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Generalization of Stirling Number of the Second Kind and Combinatorial Identity

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Abstract

The Stirling numbers of second kind and related problems are widely used in combinatorial mathematics and number theory, and there are a lot of research results. This article discuss the function: $\sum A_1^{C_1} A_2^{C_2} \cdots A_k^{C_k}$ ($C_1 + C_2 + \cdots + C_K = N - K$, $C_i \ge 0$), obtain its calculation formula and a series of conclusions, which generalize the results of existing literature, and further obtain the combinatorial identity:
$\sum (-1)^{K-i} * C(K-1, K-i) C(A-1+i, N-1) = C(A, N-K)$.

Subject Areas

Combinatorics

Keywords

Combinatorics, Combinatorial Identity, Stirling Numbers, Calculation Formula

1. Introduction

Stirling number of the second kind $S_2(n, K)$ [1] is defined as

$$t^N = \sum_{k=0}^{N} S_2(N, k)[t]_k \tag{1*}$$

It has attributes:

[1] $S_2(N, K) = \sum 1^{C_1} 2^{C_2} \cdots K^{C_k} \quad (C_1 + C_2 + \cdots + C_K = N - K,\ C_i \ge 0)$ (2*)

[1] $S_2(N, K) = \frac{1}{K!} \sum_{i=0}^{K-1} (-1)^i \binom{K}{i} (K-i)^N$ (3*)

Let $j = K - i \rightarrow S_2(N, K) = \frac{1}{(K-1)!} \sum_{j=1}^{K} (-1)^{K-j} \binom{K-1}{K-j} j^{N-1}$ (4*)

It is similar to the expansion of

$$\left(X-Y\right)^{N-1} \tag{5*}$$

$$S_2\left(0,0\right)=1,\quad S_2\left(N,0\right)=0\quad\left(N>0\right)$$

$$S_2\left(N,1\right)=1$$

$$S_2\left(N,2\right)=\left(2^{N-1}-1^{N-1}\right)/1!$$

$$S_2\left(N,3\right)=\left(3^{N-1}-2*2^{N-1}+1^{N-1}\right)/2!$$

$$S_2\left(N,4\right)=\left(4^{N-1}-3*3^{N-1}+3*2^{N-1}-1^{N-1}\right)/3!$$

$$S_2\left(N,5\right)=\left(5^{N-1}-4*4^{N-1}+6*3^{N-1}-4*2^{N-1}+1^{N-1}\right)/4!$$

$$S_2\left(N,6\right)=\left(6^{N-1}-5*5^{N-1}+10*4^{N-1}-10*3^{N-1}+5*2^{N-1}-1^{N-1}\right)/5!$$

$$S_2\left(N,N-1\right)=\binom{N}{2} \tag{6*}$$

2. Main Conclusion and Proof

Definition: The generalization of Stirling number of the second kind

If $\{a\}=\{A_1,A_2,\cdots,A_k\}$, $A\in\mathbb{Z}$, $A_i<A_j$, **(** $i<j$ **), then**

$$G\left(N,K,\{a\}\right)=\sum A_1^{C_1}A_2^{C_2}\cdots A_k^{C_k}\quad\left(C_1+C_2+\cdots+C_K=N-K,\ C_i\ge 0\right)$$

$$G_1\left(N,K,A\right)=G\left(N,K,\{A,A+1,\cdots,A+K-1\}\right)\ \rightarrow\ S_2\left(N,K\right)=G_1\left(N,K,1\right)$$

The function has been discussed by many papers [2] [3] [4], including definition, recursive relation, generating function and so on. This article will not narrate.

1) $$\begin{aligned}G\left(N,K,\{a\}\right)&=G\left(N-1,K-1,\{A_1,\cdots,A_{k-1}\}\right)+A_k*G\left(N-1,K,\{a\}\right)\\&=G\left(N-1,K-1,\{A_2,\cdots,A_k\}\right)+A_1*G\left(N-1,K,\{a\}\right)\end{aligned}$$

Proof: By definition.

The first equation corresponds to $S_2\left(n,K\right)=S_2\left(n-1,k-1\right)+k*S_2\left(n-1,K\right)$.

2) $$G\left(N,K,\{a\}\right)=\frac{G\left(N,K-1,\{A_2,\cdots,A_k\}\right)-G\left(N,K-1,\{A_1,\cdots,A_{k-1}\}\right)}{A_k-A_1}$$

Proof: From the second equation of 1).

3) $G\left(N,1,\{A\}\right)=A^{N-1}$, corresponds to $S_2\left(N,1\right)=1$

4) $G\left(N,2,\{A_1,A_2\}\right)=\frac{A_2^{N-1}-A_1^{N-1}}{A_2-A_1}=\frac{A_1^{N-1}}{A_1-A_2}+\frac{A_2^{N-1}}{A_2-A_1}$, corresponds to $S_2\left(N,2\right)=2^{N-1}-1=2^{N-1}-1^{N-1}$.

Proof: $G\left(N,2,\{A_1,A_2\}\right)=A_1^{N-2}+A_1^{N-3}A_2+\cdots+A_2^{N-2}$.

5) $G\left(N,K,\{a\}\right)=\sum_{i=1}^{K}\frac{\left(A_i\right)^{N-1}}{\prod_{i\ne j}\left(A_i-A_j\right)}$, this is the calculation formula.

Proof: Induce by 2), 3), 4).

The form is symmetrical, for example:

$$G(N,3,\{a\}) = \frac{A_1^{N-1}}{(A_1 - A_2)(A_1 - A_3)} + \frac{A_2^{N-1}}{(A_2 - A_1)(A_2 - A_3)} + \frac{A_3^{N-1}}{(A_3 - A_1)(A_3 - A_2)}$$

[2] obtains it by generating function.

Lemma 1: if {*a*} is an equal difference sequence $\{A, A+d, \cdots, A+(K-1)d\}$,

$$\frac{1}{\prod_{i=m, i\neq j}(A_i - A_j)} = \frac{(-1)^{K-m}}{d^{K-1}(K-1)!}\binom{K-1}{K-m}.$$

Proof:

$$\begin{aligned}\frac{1}{\prod_{i=m, i\neq j}(A_i - A_j)} &= \frac{1}{\prod_{i=m, j<m}(A_i - A_j)}\frac{1}{\prod_{i=m, j>m}(A_i - A_j)} \\ &= \frac{1}{d^{K-1}}\frac{1}{(m-1)!}\frac{(-1)^{K-m}}{(K-m)!} \\ &= \frac{(-1)^{K-m}(K-1)!}{d^{K-1}(K-1)!(m-1)!(k-m)!} \\ &= \frac{(-1)^{K-m}}{d^{K-1}(K-1)!}\binom{K-1}{K-m}\end{aligned}$$

6) If $\{a\} = \{A, A+d, \cdots, A+(K-1)d\}$,

$$G(N,K,\{a\}) = \frac{1}{d^{K-1}(K-1)!}\sum_{j=1}^{K}(-1)^{K-j}\binom{K-1}{K-j}A_j^{N-1}.$$

Proof: By 5) and **Lemma 1**.

It is similar to the expansion of $(X-Y)^{N-1}$, in particular:

7) $G1(N,K,A) = \frac{1}{(K-1)!}\sum_{i=1}^{K}(-1)^{K-i}\binom{K-1}{K-i}(A-1+i)^{N-1}$ similar to (4*), (5*)

8) $G_1(N,K,1) = S_2(N,K) = \frac{1}{(K-1)!}\sum_{i=1}^{K}(-1)^{K-i}\binom{K-1}{K-i}i^{N-1}$ equal to (4*), (5*)

9) $G(N,K,\{d,2d,\cdots,Kd\}) = \frac{d^{N-K}}{(K-1)!}\sum_{i=1}^{K}(-1)^{K-i}\binom{K-1}{K-i}i^{N-1} = d^{N-K}S_2(N,K)$

10) $G_1(K+1,K,A) = A + (A+1) + \cdots + (A+K-1) = K*A + \binom{K}{2}$ corresponds to (6*)

Theorem 1: $G_1(N<K,K,\{a\}) = 0$; $G_1(K,K,\{a\}) = 1$.

Proof:

7) $\rightarrow$ $G_1(1,K\geq 1,A) = \frac{1}{(K-1)!}\sum_{i=1}^{K}(-1)^{K-i}\binom{K-1}{K-i} = 0$

4) $\rightarrow$ $G_1(2,2,A) = 1$

Suppose $G_1(X,K-1,A)$ match the theorem:

$$G_1(N<K-1,K,A)$$

3) $\rightarrow$ $$= \frac{G_1(N,K-1,\{A_2,\cdots,A_k\}) - G_1(N,K-1,\{A_1,\cdots,A_{k-1}\})}{A_k - A_1} = \frac{0-0}{K-1} = 0$$

$$G_1(N=K-1,K,A)$$

$$=\frac{G_1(N,K-1,\{A_2,\cdots,A_k\})-G_1(N,K-1,\{A_1,\cdots,A_{k-1}\})}{A_k-A_1}=\frac{1-1}{K-1}=0$$

$$G_1(N=K,K,A)$$

$$=\frac{G_1(N,K-1\{A_2,\cdots,A_k\})-G_1(N,K-1,\{A_1,\cdots,A_{k-1}\})}{A_k-A_1}$$

$$=\frac{(K-1)(A+1)+\binom{K}{2}-(K-1)*A-\binom{K}{2}}{K-1}=1$$

Induction proved.

q.e.d.

The theorem verify the definition, A can be any integer.

Definition: $A\in Z$, $G_2(N,K,A)=\sum_{i=1}^{K}(-1)^{K-i}\binom{K-1}{K-i}\binom{A-1+i}{N-1}$

Theorem 2: $G_2(N+K,K,A)=\binom{A}{N}$

Proof:

Let $F(N)=(N-1)!G_2(N,K,A)=\sum_{i=1}^{K}(-1)^{K-i}\binom{K-1}{K-i}[A-1+i]_{N-1}$.

Substitution (1*) to 7), use Theorem 1:

$$G_1(1,K,A)=0,K>1\ \rightarrow\ F(1)=0\ \rightarrow\ G_2(1,K>1,A)=0$$

$$G_1(2,K,A)=0,K>2,F(1)=0\ \rightarrow\ F(2)=0\ \rightarrow\ G_2(2,K>2,A)=0$$

$$\cdots$$

$$G_1(K,K,\{a\})=1\ \rightarrow\ F(K)=(K-1)!\ \rightarrow\ G_2(K,K,A)=1=\binom{A}{0}$$

10) $\rightarrow$

$$G_1(1+K,K,A)=K*A+\binom{K}{2}=\frac{S_2(K,K)F(K+1)+S_2(K,K-1)F(K)}{(K-1)!}\ \rightarrow$$

$$F(1+K)=A*K!\rightarrow\ G_2(1+K,K,A)=A=\binom{A}{1}$$

$$\binom{A}{N+1}=\binom{A-1}{N}+\binom{A-1}{N+1}\ \rightarrow$$

$$G_2(N+1+K,K,A)=G_2(N+K,K,A-1)+G_2(N+1+K,K,A-1)\ \rightarrow$$

$$G_2(N+K,K,A)=\binom{A}{N}$$

q.e.d.

Record in [5]:

$$\sum_{k=0}^{m-1}(-1)^k\binom{m}{k}\binom{m+n-k-1}{n}=\binom{n-1}{m-1}\qquad(**)$$

Let $A = n-1$, $m = K-1$, $i = K-k$ $\rightarrow$ left $= \sum_{i=0}^{K}(-1)^{K-i}\binom{K-1}{K-i}\binom{A-1+i}{A+1}$.

Let $K+N-1 = A+1$ $\rightarrow$

$$\text{left} = \sum_{i=0}^{K}(-1)^{K-i}\binom{K-1}{K-i}\binom{A-1+i}{K+N-1} = \binom{n-1}{m-1} = \binom{A}{A-N} = \binom{A}{N}.$$

(**) has 2 variables (m, n), it is $G_2(K+A+2,K,A)$ actually.

Theorem 2 has 3 variables, is promotion of (**).

11) $G_2(N+K,K,A)$: The Inclusion-Exclusion Principle.

$G_2(N+K,K,A) = \binom{A}{N}$ is independent of K.

Choice N from A, one way is:

$$\binom{A}{N} = \binom{A-1}{N-1} + \binom{A-1}{N} = \binom{A-1}{N-1} + \binom{A-2}{N-1} + \cdots + \binom{N-1}{N-1}$$

$$\binom{A}{N} = \binom{A+1}{N+1} - \binom{A}{N+1} = G_2(N+2,2,A)$$

Another way is:

$$= \left\{\binom{A+2}{N+2} - \binom{A+1}{N+2}\right\} - \left\{\binom{A+1}{N+2} - \binom{A}{N+2}\right\}$$
$$= G_2(N+3,3,A)$$

$$\cdots$$

12)
$$\begin{aligned} &G(N+K,K,\{a\}) - \left(\sum A_i\right)G(N+K-1,K,\{a\}) \\ &+ \left(\sum A_i A_j\right)G(N+K-1,K,\{a\}) + \cdots \\ &+ (-1)^K (A_1A_2\cdots A_k)G(N,K,\{a\}) = 0 \end{aligned}$$

Proof:

$$\begin{aligned} 0 &= G_1(N,0,\{a\}) = G(N+1,1,\{a\}) - A_1G(N,1,\{a\}) \\ &= \{G(N+2,2,A) - A_2G_1(N+1,2,A)\} - A_1\{G(N+1,2,A) - A_2G_1(N,2,A)\} \\ &= G(N+2,2,A) - (A_1 + A_2)G_1(N+1,2,A) + A_1A_2G_1(N,2,A) \end{aligned}$$

Induction proved.

q.e.d.

This is similar to the Inclusion-Exclusion Principle,in particular:

13)

$$S_2(N,K) - S_1(K+1,K)S_2(N-1,K) + \cdots + (-1)^K S_1(K+1,1)S_2(N-K,K) = 0$$

S_1 is unsigned Stirling number of the first kind.

14) $G_1(N,K,A) = \sum_{t=K-1}^{N-1} S_2(N-1,t)\binom{A}{t+1-K}[K]^{t+1-K}$

Proof:

Substitution (1*) to 7), use Theorem 2:

$$
\begin{aligned}
G_1(N,K,A) &= \frac{1}{(K-1)!}\sum_{i=1}^{K}(-1)^{K-i}\binom{K-1}{K-i}(A-1+i)^{N-1} \\
&= \frac{1}{(K-1)!}\sum_{i=1}^{K}(-1)^{K-i}\binom{K-1}{K-i}\sum_{t=0}^{N-1}S_2(N-1,t)[A-1+i]_t \\
&= \sum_{t=0}^{N-1}S_2(N-1,t)\sum_{i=1}^{K}(-1)^{K-i}\binom{K-1}{K-i}\binom{A-1+i}{t}\frac{t!}{(K-1)!} \\
&= \sum_{t=0}^{N-1}S_2(N-1,t)G_2(t+1,K,A)[K]^{t+1-K} \\
&= \sum_{t=0}^{N-1}S_2(N-1,t)\binom{A}{t+1-K}[K]^{t+1-K}
\end{aligned}
$$

q.e.d.

$$\rightarrow S_2(N,K) = G_1(N,K,1) = K * S_2(N-1,K) + S_2(N-1,K-1)$$

3. Conclusions

This paper starting from (4*), (5*), discusses the problems from different perspectives.

The introduced function has good characteristics, can be further studied.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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