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Optimal Inequality in the One-Parameter Arithmetic and Harmonic Means

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Abstract

This research work considers the inequalities: (Ieq). The researchers attempt to find an answer as to what are the best possible parameters α,β that (Ieq) can be hold? The main tool is the optimization of some suitable functions that we seek to find out.

Subject Areas

Mathematical Analysis

Keywords

One-Parameter, Arithmetic, Harmonic Means

1. Introduction

In this paper we consider the following inequalities:

$$\alpha A(a,b)+(1-\alpha)H(a,b)\le J_p(a,b)\le \beta A(a,b)+(1-\beta)H(a,b) \quad \text{(Inq) (1)}$$

with $A(a,b)=\dfrac{a+b}{2}$; $H(a,b)=\dfrac{2ab}{a+b}$

$$J_p(a,b)=\begin{cases}\dfrac{p\left(a^{P+1}-b^{P+1}\right)}{(P+1)\left(a^P-b^P\right)}; & a\ne b;\quad P\ne 0,-1\\[2mm] \dfrac{a-b}{\ln a-\ln b}, & a\ne b;\quad P=0\\[2mm] \dfrac{ab(\ln a-\ln b)}{a-b}, & a\ne b;\quad P=-1\\[2mm] a, & a=b\end{cases} \tag{1.1}$$

Our motivation of this study is to find out such inequality that arises in the search for determination of a point of reference about which some function of variants would be minimum or maximum. Since very early times, people have been interested in the problem of choosing the best single quantity, which could summarize the whole information contained in a number of observations (measurements). Moreover, the theory of means has its roots in the work of the Pythagorean who introduced the harmonic, geometric, and arithmetic means. Peter *et al.* [1] introduced seven other means and gave the well-known elegant geometric proof of the celebrated inequalities among the harmonic, geometric, and arithmetic means. The strong relations and introduction of the theory of means with the theories of inequalities, function equations, probability and statistics add greatly to its importance. This single element is usually called a means or averages. The term "means" or "average" (middle value) has for a long time been used in all branches of human activity. The main objective of this research work is to present optimization of inequality in the one-parameter, arithmetic and harmonic means.

The basic function of mean value is to represent a given set of many values by some single value. In [2], the author was the first time introduced power means defined the meaning of the term "representation" as determination of appoint of reference about which some function of variants would be minimum. More recently the means were the subject of research and study whereas essential areas in several applications such as: physics, economics, electrostatics, heat conduction, medicine and even in meteorology. It can be observed that the power mean $M_p(a,b)$ of order p can be rewritten as (see as [3])

$$M_p(a,b)=\begin{cases}\left(\dfrac{a^p+b^p}{2}\right)^{\frac{1}{p}}; & p\neq 0\\ \sqrt{ab}; & p=0\end{cases}$$

If we denote by

$$A(a,b)=\frac{1}{2}(a+b),\ G(a,b)=\sqrt{ab}\ \text{ and }\ H(a,b)=\frac{2ab}{a+b},$$

the arithmetic means, geometric means and harmonic means of two positive numbers a and b, respectively. In addition, the logarithmic and identric means of two positive real numbers a and b defined by [4]

$$L(a,b)=\begin{cases}\dfrac{b-a}{\log b-\log a} & a\neq b\\ a & a=b\end{cases}$$

$$I(a,b)=\begin{cases}\dfrac{1}{\mathrm{e}}\left(\dfrac{b^b}{a^a}\right)^{1l(b-a)} & a\neq b\\ a & a=b\end{cases}$$

Several authors investigated and developed a relationship of optimal inequalities between the various means.

The well-known inequality that:

$$\min\{a,b\} \le H(a,b) = M_{-1}(a,b) \le G(a,b) = M_0(a,b)$$
$$\le L(a,b) \le I(a,b) \le A(a,b) = M_1(a,b) \le \max\{a,b\}$$

and all inequalities are strict for $a \ne b$.

In [4], researchers studied what are the best possible parameters $\alpha_1, \alpha_2, \beta_1$ and β_2 by two theorems:

Theorem (1) the double inequality: -

$$\alpha_1 A(a,b) + (1-\alpha_1) H(a,b) \le L(a,b) \le \beta_1 A(a,b) + (1-\beta_1) H(a,b)$$

holds for all $a,b>0$ if and only if $\alpha_1 \le 0$ and $\beta_1 \ge \frac{2}{3}$ when proved that the parameters $\alpha_1 \le 0$ and $\beta_1 \ge \frac{2}{3}$ cannot be improved.

Theorem (2) the double inequality: -

$$\alpha_2 A(a,b) + (1-\alpha_2) H(a,b) \le L(a,b) \le \beta_2 A(a,b) + (1-\beta_2) H(a,b)$$

holds for all $a,b>0$ if and only if $\alpha_2 \le \frac{2}{\mathrm{e}}$ and $\beta_2 \ge \frac{5}{6}$ when proved that the parameters $\alpha_2 \le \frac{2}{\mathrm{e}}$ and $\beta_2 \ge \frac{5}{6}$ cannot be improved.

Interestingly in [1] B. Long *et al.*, proved that the following results: $M_0(a,b)$ and $M_{t/3}(a,b)$ are the best possible lower and upper power bounds for the generalized logarithmic mean $L_t(a,b)$ for any fixed $t>0$ the double inequalities

$$M_0(a,b) < L_t(a,b) < M_{t/3}(a,b)$$

holds for all $a,b>0$ with $a \ne b$, and they found $L_2(a,b)$ the optimal lower generalized logarithmic means bound for the identric means $I(a,b)$ for inequalities $L_2(a,b) < I(a,b)$ holds for all *a*, *b* are positive numbers with $a \ne b$. Pursuing another line of investigation, in [5] the authors showed the sharp upper and lower bounds for the Neuman-sandor $NS(a,b)$ [6] in terms of the liner convex combination of the logarithmic means $L(a,b)$ and second seiffert means $T(a,b)$ [7] of two positive numbers *a* and *b*, respectively for the double inequalities

$$\alpha L(a,b) + (1-\alpha) T(a,b) \le NS(a,b) \le \beta L(a,b) + (1-\beta) T(a,b)$$

holds for all $a,b>0$ with $a \ne b$ is true if and only if $\alpha \ge \frac{1}{4}$ and $\beta \le 1-\pi l\left[4\log\left(1+\sqrt{2}\right)\right]$.

In [8] have improvements and refinements by H.Z. Xu *et al.*, for they found several sharp upper and lower bounds for the Sandor-yang means $R_{QA}(a,b)$ and $R_{AQ}(a,b)$ [9] [10] in terms of combinations of the arithmetic means $A(a,b)$ and the contra-harmonic mean $C(a,b)$ [11] [12].

The authors have to proven our main results several lemmas find the best possible parameters $\alpha_i, \beta_i \in (i=1,2,3,4)$ such that the double inequalities

$$c^{\alpha_1}(a,b)A^{1-\alpha_1}(a,b) < R_{QA}(a,b) < c^{\beta_1}(a,b)A^{1-\beta_1}(a,b)$$
$$c^{\alpha_2}(a,b)A^{1-\alpha_2}(a,b) < R_{QA}(a,b) < c^{\beta_2}(a,b)A^{1-\beta_2}(a,b)$$
$$\alpha_3\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right]+(1-\alpha_3)C^{1/3}(a,b)A^{2/3}(a,b)$$
$$< R_{QA}(a,b) < \beta_3\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right]+(1-\beta_3)C^{1/3}(a,b)A^{2/3}(a,b)$$
$$\alpha_4\left[\frac{1}{6}C(a,b)+\frac{5}{6}A(a,b)\right]+(1-\alpha_4)C^{1/6}(a,b)A^{5/6}(a,b)$$
$$< R_{AQ}(a,b) < \beta_4\left[\frac{1}{6}C(a,b)+\frac{5}{6}A(a,b)\right]+(1-\beta_4)C^{1/6}(a,b)A^{5/6}(a,b)$$

holds for all $a,b>0$ with $a \neq b$.

2. Main Results

Our main results are set in the following theorem:

Theorem 1

1) Assume $a>0, b>0$ with $\frac{a}{b}>1$ then,

a) if $p \in (-1, p_1)$ where $p_1 = \frac{-9+\sqrt{73}}{2} < 0$. There exist α_* and α^* reals such that, if $\alpha_* < \alpha < \alpha^* < \beta$ then the double inequality **(Inq)** holds.

b) if $p=0$. If $\alpha<0$ and $\frac{3}{2}<\beta<2$ then the double inequality **(Inq)** holds.

c) if $p=-1$. If $\alpha<0$ and $\beta>\frac{1}{3}$ then the double inequality **(Inq)** holds.

2) If $a=b$ then then the double inequality **(Inq)** holds for all α and β reals.

Proof. 1) Assuming $a>0, b>0$ with $\frac{a}{b}>1$

First case a): we have

$$\alpha\left(\frac{a+b}{2}\right)+(1-\alpha)\left(\frac{2ab}{a+b}\right) \le \frac{p\left(a^{p+1}-b^{p+1}\right)}{(p+1)\left(a^p-b^p\right)} \le \beta\left(\frac{a+b}{2}\right)+(1-\beta)\left(\frac{2ab}{a+b}\right)$$
$$a \neq b; \quad p \neq 0,-1; \quad a>b.$$

Set $t=\frac{a}{b}>1$. Then, we obtain

$$\alpha\left(\frac{b(t+1)}{2}\right)+(1-\alpha)\left(\frac{2tb}{t+1}\right) \le \frac{pb\left(t^{p+1}-1\right)}{(p+1)\left(t^p-1\right)} \le \beta\left(\frac{b(t+1)}{2}\right)+(1-\beta)\left(\frac{2tb}{t+1}\right)$$

We start by showing that

$$\alpha\left(\frac{b(t+1)}{2}\right)+(1-\alpha)\left(\frac{2tb}{t+1}\right)-\frac{pb\left(t^{p+1}-1\right)}{(p+1)\left(t^p-1\right)} \le 0,$$

$$\Leftrightarrow \frac{\alpha b(t+1)^2(p+1)(t^p-1)+4(1-\alpha)tb(p+1)(t^p-1)}{2(t+1)(p+1)(t^p-1)} + \frac{-2(t+1)pb(t^{p+1}-1)}{2(t+1)(p+1)(t^p-1)} \le 0$$

Because $p > -1$, we have $2(t+1)(p+1)(t^p-1) > 0$ therefore the study amounts to proving that

$$\alpha b(t+1)^2(p+1)(t^p-1)+4(1-\alpha)tb(p+1)(t^p-1) - 2(t+1)pb(t^{p+1}-1) \le 0.$$

Let

$$f(t) = \alpha b(t+1)^2(p+1)(t^p-1)+4(1-\alpha)tb(p+1)(t^p-1) - 2(t+1)pb(t^{p+1}-1)$$

We have to prove that the function *f* is negative under certain conditions on the parameters α, β and *p*, a.e: $f(t) \le 0$. So

$$f(t) = \alpha b(t+1)^2(p+1)(t^p-1)+4(1-\alpha)tb(p+1)(t^p-1) - 2(t+1)pb(t^{p+1}-1) \le 0$$

Because $f(1) = 0$, it will suffice to show that *f* is decreasing for all $t > 1$. Which amounts to studying the sign of the derivative f' of *f*. We have:

$$\begin{aligned} f'(t) &= \left[\alpha b(p+1)(p+2) - 2bp(p+2)\right]t^{p+1} \\ &+ \left[2\alpha b(p+1)^2 + 4(1-\alpha)b(p+1)^2 - 2bp(p+1)\right]t^P \\ &+ \left[\alpha bp(p+1)\right]t^{p-1} + \left[-2\alpha b(p+1)\right]t \\ &+ \left[-2\alpha b(p+1) - 4(1-\alpha)b(p+1) + 2bp\right] \end{aligned}$$

Because $f'(1) = 0$, it will suffice to show that f' is decreasing for all $t > 1$. Which amounts to studying the sign of the derivative f'' of f'. We have:

$$\begin{aligned} f''(t) &= \left[\alpha b(p+1)^2(p+2) - 2bp(p+2)(p+1)\right]t^p \\ &+ \left[2\alpha bp(p+1)^2 + 4(1-\alpha)bp(p+1)^2 - 2bp^2(p+1)\right]t^{p-1} \\ &+ \left[\alpha bp(p+1)(p-1)\right]t^{p-2} + \left[-2\alpha b(p+1)\right] \end{aligned}$$

Likewise we find that $f''(1) = 0$ so it will suffice to show that f'' is decreasing for all $t > 1$. Which amounts to studying the sign of the derivative f''' of f''. We have:

$$\begin{aligned} f'''(t) &= \left[\alpha bp(p+1)^2(p+2) - 2bp^2(p+2)(p+1)\right]t^{p-1} \\ &+ \left[2\alpha bp(p+1)^2(p-1) + 4(1-\alpha)bp(p+1)^2(p-1) \right. \\ &\left. - 2bp^2(p+1)(p-1)\right]t^{p-2} + \left[\alpha bp(p+1)(p-1)(p-2)\right]t^{p-3} \end{aligned}$$

and we get

$$f'''(1) = 6\alpha bp(p+1) - 2pb(p+1)[p+2].$$

Since $p \in (-1, p_1)$ where $p_1 = \frac{-9+\sqrt{73}}{2} < 0$ so, we will have the following equivalence

$$f'''(1) \le 0 \Leftrightarrow \alpha \ge \frac{2pb(p+1)[-p-2]}{6bp(p+1)} = \frac{-(p+2)}{3} = \alpha_1$$

Now, we can put

$$f'''(t) = t^{p-3}\left(At^2 + Bt + C\right) \Leftrightarrow f'''(t) = t^{p-3} f_1(t),$$

with

$$A = \alpha bp(p+1)^2(p+2) - 2bp^2(p+2)(p+1)$$

$$B = 2\alpha bp(p+1)^2(p-1) + 4(1-\alpha)bp(p+1)^2(p-1) - 2bp^2(p-1)(p+1)$$

then, we obtain

$$f_1'(t) = 2At + B = 0 \Leftrightarrow t_0 = \frac{-B}{2A} > 0$$

We must have

$$A < 0, \text{ for } \alpha > \frac{2bp^2(p+2)(p+1)}{bp(p+1)^2(p+2)} = \frac{2p}{p+1} = \alpha_2, \text{ with } p \in (-1, p_1)$$

and

$$B > 0, \text{ for } \alpha < \frac{p+2}{p+1} = \alpha_3, \text{ with } p \in (-1, p_1)$$

such that

$$t_0 = \frac{-B}{2A} < 1 < t, \text{ for } \alpha > \frac{p+2}{3} = \alpha_4, \text{ with } p \in (-1, p_1),$$

so that f_1 is decreasing for $t > 1$ and therefore, we obtain that $f'''(t) < 0$ because $f'''(1) \le 0$. By the same process we find that $f''(t)$ then that $f'(t)$ and $f(t)$.

Finally in this part for $p \in (-1, p_1)$, we obtain that there exists $\alpha_* = \max(\alpha_1, \alpha_2, \alpha_4)$ and α_3 such that for all $\alpha \in (\alpha_*, \alpha_3)$ we have:

$$\alpha\left(\frac{a+b}{2}\right) + (1-\alpha)\left(\frac{2ab}{a+b}\right) \le \frac{p\left(a^{p+1} - b^{p+1}\right)}{(p+1)\left(a^p - b^p\right)}.$$

To show the second inequality in this first case, we proceed by similar calculations. This is done by considering the function g defined by

$$\begin{aligned} g(t) = {} & \left[\beta b(p+1) - 2bp\right]t^{p+2} + \left[2\beta b(p+1) + 4(1-\beta)b(p+1) - 2bp\right]t^{P+1} \\ & + \left[\beta b(p+1)\right]t^p + \left[-\beta b(p+1)\right]t^2 \\ & + \left[-2\beta b(p+1) - 4(1-\beta)b(p+1) + 2bp\right]t + \left[-\beta b(p+1) + 2bp\right] \end{aligned}$$

So, after all the calculations, we get that for $p \in (-1, p_1)$, there exists

$\alpha^* = \max(\beta_1, \beta_2, \beta_3, \beta_4) = \beta_3 = \alpha_3 = \frac{p+2}{p+1}$ such that $g(t) \ge 0$, for all $\beta > \alpha^*$.

a.e:

$$\frac{p\left(a^{p+1} - b^{p+1}\right)}{(p+1)\left(a^p - b^p\right)} \le \beta\left(\frac{a+b}{2}\right) + (1-\beta)\left(\frac{2ab}{a+b}\right)$$

Second case b):

With similar calculations and by the same idea we obtain that for all $\alpha < 0$ and $\beta \in \left(\frac{3}{2}, 2\right)$ then,

$$\alpha\left(\frac{a+b}{2}\right) + (1-\alpha)\left(\frac{2ab}{a+b}\right) \le \frac{a-b}{\ln a - \ln b} \le \beta\left(\frac{a+b}{2}\right) + (1-\beta)\left(\frac{2ab}{a+b}\right).$$

Third case c):

By the method above and similar calculations, we also find that for all $\alpha < 0$ and $\beta > \frac{1}{3}$ then,

$$\alpha\left(\frac{a+b}{2}\right) + (1-\alpha)\left(\frac{2ab}{a+b}\right) \le \frac{ab(\ln a - \ln b)}{a-b} \le \beta\left(\frac{a+b}{2}\right) + (1-\beta)\left(\frac{2ab}{a+b}\right).$$

2) Assuming $a = b$.

We easily get:

$$\alpha A(a,b) + (1-\alpha)H(a,b) = a = J_p(a,b)$$

$$\beta A(a,b) + (1-\beta)H(a,b) = a = J_p(a,b),$$

which shows that the double inequality holds for all of the parameters the α and β.

3. Conclusions

In our work, we studied the following double inequality

$$\alpha A(a,b) + (1-\alpha)H(a,b) \le J_p(a,b) \le \beta A(a,b) + (1-\beta)H(a,b)$$

by searching the best possible parameters such that (Inq) can be held.

Firstly, we have inserted

$$f(t) = \alpha A(a,b) + (1-\alpha)H(a,b) - J_p(a,b)$$

Without loss of generality, we have assumed that $a > b$ and let $t = \frac{a}{b} > 1$ to determine the condition for α and β to become $f(t) \le 0$.

Secondly, have inserted

$$g(t) = \beta A(a,b) + (1-\beta)H(a,b) - J_p(a,b)$$

Without loss of generality, we assume that $a > b$ and let $t = \frac{a}{b} > 1$ to determine the condition for α and β to become $g(t) \geq 0$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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