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Multiple Solutions for Nonhomogeneous Kirchoff-Type Problem with Hardy-Sobolev Critical Exponent

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Abstract

In this work, we show the existence of multiple solutions for nonhomogeneous Kirchoff-type problem with Hardy-Sobolev critical exponent, by using Ekeland's variational principle and mountain pass theorem without Palais-Smale conditions.

Subject Areas

Mathematical Analysis

Keywords

Kirchoff-Type Problem, Hardy-Sobolev Critical Exponent, Ekeland's Variational Principle, Mountain Pass Theorem

1. Introduction

This paper deals with the existence and multiplicity of solutions to the following problem

$$(\mathcal{P}_\lambda)\begin{cases} -\left(a - b\int_\Omega |\nabla u|^2\,dx\right)\Delta u = h(x)\frac{|u|^3}{|x|} + \lambda g(x), & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases}$$

where $\Omega \in \mathbb{R}^3$, $a, b, \lambda > 0$, $-\infty < a < 1/2, a \leq b < a + 1/4$, g is a continuous function on $\mathbb{R}^3$ and h is a bounded positive function on $\mathbb{R}^3$.

By $\mathcal{H} = \mathcal{H}_0^1(\Omega)$, we denote the closure of $C_0^\infty(\Omega)$ with respect to the norms

$$\|u\| = \left(\int_\Omega |\nabla u|^2\right)^{1/2}$$

The original one-dimensional Kirchhoff equation was introduced by Kirchhoff [1] in 1883. His model takes into account the changes in length of the strings produced by transverse vibrations.

In recent years, the existence and multiplicity of solutions to the nonlocal problem

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^{2}dx\right)\Delta u=g(x;u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

has been studied by various researchers and many interesting and important results can be found. For instance, positive solutions could be obtained in [2–4]. Especially, Chen *et al.* [5] discussed a Kirchhoff type problem when $g(x;u)=f(x)u^{p-2}u+\lambda g(x)|u|^{q-2}u$, where $1<q<2<p<2^{*}=2N/(N-2)$ if $N\geq 3$, $2^{*}=\infty$ if $N=1,2$, $f(x)$ and $g(x)$ with some proper conditions are sign-changing weight functions. And they have obtained the existence of two positive solutions if $p>4, 0<\lambda<\lambda_0(a)$.

Researchers, such as Mao and Zhang [6], Mao and Luan [7], found sign-changing solutions. As for in nitely many solutions, we refer readers to [8,9]. He and Zou [10] considered the class of Kirchhoff type problem when $g(x;u)=\lambda f(x;u)$ with some conditions and proved a sequence of a.e. positive weak solutions tending to zero in $L^{\infty}(\Omega)$.

In the case of a bounded domain of $\mathbb{R}^N$ with $N\geq 3$, Tarantello 8 proved, under a suitable condition on f, the existence of at least two solutions to (1.2) for $a=0, b=1$ and $g(x;u)=|u|^{\frac{4}{N-2}}u+f$.

Since our approach is variational, we define the functional I_λ on $\mathcal{H}$ by

$$I_\lambda(u):=(a/2)\|u\|^{2}-(a/4)\|u\|^{4}-(1/4)\int_{\Omega}h(x)|x|^{-1}|u|^{4}dx \\ -\lambda\int_{\Omega}g(x)udx.$$

We say that $u\in\mathcal{H}$ is a weak solution of the problem $(\mathcal{P}_\lambda)$ if it is a nontrivial nonnegative function and satisfies

$$\left(a-b\|u\|^{2}\right)\int_{\Omega}(\nabla u\nabla v)dx-\int_{\Omega}h(x)|x|^{-1}|u|^{3}uvdx \\ -\lambda\int_{\Omega}g(x)vdx=0,\ \text{for } v\in\mathcal{D}.$$

Throughout this work, we consider the following assumptions

$$\begin{array}{lccl} (G) & g & \in & \mathcal{H}' \text{ (dual of } \mathcal{H}),\\ (H) & \lim\limits_{|x|\longrightarrow 0}h(x) & = & \lim\limits_{|x|\longrightarrow\infty}h(x)=h_0>0,\ h(x)\geq h_0,\ x\in\ \Omega. \end{array}$$

In our work, we prove the existence of at least two distinct critical points of I_λ.

Our main result is given as follows

Theorem 1. *Sppose that $a,b>0$, (H) holds, $0\neq g\in\mathcal{H}'\cap C(\Omega)$. Then there exists $\lambda_*>0$ such that $(\mathcal{P}_\lambda)$ has at least two nontrivial solutions for any $\lambda\in(0,\lambda_*)$.*

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the proof of Theorem 1.

2. Preliminaries

Definition 1. *An entire solution v to $(\mathcal{P}_{\lambda,\mu})$ is a ground state solution if it achieves the best constant*

$$A := \inf_{u\in\mathcal{H}(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\left(\int_\Omega |x|^{-1}|u|^4\,dx\right)^{1/2}}, \tag{2.1}$$

for $k=1$,

$$S := \inf_{u\in\mathcal{H}(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\left(\int_\Omega |u|^6\,dx\right)^{1/3}}.$$

Lemma 1. *Let $(u_n)\subset\mathcal{H}$ be a Palais-Smale sequence $[(PS)_c$ in short$]$ of I_λ, i.e.,*

$$I_\lambda(u_n)\longrightarrow c \text{ and } I_\lambda^{'}(u_n)\longrightarrow c \text{ in } \mathcal{H}^{'} \text{ (dual of } \mathcal{H}) \text{ as } n\longrightarrow\infty, \tag{2.2}$$

for some $c\in\mathbb{R}$. Then, $u_n \rightharpoonup u$ in $\mathcal{H}$ and $I_\lambda^{'}(u)=0$.

Proof. From (2.2) we have

$$(a/2)\|u_n\|^2-(b/4)\|u_n\|^4-(1/4)\int_\Omega h(x)|x|^{-1}|u_n|^4\,dx$$
$$-\lambda\int_\Omega g(x)u_n dx = c+o_n(1)$$

and

$$a\|u_n\|^2-b\|u_n\|^4-\int_\Omega h(x)|x|^{-1}|u_n|^4\,dx$$
$$-\lambda\int_\Omega g(x)u_n dx = o_n(1), \text{ for } n \text{ large},$$

where $o_n(1)$ denotes $o_n(1)\longrightarrow 0$ as $n\longrightarrow\infty$. Then,

$$\begin{aligned} c+o_n(1) &= I_\lambda(u_n)-(1/4)\left\langle I_\lambda^{'}(u_n),u_n\right\rangle \\ &\geq (a/4)\|u_n\|^2-(3/4)\lambda\|g\|_{\mathcal{H}'}\|u_n\|, \end{aligned}$$

(u_n) is bounded in $\mathcal{H}$. Up to a subsequence if necessary, we obtain that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } \mathcal{H} \\ u_n &\longrightarrow u \text{ in } L_p(\Omega), 1\leq p<6 \\ |x|^{-1}|u_n|^4 &\rightharpoonup |x|^{-1}|u|^4 \text{ in } L^1(\Omega) \\ u_n &\longrightarrow u \text{ a.e in } \Omega. \end{aligned}$$

Consequently, we get for all $v\in C_0^\infty(\Omega)$,

$$\left(a-b\|u\|^2\right)\int_\Omega(\nabla u\nabla v)\,dx-\int_\Omega h(x)|x|^{-1}|u|^3uvdx$$
$$-\lambda\int_\Omega g(x)vdx=0, \text{ for } v\in\mathcal{D}.$$

which means that

$$I_\lambda^{'}(u)=0.$$

□

Lemma 2. *Let $(u_n) \subset \mathcal{H}$ be a Palais-Smale sequence $[(PS)_c$ in short$]$ of I_λ for some $c \in \mathbb{R}$. Then, $u_n \rightharpoonup u$ in $\mathcal{H}$ and either*

$$u_n \longrightarrow u \text{ or } I_\lambda(u) \le c.$$

Proof. We know that (u_n) is bounded in $\mathcal{H}$. Up to a subsequence if necessary, we have that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } \mathcal{H} \\ u_n &\longrightarrow u \text{ a.e in } \Omega. \end{aligned}$$

Denote $v_n = u_n - u$, then $v_n \rightharpoonup 0$. As in Brézis and Lieb [5], we have

$$\|v_n\|^2 = \|u_n\|^2 - \|u\|^2$$

and

$$\lim_{n\longrightarrow\infty} \int_\Omega h(x)\left(|x|^{-1}|u_n|^2 - |x|^{-1}|u_n-u|^2\right)dx$$
$$= \int_\Omega h(x)|x|^{-1}|u|^2\,dx.$$

On the other hand, by using the assumption (H), we obtain

$$\lim_{n\longrightarrow\infty}\int_\Omega h(x)|x|^{-1}|v_n|^2\,dx = h_0 \lim_{n\longrightarrow\infty}\int_\Omega |x|^{-1}|v_n|^2\,dx.$$

Then, we get

$$I_\lambda(u_n) = I_\lambda(u) + (1/2)\|v_n\|^2 - (h_0)\int_\Omega |x|^{-1}|v_n|^2\,dx + o_n(1)$$

and

$$\left\langle I'_\lambda(u_n), u_n\right\rangle = \|v_n\|^2 - h_0\int_\Omega |x|^{-1}|v_n|^2\,dx + o_n(1).$$

Then we can assume that

$$\lim_{n\longrightarrow\infty}\|v_n\|^2 = h_0\lim_{n\longrightarrow\infty}\int_\Omega |x|^{-1}|v_n|^2\,dx = l \ge 0.$$

Assume $l > 0$, we have by definition of S

$$l \ge S\left(lh_0^{-1}\right),$$

Thus we get

$$c = I_\lambda(u).$$

□

3. Proof of Theorem 1

The proof of Theorem 1 is given in two parts.

3.1. Existence of a Local Minimizer

We prove that there exists $\lambda_* > 0$ such that for any $\lambda \in (0, \lambda_*)$, I_λ can achieve a local minimizer.

First, we establish the following result.

Proposition 1. *Sppose that* $a, b > 0$, (H) *holds,* $0 \neq g \in \mathcal{H}' \cap C(\Omega)$. *Then there exists* $\lambda_*, \varrho, \delta > 0$ *such that for all* $\lambda \in (0, \lambda_*)$*:* $I_\lambda(u) \geq \delta > 0$ *for* $\|u\| = \varrho$.

Proof. By the Holder inequality and the definition of S, we get for all $u \in \mathcal{H} \backslash \{0\}$ and $\varepsilon > 0$

$$\begin{aligned} I_\lambda(u) &:= (a/2)\|u\|^2 - (b/4)\|u\|^2 - (1/4)\int_\Omega h(x)|x|^{-1}|u|^4 dx \\ &\quad - \lambda \int_\Omega g(x) u dx, \\ &\geq (a/2 - \varepsilon)\|u\|^2 - (b/4)\|u\|^2 - (|h|_\infty/4) S\|u\|^4 - C_\varepsilon \|\lambda g\|_{\mathcal{H}'_\mu}. \end{aligned}$$

Taking $\varepsilon < a/2$ and $\varrho = \|u\|$, then there exist $\varrho_0 > 0$ small enough and a positive constant λ_* such that

$$I_\lambda(u) \geq \delta > 0 \text{ for } \|u\| = \varrho_0 \text{ and } \lambda \in (0, \lambda_*). \tag{3.1}$$

Since g is a continuous function on Ω, not identically zero, we can choose $\phi \in C_0^\infty(\Omega \backslash \{0\})$ such that $\int_\Omega g(x)\phi dx > 0$. It follows that for $t > 0$ small,

$$\begin{aligned} I_\lambda(t\phi) &:= \left(at^2/2\right)\|\phi\|^2 - \left(bt^4/4\right)\|\phi\|^2 \\ &\quad - \left(t^4/4\right)\int_\Omega h(x)|x|^{-1}|\phi|^4 dx + \quad (3.2) \\ &\quad - \lambda t \int_\Omega g(x)\phi dx < 0. \quad (3.3) \end{aligned}$$

We also assume that t is so small enough such that $\|t\phi\| < \varrho_0$. Thus, we have

$$c_1 = \inf\{I(u) : u \in B_{\varrho_0}\} < 0, \text{ where } B_{\varrho_0} = \{u \in \mathcal{H}, \ \|u\| \leq \varrho_0\}. \tag{3.4}$$

Using the Ekeland's variational principle, for the complete metric space $\overline{B}_{\rho_0}$ with respect to the norm of $\mathcal{H}$, we can prove that there exists a $(PC)_{c_1}$ sequence $(u_n) \subset \overline{B}_{\rho_0}$ such that $u_n \rightharpoonup u_1$ for some u_1 with $\|u_1\| \leq \rho_0$.

Now, we claim that $u_n \longrightarrow u_1$. If not, by Lemma 2, we have

$$c_1 \geq I_\lambda(u_1) \geq c_1$$

Then we obtain a critical point u_1 of I_λ for all $\lambda \in (0, \lambda_*)$ satisfying

$$c_1 = I_\lambda(u_1) < 0.$$

Thus u_1 is a nontrivial solution of our problem with negative energy. □

3.2. Existence of Mountain Pass Type Solution

We use the mountain pass theorem without Palais-Smale conditions to prove the existence of a nontrivial solution with positive energy. For this, we need the following Lemma.

Lemma 3. *Let* $\lambda^* > 0$ *such that* $C_\lambda^* > 0$ *for all* $\lambda \in (0, \lambda^*)$. *Then there exist* $\Lambda \in (0, \lambda^*)$ *and* $\varphi_\varepsilon \in \mathcal{H}$, $\varepsilon > 0$ *such that* $\sup_{t \geq 0} I_\lambda(t\varphi_\varepsilon) < C_\lambda^*$ *for all* $\lambda \in (0, \Lambda)$.

Proof. Let

$$\varphi_\varepsilon(x) = \begin{cases} \omega_\varepsilon(x) & \text{if } g(x) \geq 0 \text{ for all } x \in \Omega \\ \omega_\varepsilon(x - x_0) & \text{if } g(x_0) > 0 \text{ for } x_0 \in \Omega \\ -\omega_\varepsilon(x) & \text{if } g(x) \leq 0 \text{ for all } x \in \Omega \end{cases} \tag{3.5}$$

where ω_ε verifies 2.1.

Then, we claim that there is an ε_0 such that

$$\int_\Omega g(x)\varphi_\varepsilon(x) > 0, \text{ for any } \varepsilon \in (0, \varepsilon_0). \tag{3.6}$$

In fact, $g(x) \geq 0$ or $g(x) \leq 0$ for all $x \in \Omega$, and (3.6) holds obviously. If there exists an $x_0 \in \Omega$ such that $g(x_0) > 0$, by the continuity of $g(x)$ there is an $\eta > 0$ such that $g(x) > 0$ for all $x \in B_\eta(x_0)$. Then, by the definition of $\omega_\varepsilon(x - x_0)$, it is easy to see that there exists an ε_0 small enough such that

$$\int_\Omega g(x)\omega_\varepsilon(x - x_0) > 0, \text{ for any } \varepsilon \in (0, \varepsilon_0). \tag{3.7}$$

Now, we consider the following functions

$$H(t) = I_\lambda(t\varphi_\varepsilon) \text{ and } \tilde{H}(t) = H(t) + \lambda \int_\Omega g(x)\varphi_\varepsilon(x).$$

By the continuity of $H(t)$, there exists t_1 a sufficiently small positive quantity such that

$$\max_{t \geq 0} H(t) = H(t_1) > 0,$$

for all $\lambda \in (0, \lambda^*)$. On the other hand, we have

$$0 = H(0) < \lambda t_1 \int_\Omega g(x)\varphi_\varepsilon(x) = C_\lambda^*,$$

then, we obtain

$$\sup_{t \geq 0} I_\lambda(t\varphi_\varepsilon) < H(t_1) - C_\lambda^* > 0.$$

By (3.6), we get

$$0 < \lambda < H(t_1)\left[t_1 \int_\Omega g(x)\varphi_\varepsilon(x)\right] = \Lambda_{**}.$$

Set

$$\Lambda = \min\{\lambda^*, \Lambda_{**}\}.$$

We deduce that

$$\sup_{t \geq 0} I_\lambda(t\varphi_\varepsilon) < C_\lambda^*,$$

for all $\lambda \in (0, \Lambda)$.

Since $\lim_{t \longrightarrow \infty} I_\lambda(t\varphi_\varepsilon) = -\infty$, we can choose $T > 0$ large enough such that $I_\lambda(T\varphi_\varepsilon) < 0$. From Proposition 1, we have $I_{\lambda|\partial B_{\varrho_0}} \geq \delta > 0$ for all $\lambda \in (0, \lambda_*)$. By mountain pass theorem without the Palais-Smale condition, there exists a $(PC)_{c_2}$ sequence (u_n) in $\mathcal{H}$ which is characterized by

$$c_2 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)),$$

with

$$\Gamma = \{\gamma \in C([0,1], \mathcal{H}), \gamma(0) = 0, \gamma(1) = T\varphi_\varepsilon\}.$$

Then, (u_n) has a subsequence, still denoted by (u_n) such that $u_n \rightharpoonup u_2$ in $\mathcal{H}$. By Lemma 2, if u_n doesn't converge to u_2, we get

$$\begin{aligned} c_2 &\geq I_\lambda(u_2) + H(t_1) \\ &\geq C_\lambda^*, \end{aligned}$$

what contradicts the fact that, by Lemma 3, we have

$$\sup_{t\geq 0} I_\lambda(t\varphi_\varepsilon) < C_\lambda^*.$$

for all $\lambda \in (0, \Lambda)$. Then

$$u_n \longrightarrow u_2 \text{ in } \mathcal{H}.$$

Thus, we obtain a critical point u_2 of I_λ for all $\lambda \in (0, \Lambda_*)$ with

$$\Lambda_* := \min\{\lambda_*, \Lambda\}$$

satisfying

$$I_\lambda(u_2) > 0.$$

□

4. Conclusion

In our work, we prove the existence of at least two distinct critical points of $I_{\lambda,\mu}$. One by the Ekeland variational principle with negative energy, and the other by mountain pass theorem without Palais-Smale conditions with positive energy.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Kirchhoff, G. (1883) Mechanik. Teubner, Leipzig.

[2] Alves, C.O., Corrêa, F.J.S.A. and Ma, T.F. (2005) Positive Solutions for a Quasilinear Elliptic Equation of Kirchhoff Type. *Computers & Mathematics with Applications*, **49**, 85-93. https://doi.org/10.1016/j.camwa.2005.01.008

[3] Cheng, C.T. and Wu, X. (2009) Existence Results of Positive Solutions of Kirchhoff Type Problems. *Nonlinear Analysis: Theory, Methods & Applications*, **71**, 4883-4892. https://doi.org/10.1016/j.na.2009.03.065

[4] Ma, T.F. and Rivera, J.E.M. (2003) Positive Solutions for a Nonlinear Nonlocal Elliptic Transmission Problem. *Applied Mathematics Letters*, **16**, 243-248.
https://doi.org/10.1016/S0893-9659(03)80038-1

[5] Chen, C., Kuo, Y. and Wu, T. (2011) The Nehari Manifold for a Kirchhoff Type Problem Involving Sign-Changing Weight Functions. *Journal of Differential Equations*, **250**, 1876-1908.
https://doi.org/10.1016/j.jde.2010.11.017

[6] Mao, A.M. and Zhang, Z.T. (2009) Sign-Changing and Multiple Solutions of Kirchhoff Type Problems without the P.S. Condition. *Nonlinear Analysis: Theory, Methods & Applications*, **70**, 1275-1287. https://doi.org/10.1016/j.na.2008.02.011

[7] Mao, A.M. and Luan, S.X. (2011) Sign-Changing Solutions of a Class of Nonlocal Quasilinear Elliptic Boundary Value Problems. *Journal of Mathematical Analysis and Applications*, **383**, 239-243. https://doi.org/10.1016/j.jmaa.2011.05.021

[8] Jin, J.H. and Wu, X. (2010) Infinitely Many Radial Solutions for Kirchhoff-Type Problems in RN. *Journal of Mathematical Analysis and Applications*, **369**, 564-574.
https://doi.org/10.1016/j.jmaa.2010.03.059

[9] Wei, L. and He, X.M. (2012) Multiplicity of High Energy Solutions for Superlinear Kirchho Equations. *Journal of Applied Mathematics and Computing*, **39**, 473-487.
https://doi.org/10.1007/s12190-012-0536-1

[10] He, X.M. and Zou, W.M. (2009) Infinitely Many Positive Solutions for Kirchhoff-Type Problems. *Nonlinear Analysis: Theory, Methods & Applications*, **70**, 1407-1414.
https://doi.org/10.1016/j.na.2008.02.021