH(PS1)}{i}$$
$$=\sum_{i=IDX(BS)-M}^{IDX(BS)}\left(\sum A_i-\sum B_i\right)\binom{N-PH(PS1)}{i}+\sum A_i\binom{N-PH(PS1)}{i-1}$$
$$=\sum_{i=IDX(BS)-M}^{IDX(BS)}\left(\sum A_i-\sum B_i\right)\binom{N-PH(PS)-1}{i}+\sum A_i\binom{N-PH(PS)-1}{i-1}$$
$$=\sum_{i=IDX(BS)-M-1}^{IDX(BS)}\sum C_J\binom{N-PH(PS)-1}{J},J=i-1,\sum C_{J-1}=\sum A_{i+1}+\sum A_i-\sum B_i$$
$$=\sum_{J=IDX(PT)-M}^{IDX(PT)+1}\sum C_J\binom{N-PH(PS)-1}{J}$$

When $i=IDS(BS)$, $\sum C_i=MIN(BS)-MIN(BS)=0$

$$=\sum_{J=IDX(PT)-M}^{IDX(PT)}\sum C_J\binom{N-PH(PS)-1}{J}$$

Use the symbol of (1.3)

$$i=IDX(BS)-CNT\ ,\quad CNT=\text{Count of }X\in K$$

Let $R(E)=\sum\prod_{L=1}^{M-1}(X_L+D_L)$, $E=\text{Count of }\{X_1,\cdots,X_{M-1}\}\in K$

$$\sum A_{i+1}=R(CNT-1)(G_M-(CNT-1))$$
$$+R(CNT-2)(K_M+(M-1)-(CNT-2))$$

$$\sum A_i=R(CNT)(G_M-CNT)+R(CNT-1)(K_M+(M-1)-(CNT-1))$$

$$\sum B_i=R(CNT)(G_M-CNT)+R(CNT-1)((K_M+1)+(M-1)-(CNT-1))$$

$$\sum C_{J-1}=\sum A_{i+1}+\sum A_i-\sum B_i$$
$$=R(CNT-1)(G_M-CNT)+R(CNT-2)(K_M+M-CNT+1)$$
$$=R(CNT-1)(T_M-\{CNT\ -1\})+R(CNT-2)(K_M+\{(M-1)-(CNT-2)\})$$

$\rightarrow$ Match of the form $(T_1+K_1)(T_2+K_2)\cdots(T_M+K_M)$

4) If $M>1$ and PT only has 1 change at $T_{i<M}$, Let $R(E)=\sum\prod_{L=1,l\neq i}^{M}(X_L+D_L)$, use the same method of (3).

5) if $PCHG(PS,PT)>1$, Use 2.3) $\rightarrow$ divide the Items into subset $\rightarrow$ deducing by induction.

q.e.d.

Example:

$$N-PH([1,3,5,7])-PCHG([1,3,5,7],[1,2,3,4])=N-(8-3-2)-3=N-6$$

$$SUM\_SUBSET(N,[1,3,5,7],[1,2,3,4])$$

$\rightarrow$

$$\text{form}=(2+3)(3+5)(4+7)$$
$$=24\binom{N-6}{5}+108\binom{N-6}{4}+174\binom{N-6}{3}+105\binom{N-6}{2}$$
$$=1\times3\times5\times7+2\times4\times6\times8+3\times5\times7\times9+\cdots$$

Among:

$$24=2\times3\times4;\ \ 105=3\times5\times7$$

$$108=2\times3\times(7+2)+2\times(5+1)\times(4-1)+3\times(3-1)\times(4-1)$$

$$174=2\times(5+1)\times(7+1)+3\times(3-1)\times(7+1)+3\times5\times(4-2)$$

Use the same method of 3.1)

3.2) Calculation formula of *SUM_K*(*SET*(*N*, *PS*, *PT*), *PF*) is similar to 1.4).

Example:

$$SUM\_SUBSET(N,[1,3,7],[1,2,3])$$
$$=6\binom{N-6}{4}+2\times(\{7\}+\{1\})\binom{N-6}{3}+3\times(\{3-1\})\binom{N-6}{3}+3\times7\binom{N-6}{2}$$

$$SUM\_SUBSET(10,[1,3,7],[1,2,3])$$
$$=1\times3\times7+2\times4\times8+3\times5\times9$$
$$=1\times3\times7+2\times(3+1)\times(7+1)+3\times(3+2)\times(7+2)$$
$$=\{1\times3\times7+2\times3\times7+3\times3\times7\}+\{2\times3\times1+3\times3\times2\}$$
$$+\{2\times1\times7+3\times2\times7\}+\{2\times1\times1+3\times2\times2\}$$

$\rightarrow$

$$\{1\times3\times7+2\times3\times7+3\times3\times7\}=3\times7\binom{10-6}{2}$$

$$\{2\times3\times1+3\times3\times2\}=3\times(\{3-1\})\binom{N-6}{3}=6\binom{10-6}{3}$$

$$\{2\times1\times7+3\times2\times7\}=2\times7\binom{N-6}{3}=14\binom{N-6}{3}$$

$$\{2\times1\times1+3\times2\times2\}=6\binom{10-6}{4}+2\times\{1\}\binom{10-6}{3}$$

3.3) Use the form $(T_1+K_1)(T_2+K_2)\cdots(T_M+K_M)=\sum X_1X_2\cdots X_M$

$$SUM\_K(SET(N,PS,PT),E_1E_2\cdots E_M)$$
$$=\sum A_q\binom{N-PH(PS)-PCHG(PS,PT)}{IDX(PT)-q}$$
$$A_q=\prod_{i=1}^{M}D_i\,,\quad D_i=\begin{cases}T_i-m: X_i=T_i, m=\text{count of }\{X_2,\cdots,X_{i-1}\}\in K\\ +m: X_i=K_i, m=\text{count of }\{X_2,\cdots,X_{i-1}\}\in T\end{cases}$$

$q=\text{count of } X\in K$ **,** $X_1=T_1$ **, 2^{M-1} Items in total.**

In particular:

If $PT1=[1,T_1,\cdots,T_M]=[1,2,\cdots,M+1]$, then $PB(PS)=PCHG(PS,PT1)$

$$N-PH(PS)-PCHG(PS,PT1)$$
$$=N-[IDX(PS)-2-PB(PS)]-PB(PS)$$
$$=N-(K_M-1)$$
$$IDX(PT1)=M+2$$
$$SUM(SET(N,PS,PT1),E_1\cdots E_M)$$
$$=2\times1^M+3\times2^M+\cdots+(N-K_M)\times(N-K_M-1)^M$$
$$\rightarrow$$
$$SUM(SET(N+K_M+1,PS,PT1),E_1\cdots E_M)$$
$$=2\times1^M+3\times2^M+\cdots+(N+1)\times N^M$$

3.4) $\sum_{n=1}^{N}(n+1)n^M=SUM\_K(SET(N+K_M+1,PS,PT1),E_1\cdots E_M)$

4. Analysis of *SUM_SUBSET*(*N*, *PS*, [1, 2, ···, *M* + 1])

$PS=[1,K_1,K_2,\cdots,K_M]$, $PT1=[1,2,3,\cdots,M+1]$. The simplest subset of *PS* is *SET*(*N*, *PS*, *PT*1).

$$SUM\_SUBSET(N,PS,PT1)$$
$$=\sum_{q=0}^{M}C_q\binom{N-(K_M-1)}{M+2-q}=\sum_{n=1}^{N-K_M}\sum_{q=0}^{M}C_q\binom{n}{M+1-q}$$
$$=1\times K_1\times\cdots\times K_M+2\times(1+K_1)\times\cdots\times(1+K_M)+\cdots \tag{1*}$$
$$+(N-K_M)\times([N-K_M-1]+K_1)\times\cdots\times(N-1)$$
$$=\sum_{n=1}^{N-K_M}n(n+K_1-1)(n+K_2-1)\cdots(n+K_M-1)$$

Solve (1*) in a normal way:

Decompose $n(n+K_1-1)\cdots(n+K_M-1)$ to $\sum_{j=1}^{M+1} D_j \binom{n}{j} \rightarrow D_j = C_q$, $q = M+1-j$

4.1) $1 < K_1 < \cdots < K_M$ **, 3.1) can decompose** $n(n+K_1-1)\cdots(n+K_M-1)$ **to** $\sum_{j=1}^{M+1} D_j \binom{n}{j}$

In particular, 1.5) $\rightarrow C_q = (M+1)!\binom{M}{M-q} \rightarrow D_j = (M+1)!\binom{M}{j-1} \rightarrow$

4.2) $[x]^M = \frac{M!}{M!}\binom{M-1}{M-1}[x]_M + \frac{M!}{(M-1)!}\binom{M-1}{M-2}[x]_{M-1} + \cdots + \frac{M!}{1!}\binom{M-1}{0}[x]_1$

4.3) ***P* is a prime number,** $N-(K_M-1)=P$

1) $|SET(N,PS,PT1)| = P-1$**,**

2) $SET(N,PS,PT1) = \{(1,K_1,\cdots,K_M),(2,1+K_1,\cdots,1+K_M)\cdots(P-1,\cdots,N-1)\}$

3) if $K_M \le P-1$ **and** $PS \ne [1,2,\cdots,P-1]$**, then** $SUM\_SUBSET(N,PS,PT1) \equiv 0\ MOD\ P$

[Proof]

$$|SET(N,PS,PT1)| = \binom{N-PH(PS)-1-PCHG(PS,PT1)}{PB(PT1)+1}$$

$$= \binom{(P+K_M-1)-(K_M+1-PB(PS)-2)-1-PCHG(PS,PT1)}{1}$$

$$\xrightarrow{PB(PS)=PCHG(PS,PT1)} P-1 \rightarrow (1)(2)$$

$$SUM\_SUBSET(N,PS,PT1) = \sum_{q=0}^{M} C_q \binom{P}{IDX(PT1)-q}$$

$$K_M \le P-1 \text{ and } PS \ne [1,2,\cdots,P-1] \rightarrow IDX(PT1) < P \rightarrow (3)$$

q.e.d.

When $K_M = P-1$ and $PS \ne [1,2,\cdots,P-1]$

$$\begin{aligned} &SUM\_SUBSET(N,PS,PT1) \\ &= 1\times K_1\times\cdots\times K_M + 2\times(1+K_1)\times\cdots\times P \\ &\quad + 3\times(2+K_1)\cdots(2+K_{M-1})(2+K_M) \\ &\quad + 4\times(3+K_1)\cdots(3+K_{M-1})(3+K_M)+\cdots \\ &\quad + (P-1)\times(P-2+K_1)\times\cdots\times(P-2+K_M) \end{aligned}$$

$$3\times(2+K_1)\cdots(2+K_{M-1})(2+K_M) \equiv 1\times3\times(2+K_1)\cdots(2+K_{M-1})$$

$$\begin{aligned} &4\times(3+K_1)\cdots(3+K_{M-1})(3+K_M) \\ &\equiv 2\times(1+[3])\times(1+[2+K_1])\cdots(1+[2+K_{M-1}]) \end{aligned}$$

…

$$(P-1)\times(P-2+K_1)\cdots(P-2+K_M)$$
$$\equiv(P-1)\times(P-2+K_1)\cdots(P-2+K_{M-1})(2P-3)$$
$$\equiv(P-4+[1])(P-4+[3])(P-4+[2+K_1])\cdots(P-4+[2+K_{M-1}])$$

$$1\times K_1\times\cdots\times K_M\equiv 1\times K_1\times\cdots\times(P-1)\equiv(P+1)(P+K_1)\cdots(P+P-1)$$
$$\equiv(P-1)(P-2+3)([P-2]+[2+K_1])\cdots([P-2]+[2+K_{M-1}])$$

$\rightarrow$

$$SUM\_SUBSET(N,PS,PT1)$$
$$\equiv SUM\_SUBSET(N,[1,3,2+K_1,\epsilon,2+K_{M-1}],PT1)\ MOD\ P$$

$PS=[1,K_1\cdots K_M]$ can be slided to $[1,3,2+K_1,\cdots,2+K_{M-1}]$ by $MOD\ P$

If $K_M=P-1$ and exists $K_{i+1}-K_i>2$, *PS* can be slided to $[1,\cdots,X<P-1]$

If $K_M=P-1$ and not exists $K_{i+1}-K_i>2$, *PS* must be a Basic Shape, can only be slided to $[1,\cdots,X=P-1]$

Define:

A Basic Shape $PS=[1,K_1\cdots K_M]=[\![L_1L_2\cdots L_Q]\!]$**, among:**

L_i **= count of continuity. (**L_i**,** L_{i+1}**) means a discontinuity, there are** Q**-1 discontinuities.**

Example: $[1,3,5]=[\![1,1,1]\!]$, $[1,2,3,5]=[\![3,1]\!]$, $[1,2,4,5]=[\![2,2]\!]$

Obvious: $[\![L_1L_2\cdots L_Q]\!]$ can been slided to $[L_Q,L_1,L_2,\cdots]$, $[L_{Q-1},L_Q,L_1,L_2,\cdots]$, $\cdots$

4.4) ***PS*** **is a Basic Shape,** $IDX(PS)=P$, $PS\neq[1,2,\cdots,P-1]$, $\{PS1,PS2,\cdots\}$ **are all shapes that** ***PS*** **can scroll to, then** $MIN(PS1)+MIN(PS2)+\cdots\equiv 0\ MOD\ P$**. This is a promotion of 1.6)**

[Proof]

$$3\times(2+K_1)\cdots(2+K_{M-1})(2+K_M)\equiv 1\times 3\times(2+K_1)\cdots(2+K_{M-1})MOD\ P$$

If $2+K_{M-1}\neq P$, $[1,3,(2+K_1),\cdots,(2+K_{M-1})]\in\{PS1,PS2,\cdots\}$

…

$$SUM\_SUBSET(N,PS1,PT1)\equiv MIN(PS1)+MIN(PS2)+\cdots\equiv 0\ MOD\ P$$

q.e.d.

Example:

$$1\times(3\times4\times5)\times(7\times8)\times10+1\times3\times(5\times6\times7)\times(9\times10)$$
$$+(1\times2)\times4\times6\times(8\times9\times10)+(1\times2\times3)\times(5\times6)\times8\times10$$
$$=139260\equiv 0\ MOD\ 11$$

5. Calculation Formula of $1^M+2^M+3^M+\cdots+N^M$

Use the form $(T_1+K_1)(T_2+K_2)\cdots(T_M+K_M)$.

In general, K_i and K_j cannot be exchanged, but when $PT=PT1=[1,2,\cdots,M+1]$,

$$SUM\_SUBSET(N,PS,PT1)=\sum_{q=0}^{M}C_q\binom{N-PH(PS)-PCHG(PS,PT1)}{IDX(PT1)-q}$$

$$C_q = \sum\prod_{i=1}^{M}\left(X_i + D_i\right) = \sum\prod_{\text{choice } M-q \text{ from } T}\left(T - D\right)\prod_{\text{choice } q \text{ from } K}\left(K + D\right)$$

Easy to see: $\prod_{\text{choice } M-q \text{ from } T}\left(T-D\right) = \left(M+1-q\right)!$

Can prove: $\left(K_1, K_2, \cdots, K_M\right)$ **is permutable in** $\sum\prod_{\text{choice } q \text{ from } K}\left(K+D\right)$

→ $\left(K_1, K_2, \cdots, K_M\right)$ is permutable in the form.

$$SUM\_SUBSET\left(N, PS, PT1\right)$$

$$= \sum_{q=0}^{M} A_q \binom{N - K_M + 1}{M + 2 - q} = \sum_{n=K_M+1}^{N}\sum_{q=0}^{M} A_q \binom{n - K_M}{M + 1 - q}$$

$$= 1 \times K_1 \times \cdots \times K_M + 2 \times \left(1 + K_1\right) \times \cdots \times \left(2 + K_M\right) + \cdots$$

Add one more factor K_i to the end:

$$1 \times K_1 \times \cdots \times K_M \times K_i + 2 \times \left(1 + K_1\right) \times \cdots \times \left(1 + K_M\right) \times \left(1 + K_i\right) + \cdots$$

$$= \sum_{n=K_M+1}^{N}\sum_{q=0}^{M} A_q \binom{n - K_M}{M + 1 - q}\left(n - 1 - K_M + K_i\right)$$

$$= \sum_{n=K_M+1}^{N}\sum_{q=0}^{M} A_q \binom{n - K_M}{M + 1 - q}\left(\left[n - K_M + 1\right] + \left[K_i - 2\right]\right)$$

$$= \sum_{n=K_M+1}^{N}\sum_{q=0}^{M}\left\{A_q\left(M + 2 - q\right)\binom{n - K_M + 1}{M + 2 - q} + A_q\left(K_i - 2\right)\binom{n - K_M}{M + 1 - q}\right\}$$

$$= \sum_{q=0}^{M} A_q\left(M + 2 - q\right)\binom{N - K_M + 2}{M + 3 - q} + \sum_{q=0}^{M} A_q\left(K_i - 2\right)\binom{N - K_M + 1}{M + 2 - q}$$

$$= \sum_{q=0}^{M} A_q\left(M + 2 - q\right)\binom{N - K_M + 1}{M + 3 - q}$$

$$+ \sum_{q=0}^{M}\left\{A_q\left(M + 2 - q\right) + A_q\left(K_i - 2\right)\right\}\binom{N - K_M + 1}{M + 2 - q}$$

$$= \sum_{q=0}^{M} A_q\left(M + 2 - q\right)\binom{N - K_M + 1}{M + 3 - q}$$

$$+ \sum_{q=0}^{M} A_q\left(K_i + \left(M - q\right)\right)\binom{N - K_M + 1}{M + 3 - \left(q + 1\right)}$$

Let $PS2 = \left[1, K_1, \cdots, K_M, K_{M+1} = K_i\right]$,

$PT2 = \left[1, T_1, \cdots, T_M, T_{M+1}\right] = \left[1, 2, \cdots, M + 1, M + 2\right]$

$IDX\left(PT2\right) = M + 3$, $N - PH\left(PS2\right) - PCHG\left(PS2, PT2\right) = N - K_M + 1$

$q =$ count of $\left\{X_1, \cdots, X_M\right\} \in K$

$A_q\left(M + 2 - q\right) = A_q\left(T_{M+1} - q\right)$ means $X_{M+1} = T_{M+1}$, $A_q\left(K_i + \left(M - q\right)\right)$ means $X_{M+1} = K_{M+1}$

It's match the form $\left(T_1 + K_1\right)\left(T_2 + K_2\right)\cdots\left(T_M + K_M\right)\left(T_{M+1} + K_i\right)$

Recursion →

5.1) $K_i < K_j$, $K_i = K_j$, $K_i > K_j$ **are allowed,**

$SUM\_SUBSET\left(N, PS, PT1\right)$ **can use the form** $\left(T_1 + K_1\right)\left(T_2 + K_2\right)\cdots\left(T_M + K_M\right)$

Example:

The form = $\left(2 + 3\right)\left(3 + 5\right)\left(4 + 3\right)$ →

$SUM\_SUBSET\left(N, \left[1, 3, 5, 3\right], \left[1, 2, 3, 4\right]\right)$

$$= 24\binom{N - 4}{5} + 84\binom{N - 4}{4} + 102\binom{N - 4}{3} + 45\binom{N - 4}{2}$$

Among:

$45 = 3\times5\times3$

$102 = 3\times5\times(4-2)+3\times(3-1)\times(3+1)+2\times(5+1)\times(3+1)$

$84 = 2\times3\times(3+2)+2\times(5+1)\times(4-1)+3\times(3-1)\times(4-1)$

$$SUM\_SUBSET(9,[1,3,5,3],[1,2,3,4])$$
$$=1\times3\times3\times5+2\times4\times4\times6+3\times5\times5\times7+4\times6\times6\times8=1914$$
$$=24\binom{5}{5}+84\binom{5}{4}+102\binom{5}{3}+45\binom{5}{2}$$

The form = $(2+4)(3+7)(4+7) \rightarrow$

$$SUM\_SUBSET(N,[1,4,7,7],[1,2,3,4])$$
$$=24\binom{N-6}{5}+126\binom{N-6}{4}+248\binom{N-6}{3}+196\binom{N-6}{2}$$

Among:

$196 = 4\times7\times7$

$248 = 4\times7\times(4-2)+4\times(3-1)\times(7+1)+2\times(7+1)\times(7+1)$

$126 = 2\times3\times(7+2)+2\times(7+1)\times(4-1)+4\times(3-1)\times(4-1)$

$$SUM\_SUBSET(12,[1,4,7,7],[1,2,3,4])$$
$$=1\times4\times7\times7+2\times5\times8\times8+3\times6\times9\times9+4\times7\times10\times10+5\times8\times11\times11$$
$$=9934=24\binom{6}{5}+126\binom{6}{4}+248\binom{6}{3}+196\binom{6}{2}$$

In particular:

5.2) $\sum_{n=1}^{M} n^M = SUM\_SUBSET(N+1,[1,1,\cdots,1],[1,2,\cdots,M])$

Example:

$$(N+1)-PH(PS)-PCHG(PS,PT1)=(N+1)-0-0=N+1$$

The form = $(2+1)$

$$\rightarrow\ 1^2+2^2+\cdots+N^2 = 2\binom{N+1}{3}+\binom{N+1}{2}=\frac{N(N+1)(2N+1)}{6}$$

The form = $(2+1)(3+1)$

$$\rightarrow\ 1^3+2^3+\cdots+N^3 = 6\binom{N+1}{4}+6\binom{N+1}{3}+\binom{N+1}{2}=\frac{N^2(N+1)^2}{4}$$

The form = $(2+1)(3+1)(4+1)$

$$\rightarrow\ 1^4+2^4+\cdots+N^4 = 24\binom{N+1}{5}+36\binom{N+1}{4}+14\binom{N+1}{3}+\binom{N+1}{2}$$

Among:

$$14 = 2\times(1+1)\times(1+1)+1\times1\times(4-2)+1\times(3-1)\times(1+1)$$
$$36 = 2\times3\times(1+2)+2\times(1+1)\times(4-1)+1\times(3-1)\times(4-1)$$

$S(M, K)$ is Stirling number of the second kind, Definition of $S(M, K) \to \sum_{n=1}^{N} n^{M} = \sum_{K=1}^{M} K!S(M,K)\binom{N+1}{K+1}$

It's equal to 5.2), so we have a way to calculate $S(M, K)$.

3.4) can be seen as

$$\sum_{n=1}^{N}(n+1)n^{M} = SUM\_SUBSET(N+1,[1,0,\cdots,0],[1,2,\cdots,M+1])$$

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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