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Operator Matrices on Banach Spaces

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Abstract

Since nonlinear schur theorem was proposed, it broke the limitation of linear operator matrices. And in this paper we study the summability theory for a class of matrices of nonlinear mapping, and the characterizations of a class of infinite matrix transformations are obtained. These results enrich the results on infinite matrices transformations, and have important meaning for the study of Banach space.

Subject Areas

Functional Analysis

Keywords

Infinite Matrix, Matrix Transformation, Banach Space

1. Introduction

A decisive break in the theory of matrix transformations was in 1950, when Robinson considered the action of infinite matrices of linear operators from a Banach space on sequences of elements of that space [1]. In the past years, many remarkable results [2] [3] [4] were yielded in this direction.

Let X and Y be topological vector spaces, and $\mathcal{F}_0(X,Y)=\{f\in Y^X : f(0)=0\}$. For sequence families $\lambda(X)\subseteq X^{\mathbb{N}}$ and $\mu(Y)\subseteq Y^{\mathbb{N}}$, the matrix $(T_{i,j})\in(\lambda(X),\mu(Y))$ means that $\sum_{j=1}^{\infty}T_{ij}(x_j)$ converges when $(x_j)\in\lambda(X)$, $i\in\mathbb{N}$ and $\left\{\sum_{j=1}^{\infty}T_{ij}(x_j)\right\}_{i=1}^{\infty}\in\mu(Y)$ for each $(x_j)\in\lambda(X)$.

As usual,

$$c_0=\{(x_j)\subseteq\mathbb{C}: x_j\to 0\},$$
$$c_0(X)=\{(x_j)\in X^{\mathbb{N}}: x_j\to 0\},$$

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$$c(X)=\left\{(x_j)\in X^{\mathbb{N}}:\lim x_j \text{ exists}\right\} \text{ and}$$

$$l^{\infty}(X)=\left\{(x_j)\in X^{\mathbb{N}}:x_j \text{ is bounded}\right\}.$$

In 2001, Li Ronglu depicted the nonlinear operator matrices transformation with some restrictive condition on topological vector spaces [5]. In the next year, Li Ronglu gave some clear-cut characterizations of the matrix families $(c_0(X),l^{\infty}(I,Y))$ and $(l^{\infty}(X),l^{\infty}(I,Y))$ consisted of matrices of linear and some nonlinear operators between topological vector spaces [6]. In this paper, we study the summability theory for a class of matrices of nonlinear mapping on Banach space, and discuss the characterization of the matrix classes:

$$(l^{\infty}(X),c(Y)),\ (l^{\infty}(X),l^{\infty}(Y)),\ (c_0(X),c_0(Y))\ \ (c_0(X),c_0(Y)).$$

All of the researches enrich the results on infinite matrices transformations, and have important meaning for the study of Banach space.

2. Preliminaries and Lemmas

In 1993, nonlinear Schur Theorem was given by Li Ronglu and C. Swartz, and broke the limitations of linear operator matrices.

Theorem A. [7] Let G be an Abelian topological group, $\Omega\neq\varnothing$, $(f_{ij})_{i,j\in\mathbb{N}}$ a matrix in G^{Ω} such that $f_{ij}(w_0)=0$ for some $w_0\in\Omega$ and all $i,j\in\mathbb{N}$. If $(f_{ij})_{i,j\in\mathbb{N}}\in(\Omega^{\mathbb{N}},c(G))$ *i.e.*, $\lim_i\sum_{j=1}^{\infty}f_{ij}(w_j)$ exists for each $\{w_j\}\subseteq\Omega$, then the series $\sum_{j=1}^{\infty}f_{ij}(w_j)$ converges uniformly with respect to both $i\in\mathbb{N}$ and $\{w_j\}\subseteq\Omega$, and $\lim_i f_{ij}(w)$ exists for every $w\in\Omega, j\in\mathbb{N}$. If, in addition, G is sequentially complete, then the converse implication is true.

As a special case, the following theorem is a nice result for the matrix family $(l^{\infty}(X),c(Y))$.

Theorem B. [8] Let X,Y be topological vector spaces and $T_{ij}:X\to Y$ a mapping such that $T_{ij}(0)=0$ for every $i,j\in\mathbb{N}$. If $(T_{ij})_{i,j\in\mathbb{N}}\in(l^{\infty}(X),c(Y))$, then for every bounded $B\subseteq X$, the series $\sum_{j=1}^{\infty}T_{ij}(x_j)$ converges uniformly with respect to both $\{x_j\}\subseteq B$ and $i\in\mathbb{N}$ and $\lim_i T_{ij}(x)$ exists for every $x\in X, j\in\mathbb{N}$. If, in addition, Y is sequentially complete, then the converse implication is true.

Note that theorem B exceeded the restriction of linear operators, and a characterization of $(T_{ij})_{i,j\in\mathbb{N}}\in(l^{\infty}(X),c(Y))$ was given. For Banach spaces X,Y, it is useful to discuss the characterization of a variety of matrix families, where the mapping need not be linear.

As preparation of the proves of the main results, we also need following lemma.

Lemma [9] $(x_j)\in l^\infty(X)$ if and only if $(t_j x_j)\in c_0(X)$ for all $(t_j)\in c_0$.

3. Main Results

Unless otherwise noted X,Y below are Banach spaces, and the mapping we studied in this section need not be linear.

Theorem 1. Let $T_{ij}\in\mathcal{F}_0(X,Y)$ for all $i,j\in\mathbb{N}$, then $(T_{ij})_{i,j\in\mathbb{N}}\in(l^\infty(X),c(Y))$ if and only if

(i) $\lim_i T_{ij}(x)$ exists for all $j\in\mathbb{N}$ and $x\in X$;

(ii) For any $\varepsilon>0$, $M>0$ there exists $m_0\in N$ such that $\left\|\sum_{j=m}^{\infty}T_{ij}(x_j)\right\|<\varepsilon$ for all natural number $m>m_0$, $i\in\mathbb{N}$, and $\{x_j\}\subseteq X$ with $\sup_j\|x_j\|\le M$.

Proof. Necessity of condition (i) and (ii) is easy to prove by the theorem B in Introduction.

Now suppose that (i) and (ii) are hold, and $(x_j)\in l^\infty(X)$, then for any $\varepsilon>0$, there exists $m_0\in\mathbb{N}$ such that $\left\|\sum_{j=m_0+1}^{\infty}T_{ij}(x_j)\right\|<\frac{\varepsilon}{3}$ for all $i\in\mathbb{N}$ by the condition (ii). And because of condition (i) there is $i_0\in\mathbb{N}$, such that $\left\|T_{kj}(x_j)-T_{ij}(x_j)\right\|<\frac{\varepsilon}{3m_0}$ for all $k,i>i_0$. Hence we have

$$\left\|\sum_{j=1}^{m_0}T_{kj}(x_j)-\sum_{j=1}^{m_0}T_{ij}(x_j)\right\|\le\sum_{j=1}^{m_0}\left\|T_{kj}(x_j)-T_{ij}(x_j)\right\|<\frac{\varepsilon}{3} \qquad (1)$$

for all $k,i>i_0$. Therefore

$$\begin{aligned}&\left\|\sum_{j=1}^{\infty}T_{kj}(x_j)-\sum_{j=1}^{\infty}T_{ij}(x_j)\right\|\\&=\left\|\sum_{j=1}^{m_0}T_{kj}(x_j)-\sum_{j=1}^{m_0}T_{ij}(x_j)+\sum_{j=m_0+1}^{\infty}T_{kj}(x_j)-\sum_{j=m_0+1}^{\infty}T_{ij}(x_j)\right\|\\&\le\left\|\sum_{j=1}^{m_0}T_{kj}(x_j)-\sum_{j=1}^{m_0}T_{ij}(x_j)\right\|+\left\|\sum_{j=m_0+1}^{\infty}T_{kj}(x_j)\right\|+\left\|\sum_{j=m_0+1}^{\infty}T_{ij}(x_j)\right\|\\&<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon\end{aligned}$$

So $\left\{\sum_{j=1}^{\infty}T_{ij}(x_j)\right\}_{i=1}^{\infty}$ is a Cauchy sequence in *Y*. Therefore $\left\{\sum_{j=1}^{\infty}T_{ij}(x_j)\right\}_{i=1}^{\infty}$ converges by the completeness of *Y*, and then $(T_{ij})_{i,j\in\mathbb{N}}\in(l^\infty(X),c(Y))$. The sufficiency is proved. Q.E.D.

Since $c_0(Y)\subseteq c(Y)$, we can get the next corollary by the theorem.

Corollary 1. Suppose that $i,j\in\mathbb{N},T_{ij}\in\mathcal{F}_0(X,Y)$, then $(T_{ij})_{i,j\in\mathbb{N}}\in(l^\infty(X),c_0(Y))$ if and only if $\lim_i T_{ij}(x)=0$ for all $j\in\mathbb{N},x\in X$, and for any $\varepsilon>0,M>0$, there exists $m_0\in\mathbb{N}$ such that $\left\|\sum_{j=m}^{\infty}T_{ij}(x_j)\right\|\le\varepsilon$ for all

$m > m_0, i \in \mathbb{N}$, and $\{x_j\} \subseteq X$ with $\sup_j \|x_j\| \le M$.

Proof. Necessity is clear by above theorem and the definition of $c_0(Y)$.

Conversely, let $(x_j) \in l^\infty(X)$, then for any $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that $\left\| \sum_{j=m_0+1}^{\infty} T_{ij}(x_j) \right\| < \frac{\varepsilon}{2}$ for all $i \in \mathbb{N}$. Since $\lim_i T_{ij}(x) = 0$ for all $j \in \mathbb{N}$ and $x \in X$, there is $i_0 \in \mathbb{N}$, such that $\|T_{ij}(x_j)\| < \frac{\varepsilon}{2m_0}$ for all $i > i_0$ and $j \in \mathbb{N}$.

Hence we have

$$\left\| \sum_{j=1}^{\infty} T_{ij}(x_j) - 0 \right\| \le \left\| \sum_{j=1}^{m_0} T_{ij}(x_j) \right\| + \left\| \sum_{j=m_0+1}^{\infty} T_{ij}(x_j) \right\| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad (2)$$

So column $\left\{ \sum_{j=1}^{\infty} T_{ij}(x_j) \right\}_{i=1}^{\infty}$ converges to 0, and then $(T_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), c_0(Y))$. The sufficiency is proved. Q.E.D.

Theorem 2. Let $T_{ij} \in \mathcal{F}_0(X,Y)$ with respect to $i, j \in \mathbb{N}$, then $(T_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), l^\infty(Y))$ if and only if

(i) $\sup_i \|T_{ij}(x)\| < +\infty$ for all $j \in \mathbb{N}, x \in X$;

(ii) For any $\varepsilon > 0, M > 0$ and $t_i \in c_0$, there exists $m_0 \in \mathbb{N}$ such that $|t_i| \left\| \sum_{j=m}^{\infty} T_{ij}(x_j) \right\| < \varepsilon$ for all $m > m_0, i \in \mathbb{N}$ and $\{x_j\} \subseteq X$ with $\sup_j \|x_j\| \le M$.

Proof. ⇒) Suppose that $T_{ij} \in \mathcal{F}_0(X,Y)$, the condition (i) is clear.

Since $\left\{ \sum_{j=1}^{\infty} T_{ij}(x_j) \right\} \in l^\infty(Y)$ for every $(x_j) \in l^\infty(X)$, $\left\{ t_i \sum_{j=1}^{\infty} T_{ij}(x_j) \right\} \in c_0(Y)$ for every $(t_i) \in c_0$ by lemma 1, that is $\sum_{j=1}^{\infty} (t_i T_{ij})(x_j) \in c_0(Y)$. Hence, for every $(t_i) \in c_0$, we have $(t_i T_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), c_0(Y))$. Therefore, by above corollary, for every $\varepsilon > 0, M > 0$ and $(t_i) \in c_0$ there is $m_0 \in \mathbb{N}$ such that for all $m > m_0$, $i \in \mathbb{N}$, and $\sup_j \|x_j\| \le M$, we have

$$|t_i| \left\| \sum_{j=m}^{\infty} T_{ij}(x_j) \right\| = \left\| \sum_{j=m}^{\infty} t_i T_{ij}(x_j) \right\| < \varepsilon. \quad (3)$$

condition (ii) is proved.

⇐) For every $j \in \mathbb{N}, x \in X$, and $(t_i) \in c_0$, we have $\lim_i t_i T_{ij}(x) = 0$ by the condition (i). Because of the condition (ii), we have $(t_i T_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), c_0(Y))$ by the Corollary 1, and then for $(x_j) \in l^\infty(X)$, we have $\left\{ t_i \sum_{j=1}^{\infty} T_{ij}(x_j) \right\}_{i=1}^{\infty} \in c_0(Y)$ is hold for every $(t_i) \in c_0$. Therefore $\left\{ \sum_{j=1}^{\infty} T_{ij}(x_j) \right\}_{i=1}^{\infty} \in l^\infty(Y)$ by lemma 1, and then $(T_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), l^\infty(Y))$. Q.E.D.

Theorem 3. Let $T_{ij} \in \mathcal{F}_0(X,Y)$ with respect to $i,j\in\mathbb{N}$, then $(T_{ij})_{i,j\in\mathbb{N}} \in (c_0(X), c_0(Y))$ if and only if

(i) $\lim_i T_{ij}(x) = 0$ for all $j\in\mathbb{N}$ and $x\in X$;

(ii) For any $\varepsilon>0, M>0$ and $(t_j)\in c_0$, there exists $m_0\in\mathbb{N}$ such that

$$\sup_{i\in\mathbb{N}, m>m_0, \|x_j\|\le M} \left\| \sum_{j=m}^{\infty} T_{ij}(t_j x_j) \right\| < \varepsilon \tag{4}$$

Proof. For $T_{ij}\in\mathcal{F}_0(X,Y)$, since $T_{ij}\circ t_j(x) = T_{ij}(t_j x)$, $\lim_i T_{ij}\circ t_j(x) = \lim_i T_{ij}(t_j x) = 0$ for any $j\in\mathbb{N}, x\in X$ and $t_j\in c_0$, by condition (i). So condition (i) and (ii) is equivalent to $(T_{ij}\circ t_j)_{i,j\in\mathbb{N}} \in (l^\infty(X), c_0(Y))$ by corollary 1.

Suppose that $(T_{ij})_{i,j\in\mathbb{N}} \in (c_0(X), c_0(Y))$, then $\left\{\sum_{j=1}^{\infty} T_{ij}(x_j)\right\}_{i=1}^{\infty} \in c_0(Y)$ for all $(x_j)\in c_0(X)$. By lemma 1, $\left\{\sum_{j=1}^{\infty} T_{ij}(t_j x_j)\right\}_{i=1}^{\infty} \in c_0(Y)$, for all $(x_j)\in l^\infty(X)$ and $(t_j)\in c_0$. Hence $(T_{ij}\circ t_j)_{i,j\in\mathbb{N}} \in (l^\infty(X), c_0(Y))$.

On the other hand, suppose that $(T_{ij}\circ t_j)_{i,j\in\mathbb{N}} \in (l^\infty(X), c_0(Y))$. For every $(x_j)\in c_0(X)$, there exists $(t_j)\in c_0$ and $(z_j)\in c_0(X)\subset l^\infty(X)$, such that $(x_j) = (t_j z_j)$, and so $\left\{\sum_{j=1}^{\infty} T_{ij}(x_j)\right\}_{i=1}^{\infty} = \left\{\sum_{j=1}^{\infty} (T_{ij}\circ t_j)(z_j)\right\}_{i=1}^{\infty} \in c_0(Y)$. Hence $(T_{ij})_{i,j\in\mathbb{N}} \in (c_0(X), c_0(Y))$. Q.E.D.

Theorem 4. Let $T_{ij}\in\mathcal{F}_0(X,Y)$ with respect to $i,j\in\mathbb{N}$, then $(T_{ij})_{i,j\in\mathbb{N}} \in (c_0(X), l^\infty(Y))$ if and only if

(i) $\sup_i \|T_{ij}(x)\| < +\infty$ for all $j\in\mathbb{N}$ and $x\in X$;

(ii) For any $\varepsilon>0, M>0$ and $(s_j),(t_j)\in c_0$, there exists $m_0\in\mathbb{N}$, such that $\left\| s_j \sum_{j=m}^{\infty} T_{ij}(t_j x_j) \right\| < \varepsilon$ for all $m>m_0, i\in\mathbb{N}$ and $\sup_j \|x_j\| \le M$.

Proof. By condition (i), for all $(t_j)\in c_0$ and $x\in X$,

$$\sup_i \|T_{ij}\circ t_j(x)\| = \sup_i \|T_{ij}(t_j(x))\| < +\infty. \tag{5}$$

By theorem 2, condition (i) and (ii) are equivalent to $(T_{ij}\circ t_j)_{i,j\in\mathbb{N}} \in (l^\infty(X), l^\infty(Y))$. Next, we prove that $(T_{ij}\circ t_j)_{i,j\in\mathbb{N}} \in (l^\infty(X), l^\infty(Y))$ for all $(t_j)\in c_0$ is equivalent to $(T_{ij})_{i,j\in\mathbb{N}} \in (c_0(X), l^\infty(Y))$.

In fact, If $(T_{ij})_{i,j\in\mathbb{N}} \in (c_0(X), l^\infty(Y))$, and let $(x_j)\in l^\infty(X)$, then $\sum_{j=1}^{\infty}(T_{ij}\circ t_j)(x_j) = \sum_{j=1}^{\infty} T_{ij}(t_j x_j)$. Since $(t_j x_j)\in c_0(X)$ for all $(t_j)\in c_0$ by lemma, we have $\left\{\sum_{j=1}^{\infty} T_{ij}\left(t_j x_j\right)\right\}_{i=1}^{\infty} \in l^{\infty}(Y)$. So $\left(T_{ij} \circ t_j\right)_{i,j\in\mathbb{N}} \in \left(l^{\infty}(X), l^{\infty}(Y)\right)$. On the other hand, suppose that $\left(T_{ij} \circ t_j\right)_{i,j\in\mathbb{N}} \in \left(l^{\infty}(X), l^{\infty}(Y)\right)$. Since for any $\left(x_j\right) \in c_0(X)$, there must be $\left(t_j\right) \in c_0$, $\left(z_j\right) \in c_0(X) \subset l^{\infty}(X)$, such that $\left(x_j\right) = \left(t_j z_j\right)$, we have

$$\left\{\sum_{j=1}^{\infty} T_{ij}\left(x_j\right)\right\}_{i=1}^{\infty} = \left\{\sum_{j=1}^{\infty} T_{ij}\left(t_j z_j\right)\right\}_{i=1}^{\infty} = \left\{\sum_{j=1}^{\infty} T_{ij} \circ t_j\left(z_j\right)\right\}_{i=1}^{\infty}, \quad (6)$$

and then $\left(T_{ij}\right)_{i,j\in\mathbb{N}} \in \left(c_0(X), l^{\infty}(Y)\right)$. Q.E.D.

4. Result

In this paper, we first review the research history of infinite matrix transformation, and then we mainly study the summability of a class of nonlinear mapping matrices in Banach space.

And some new results about, matrix transformation theorems are obtained: we characterize the matrix classes such as $\left(l^{\infty}(X), c(Y)\right)$, $\left(l^{\infty}(X), l^{\infty}(Y)\right)$, $\left(c_0(X), c_0(Y)\right)$, $\left(c_0(X), l^{\infty}(Y)\right)$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Robinson, A. (1985) On Functional Transformations and Summability. *Proceedings of the London Mathematical Society*, **52**, 132-160. https://doi.org/10.1112/plms/s2-52.2.132

[2] Jeribi, A. (2015) Spectral Theory and Applications of Linear Operators and Block Operator Matrices. Springer International Publishing Switzerland. https://doi.org/10.1007/978-3-319-17566-9

[3] Bani-Domi, W. and Kittaneh, F. (2008) Norm Equalities and Inequalities for Operator Matrices. *Linear Algebra and Its Applications*, **429**, 57-67. https://doi.org/10.1016/j.laa.2008.02.004

[4] Bani-Ahmad, F.A. and Bani-Domi, W. (2016) New Norm Equalities and Inequalities for Operator Matrices. *Journal of Inequalities and Applications*, **2016**, Article number: 175. https://doi.org/10.1186/s13660-016-1108-y

[5] Li, R.L., Li, L.S. and Shin, M.K. (2001) Summability Results for Operator Matrices on Topological Vector Spaces. *Science in China*, **44**, 1300-1311. https://doi.org/10.1007/BF02877019

[6] Li, R.L., Kang, S.M. and Swartz, C. (2002) Operator Matrices on Topological Vector

Spaces. *Journal of Mathematical Analysis and Applications*, **274**, 645-658. https://doi.org/10.1016/S0022-247X(02)00322-0

[7] Li, R.L. and Swartz, C. (1993) A Nonlinear Schur Theorem. *Acta Scientiarum Mathematicarum*, **58**, 497-508.

[8] Maddox, I.J. (1980) Infinite Matrices of Operators. In: *Lecture Notes in Math.*, Vol. 786, Springer-Verlag, Berlin. https://doi.org/10.1007/BFb0088196

[9] Swartz, C. (1978) Applications of the Mikusinski Diagonal Theorem. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys*, **26**, 421-424.