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Complete Arcs and Surfaces in Three Dimensional Projective Space *PG*(3,7)

Ali Ahmed A. Abdulla[1], Nada Yassen Kasm Yahya[2]

[1]Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Mosul, Iraq
[2]Department of Mathematics, College of Education for Pure Science, University of Mosul, Mosul, Iraq
Email: drnadaqasim1@gmail.com

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Abstract

The purpose of this thesis is to construct surfaces and complete arcs in the projective 3-space $PG(3,q)$ over Galois fields $GF(q)$, $q = 7$. A(k,n)-arc in $PG(3,q)$ is a set of k points; no $n + 1$ of them are coplanar. A(k,n)-arc is complete if it is not contained in a (k + 1, n)-arc. In this work the (k,ℓ)-span are constructed in $PG(3,7)$ and it is found that, it exists in $PG(3,7)$ when $k = 50$. Moreover, the maximum (k,ℓ)-span, is called a spread.

Subject Areas

Projective Geometry

Keywords

Algebraic Curves and Surfaces, Complete Arcs and Surfaces in Three Dimensional Projective Space $PG(3,q)$, (k,ℓ)-Span, Spread, Sets of Subspaces

1. Introduction

The study of finite projective spaces was at one time no more than an adjunct to algebraic geometry over the real and complex numbers. But, more recently, finite spaces have been studied both for their application to practical topics such as coding theory and design experiments, and for their illumination of more abstract mathematical topics such as finite group theory and graph theory. Perhaps the fastest growing area of modern mathematics is combinatorics that is concerned with the study of arrangement of elements into sets. These elements are usually finite in number, and the arrangement is restricted by certain boundary conditions imposed by the particular problem under investigation. Much of the growth of combinatory has gone hand in hand with the development of the computer. A major reason for this rapid growth of combinatorics is its wealth of applications, to computer science, communications, transportations, genetics, experimental design, and so on.

A recurring theme of this work is the characterization of algebraic varieties in $PG(n,q)$ as finite sets of points with certain combinatorial properties. Among these finite sets, the concept of (k,n)-arcs is in the n-dimensional projective space over Galois field $GF(q)$, $q = p^m$ for some prime number p and some integer m.

Hirschfeld, J.W.P. (1979) studies the basic definition and theorems of projective geometrics over finite fields [1], Kareem view (k,ℓ)-span in $PG(3,p)$ over Galois field $GF(p)$, p = 4 in 2013 [2]. Al-Mokhtar studies the complete arcs and surface in three-dimensional projective space over Galois field $GF(P)$, p = 2, 3 [3]. In three-dimensional projective space, the control problem is how to construct and find the whole space spread which is $(50,\ell)$-span in $PG(3,7)$ and prove it in general when $P \geq 2$ is p^2+1. Complete arcs have important connections with a number of other objects, see [4]-[19] and the references therein. Hirschfeld, J.W.P. (1998) studies the basic definition and theorems of projective geometrics over finite fields, in 2008 [20]. This paper includes three sections. First section considers the preliminaries of projective 3-space which contains some definition and theorems for the concept, whereas the second section consists of the subspace in $PG(3,p)$. Finally, the third section constructs maximum complete (k,ℓ). A span in $PG(3,7)$ is spreading, and in general proves Geometric rule in (Conclusions) $P \geq 2$. The total number of (k,ℓ)-span in $PG(3,q)$ is $p^2+1, p \geq 2$.

2. (k,ℓ)-Spread in $PG(3,p)$

2.1. Sets of Subspaces

Qualifier [19]

A (k,ℓ)-set in $PG(n,q)$ is a set of k spaces π_ℓ. A k-set is a$(k,0)$-set, that is a set of k points.

Definition [19]

A $(k, \ell; r, s, n, q)$-set is a (k,ℓ)-set in $PG(n,q)$ at most r spaces π_ℓ of which lie in any πs.

It is of great interest for applications to find maximal and particularly maximum such sets as k varies but the other parameters remain fixed. Thus a complete $(k, \ell; r, s; n, q)$-set is one not contained in any $(k+1, \ell; r, s, n, q)$-set.

The definition of $(k, \ell; r, s; n, q)$-set is specialized as follows.

A $(k; r, s; n, q)$-set is a $(k, 0; r, s; n, q)$-set;

A $(k, r; n, q)$-set is a $(k; r, r\text{-}1; n, q)$-set;

A $(k; r)$-cap is a (k; r, ℓ; n, q)-set with $n \geq 3$;

A k-cap is a $(k, 2)$-cap;

A k-arc is a $(k; n, n\text{-}1; n, q)$-set;

A plane (*k*; r)-arc is a (*k*; *r*, 1; 2, *q*)-set;

A (*k*,ℓ)-span is a (*k*, *l*; 1, 2ℓ; *n*, *q*)-set with ℓ ≥ 1.

Thus a plane *k*-arc (or just *k*-arc) is a set of *k* points in $PG(2,q)$, no three of which are collinear, a maximum plane *k*-arc is an oval.

A *k*-cap is a set of *k* points in $PG(n,q)$ with $n \geq 3$ such that no three are collinear.

A maximum *k*-cap in $PG(3,q)$ is an ovaloid, denoted by $m(n,q)$.

A (*k*,ℓ)-span is a set of *k* spaces π_ℓ no two of which intersect, a maximum (*k*,ℓ)-span is a sprea

Definition

In $PG(3,q)$, if *K* is any *k*-set, then an *n*-secant of *K* is a line (a plane) ℓ such that $|\ell \cap K| = n$. In particular, the following terminology are also used

0-secant is an external line (plane).

1-secant is a unisecant line (plane).

2-secant is a bisecant line (plane).

3-secant is a trisecant line (plane).

2.2. Algebraic Curves and Surfaces

Qualifier [6] [7]

A polynomial F in a polynomial ring $K[x_1,\cdots,x_n]$ is called homogenous or a form of degree *d* if all of its terms have the same degree *d*.

Qualifier [6] [7]

A subset *v* of $PG(n,K)$ is a variety (over *K*) if there exist forms $F_1, F_2, \cdots, F_r$ in $K[x_1,\cdots,x_n]$, such that:

$$v = \{P(A) \text{ in } PG(n,K) \mid F_1(A) = F_2(A) = \cdots = F_r(A) = 0\}$$
$$= V_{n,K}(F_1, F_2, \cdots, F_r)$$

The points *P*(*A*) are points of *v*. When *K* = *GF*(*q*), the notation $V_{n,q}(F_1, F_2, \cdots, F_r)$ is used.

A variety $V_{n,K}(F)$ in $PG(n,K)$ is a primal. A primal in $PG(2,K)$ is a plane algebraic curve, a primal in $PG(3,K)$ is a surface. The dimension of a primal is $n-1$; in particular, a curve in $PG(2,K)$ and a surface in $PG(3,K)$ have dimensions one and two respectively.

Definition

A point *N* not on a (*k*,*n*)-set *A* has index *i* if there are exactly *i* (*n*-secants) of *K* through *N*, one can denote the number of points *N* of indexi by C_i.

It is concluded that the (*k*,*n*)-set is complete iff $C_0 = 0$. Thus the *k*-set is complete iff every point of $PG(3,q)$ lies on some *n*-secant of the (*k*,*n*)-set.

Qualifier

A(*k*,*n*)-arc *A* in $PG(3,q)$ is a set of *k* points such that at most *n* points of which lie in any plane, $n \geq 3$. *n* is called the degree of the (*k*,*n*)-arc.

Definition [19]

Let T_i be the total number of the *i*-secants of a (*k*,*n*)-arc *A*, then the type of *A* w.r.t. its planes denoted by $(T_n, T_{n-1}, \cdots, T_0)$.

Qualifier [19]

Let (k_1,n)-arc A is of type $(T_n, T_{n-1}, \cdots, T_0)$ and (k_2,n)-arc B is of type $(S_n, S_{n-1}, \cdots, S_0)$, then A and B have the same type iff $T_i = S_i$, for all i, in this case, they are projectively equivalent.

Notion

Let $t(P)$ represents the number of unisecants (planes) through a point P of a (k,n)-arc A and let T_i represent the numbers of i-secants (planes) for the arc A in $PG(3,q)$, then:

1) $$t = t(P) = q^2 + q + 2 - k - \frac{(k-1)(k-2)}{2} - \cdots - \frac{(k-1)(k-2)\cdots(k-(n-1))}{(n-1)!}$$

2) $T_1 = kt$

3) $T_2 = \frac{k(k-1)}{2}$

4) $T_3 = \frac{k(k-1)(k-2)}{3!}$

5) $T_n = \frac{k(k-1)\cdots(k-n+1)}{n!}$

6) $$T_0 = q^3 + q^2 + q + 1 - kt - \frac{k(k-1)}{2} - \frac{k(k-1)(k-2)}{3!} - \cdots - \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

Proof

1) there exist $k - 1$ bisecants to A through P and there exist $\binom{k-1}{2}$ trisecants to A through P, and so there exist $\binom{k-1}{n-1}$ n-secants to A through P, and since there exist exactly $q^2 + q + 1$ planes through P, then the number of the unisecants through P:

$$\begin{aligned} t(P) &= q^2 + q + 1 - (k-1) - \binom{k-1}{2} - \cdots - \binom{k-1}{n-1} \\ &= q^2 + q + 2 - k - \frac{(k-1)(k-2)}{2} - \cdots - \frac{(k-1)(k-2)\cdots(k-n+1)}{(n-1)!} \\ &= t \end{aligned}$$

2) T_1 = the number of unisecants to A, since each point of A has t unisecants and the number of the points of A is k, then $T_1 = kt$.

3) T_2 = the number of bisecants to A, which is the number of planes passing through any two points of A. Hence $T_2 = \binom{k}{2} = \frac{k(k-1)}{2}$.

4) T_3 = the number of trisecants of A, which is the number of planes passing through any three points of A. Hence $T_3 = \binom{k}{3} = \frac{k(k-1)(k-2)}{3!}$.

5) T_n = the number of n-secants planes to A,

$$T_n = \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}.$$

6) q^3+q^2+q+1 represents the number of all planes, then in a (k,n)-arc of $PG(3,q)$, $q^3+q^2+q+1 = T_0+T_1+T_2+T_3+\cdots+T_n$

$$\begin{aligned} T_0 &= q^3+q^2+q+1-T_1-T_2-T_3-\cdots-T_n \\ &= q^3+q^2+q+1-kt-\frac{k(k-1)}{2}-\frac{k(k-1)(k-2)}{3!}-\cdots \\ &\quad -\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} \end{aligned}$$

Theorem

Let T_i represent the total number of the i-secants for a (k,n)-arc A in $PG(3,q)$, then the following equations are satisfied:

1) $\sum_{i=0}^{n} T_i = q^3+q^2+q+1$

2) $\sum_{i=1}^{n} i!T_i = kt+k(k-1)+k(k-1)(k-2)+\cdots+k(k-1)(k-n)$

3) $$\sum_{i=2}^{n} i(i-1)T_i = k(k-1)+k(k-1)(k-2)+1/2\,k(k-1)(k-2)(k-3) +\cdots+1/((n-2)!)\,k(k-1)(k-n)$$

Proof

1) $\sum_{i=0}^{n} T_i$ represents the sum of numbers of all i-secants to A, which is the number of all planes in the space. Hence $\sum_{i=0}^{n} T_i = q^3+q^2+q+1$.

2) $T_1 = kt$, $t = q^2+q+2-k-\frac{(k-1)(k-2)}{2}-\cdots-\frac{(k-1)\cdots(k-n+1)}{(n-1)!}$,

$T_2 = \frac{k(k-1)}{2}$, $T_3 = \frac{k(k-1)(k-2)}{3!}$, $T_4 = \frac{k(k-1)(k-2)(k-3)}{4!}$, $\cdots$,

$T_n = \frac{k(k-1)\cdots(k-n+1)}{n!}$

$$\begin{aligned} \sum_{i=1}^{n} i!T_i &= T_1+2!T_2+3!T_3+\cdots+n!T_n \\ &= kt+k(k-1)+k(k-1)(k-2) \\ &\quad +\cdots+k(k-1)(k-n+1) \end{aligned}$$

3)

$$\begin{aligned} \sum_{i=2}^{n} i(i-1)T_i &= 2T_2+6T_3+12T_4+\cdots+n(n-1)T_n \\ &= k(k-1)+k(k-1)(k-2)+\frac{1}{2}k(k-1)(k-2)(k-3)+\cdots \\ &\quad +\frac{1}{(n-2)!}k(k-1)(k-n+1) \end{aligned}$$

Notion

Let $R_i = R_i(P)$ represents the number of the *i*-secants (planes) through a point *P* of a (*k*,*n*)-arc *A*, in $PG(3,q)$ then the following equations are satisfied:

1) $\sum_{i=1}^{n} R_i = q^2 + q + 1$

2) $$\sum_{i=2}^{n}(i-1)!R_i = (k-1)+(k-1)(k-2)+\cdots+(k-1)(k-2)(k-n-1) = \sum_{i=1}^{n-1}(k-1)(k-i)$$

Proof

1) $\sum_{i=1}^{n} R_i = R_1 + R_2 + \cdots + R_n$, $\sum_{i=1}^{n} R_i$ represents the sum of numbers of all the *i*-secants through a point *P* of the arc *A*, which is the number of the planes through *P*. Thus,

$$\sum_{i=1}^{n} R_i = q^2 + q + 1.$$

2) $\sum_{i=2}^{n}(i-1)!R_i = R_2 + 2!R_3 + 3!R_4 + \cdots + (n-1)!R_n$

$$R_2 = k-1, R_3 = \binom{k-1}{2}, R_4 = \binom{k-1}{3}, \cdots, R_n = \binom{k-1}{n-1}$$

$$R_3 = \frac{(k-1)!}{2!(k-3)!}, R_4 = \frac{(k-1)!}{3!(k-4)!}, \cdots, R_n = \frac{(k-1)!}{(n-1)!(k-n)!}$$

$$R_3 = \frac{(k-1)(k-2)}{2}, R_4 = \frac{(k-1)(k-2)(k-3)}{3!}, \cdots, R_n = \frac{(k-1)\cdots(k-(n-1))}{(n-1)!}$$

$$\begin{aligned}\sum_{i=2}^{n}(i-1)!R_i &= k-1+\frac{2!(k-1)(k-2)}{2!}+\frac{3!(k-1)(k-2)(k-3)}{3!}+\cdots \\ &\quad+\frac{(n-1)!(k-1)(k-2)\cdots(k-(n-1))}{(n-1)!} \\ &= (k-1)+(k-1)(k-2)+(k-1)(k-2)(k-3)+\cdots \\ &\quad+(k-1)(k-2)(k-(n-1)) \\ &= \sum_{i=1}^{n-1}(k-1)(k-i)\end{aligned}$$

Theorem

Let $S_i = S_i(Q)$ represent the numbers of the *i*-secants (planes) of a (*k*,*n*)-arc *A* through a point *Q* in $PG(3,q)$ such that *Q* not in *A*, then the following equations are satisfied:

1) $\sum_{i=0}^{n} S_i = q^2 + q + 1$

2) $\sum_{i=1}^{n} iS_i = k$

Proof

1) $\sum_{i=0}^{n} S_i$ represents the sum of the total numbers of all *i*-secants to *A* through a point *Q* not in *A*, which is equal to the number of all planes through the point *Q*. Thus $\sum_{i=0}^{n} S_i = q^2 + q + 1$.

2) $\sum_{i=1}^{n} iS_i = S_1 + 2S_2 + 3S_3 + \cdots + nS_n$.

$S_1, S_2, \cdots, S_n$ represent the numbers of the *i*-secants of the arc *A* through the point *Q* not in *A*. S_1 is the number of the unisecants to *A*, each one passes through one point of *A*. S_2 is the number of the bisecants to *A*, each one passes through two points of *A*. S_3 is the number of the trisecants to *A*, each one passes through three points of *A*. Also, S_n is the number of the *n*-secants to *A*, each one passes through *n* points of *A*. Since the number of points of the (*k*,*n*)-arc *A* is *k*, then $\sum_{i=1}^{n} iS_i = k$.

Notion

Let C_i be the number of points of index *i* in $S = PG(3,q)$ which are not on a complete (*k*,*n*)-arc *A*, then the constants C_i of *A* satisfy the following equations:

i) $\sum_{\alpha}^{\beta} C_i = q^3 + q^2 + q + 1 - k$

ii) $\sum_{\alpha}^{\beta} iC_i = \frac{k(k-1)\cdots(k-n+1)}{n!}(q^2 + q + 1 - n)$

where α is the smallest *i* for which $C_i \neq 0$, β be the largest *i* for which $C_i \neq 0$.

Proof

The equations express in different ways the cardinality of the following sets

i) $\{Q \mid Q \in S \setminus A\}$

ii) $\{(Q,\pi) / Q \in \pi \setminus A, \pi \text{ an } n\text{-secant of } A\}$

for in (i) $\sum_{\alpha}^{\beta} C_i$ represents all points in the space which are not in *A*, then $\sum_{\alpha}^{\beta} C_i = q^3 + q^2 + q + 1 - k$, and in (ii) $\sum_{\alpha}^{\beta} iC_i$ represents all points in the space not in *A*, which are on *n*-secants of *A*, that is, each *n*-secant contains $q^2 + q + 1 - n$ points, and the number of the *n*-secants is $\binom{k}{n}$, then

$$\sum_{\alpha}^{\beta} iC_i = \binom{k}{n}(q^2 + q + 1 - n)$$
$$= \frac{k(k-1)\cdots(k-n+1)}{n!}(q^2 + q + 1 - n)$$

Theorem

If *P* is a point of a (*k*,*n*)-arc *A* in $PG(3,q)$, which lies on an *m*-secant (plane) of *A*, then the planes through *P* contains at most $(n-1)q(q+1)+m$ points of *A*.

Proof

If *P* in *A* lies on an *m*-secant (plane), then every other plane through *P* contains at most *n* − 1 points of A distinct from *P*. Hence the q^2+q+1 planes through *P* contain at most $(n-1)(q^2+q)+m$ points of *A*.

3. The Fundamental Theorem of Projective Geometry

In this section, the characterization of algebraic varieties in $PG(3,q)$ is given as finite sets of points with certain combinatorial properties.

Definition: "Plane *π*" [1]

A plane *π* in $PG(3,p)$ is the set of all points $P(X_1,X_2,X_3,X_4)$ satisfying a linear equation $U_1X_1+U_2X_2+U_3X_3+U_4X_4=0$. This plane is denoted by $\pi[U_1,U_2,U_3,U_4]$. space which consists of points, lines and planes with the incidence relation between them [1].

Notion [1]

A projective 3-space $PG(3,k)$ over a field *K* is a 3-dimensional projective $PG(3,k)$ satisfying the following axioms:

1) Any two distinct points are contained in a unique line.

2) Any three distinct non-collinear points, also any line and point not on the line are contained in a unique plane.

3) Any two distinct coplanar lines intersect in a unique point.

4) Any line not on a given plane intersects the plane in a uniquepoint.

5) Any two distinct planes intersect in a unique line.

A projective space $PG(3,p)$ over Galois field *GF*(*p*), where $p=q^m$ For some prime number *q* and some integer *m*, is a 3-dimensional projective space.

Any point in $PG(3,p)$ has the form of a quadruple (X_1,X_2,X_3,X_4), where X_1,X_2,X_3,X_4 are elements in *GF*(*p*) with the exception of the quadrable consisting of four zero elements. Two quadrable (X_1,X_2,X_3,X_4) and (y_1,y_2,y_3,y_4) represent the same point if there exists *λ* in *GF*(*p*)/{0} such that $(X_1,X_2,X_3,X_4)=\lambda(y_1,y_2,y_3,y_4)$. Similarly, any plane in $PG(3,p)$ has the form of a quadrable $[X_1,X_2,X_3,X_4]$, where X_1,X_2,X_3,X_4, are elements in *GF*(*p*) with the exception of the quadrable consisting off our zero elements.

Two quadrable $[X_1,X_2,X_3,X_4]$ and $[y_1,y_2,y_3,y_4]$ represent the same plane if there exists *λ* in *GF*(*p*)\{0} such that $[X_1,X_2,X_3,X_4]=\lambda[y_1,y_2,y_3,y_4]$.

Finally, a point $P(X_1,X_2,X_3,X_4)$ is incident with the plane $\pi[a_1,a_2,a_3,a_4]$ iff

$$a_1X_1+a_2X_2+a_3X_3+a_4X_4=0.$$

Theorem [1] [19]

The points of $PG(3,p)$ have a unique forms which are $(1,0,0,0),(x,1,0,0),(x,y,1,0),(x,y,z,1)$ for all x,y,z in *GF*(*p*).

There exists one point of the form $(1,0,0,0)$.

There exists p points of the form $(x,1,0,0)$.

There exists p^2 points of the form $(x,y,1,0)$.

There exists p^3 points of the form $(x,y,z,1)$.

Theorem [19]

The planes of $PG(3,p)$ have a unique forms which are:
$[1,0,0,0],[x,1,0,0],[x,y,1,0],[x,y,z,1]$ for all x,y,z in $GF(p)$.

There exists one plane of the form $[1,0,0,0]$.

There exists p planes of the form $[x,1,0,0]$.

There exists p^2 planes of the form $[x,y,1,0]$.

There exists p^3 planes of the form $[x,y,z,1]$.

Notion [1]

In $PG(3,p)$ satisfies the following:

1) Every line contains exactly $p+1$ points and every point is on exactly $p+1$ lines.

2) Every plane contains exactly p^2+p+1 points (lines) and every point is on exactly p^2+p+1 planes.

3) There exist p^3+p^2+p+1 of points and there exists p^3+p^2+p+1 of planes.

4) Any two planes intersect in exactly $p+1$ points and any line is on exactly $P+1$ planes. So, any two points are on exactly $p+1$ planes.

Notion [1]

There exists $(p^2+1)(p^2+p+1)$ of lines in $PG(3,p)$.

Proof

In $PG(3,p)$, there exist p^3+p^2+p+1 planes, and each plane contains exactly p^2+p+1 lines, then the numbers of lines is equal to $(p^3+p^2+p+1)(p^2+p+1)$, but each line is on $p+1$ planes, then there exist exactly $\frac{(p^3+p^2+p+1)(p^2+p+1)}{p+1}=(p^2+1)(p^2+p+1)$ lines in $PG(3,p)$.

Qualifier [1] [19]

A (k,ℓ)-span, $\ell \geq 1$ is a set of k spaces π_ℓ no two of which intersect.

Definition [1]

A maximum (k,ℓ)-span is a set of k spaces π_ℓ which are every points of $PG(3,p)$ lies in exactly one line of the, and every two lines of π_ℓ are disjoint

Qualifier [1] [19]

Every maximum (k,ℓ)-span is a spread.

4. The Projective Space and the (k,ℓ)-Span in $PG(3,7)$

4.1. The Projective Space in $PG(3,7)$

$PG(3,7)$ contains 400 points and 400 planes such that each point is on 57 planes and every plane contains 57 points, any line contains 8 points and it is the intersection of 8 planes, all the points, planes and lines of $PG(3,7)$ are given in Table 1 and Table 2.

Table 1. Points and plane of *PG*(3,7).

i	p_i	π_i
1	(1, 0, 0, 0)	2 9 16 23 30 37 44 51 58 65 72 79 86 93 100 107 114 121 128 135 142 149 156 163 170 177 184 191 198 205 212 219 226 233 240 247 254 261 268 275 282 289 296 303 310 317 324 331 338 345 352 359 366 373 380 387 394
2	(0, 1, 0, 0)	1 9 10 11 12 13 14 15 58 59 60 61 62 63 64 107 108 109 110 111 112 113 156 157 158 159 160 161 162 205 206 207 208 209 210 211 254 255 256 257 258 259 260 303 304 305 306 307 308 309 352 353 354 355 356 357 358
…	…	…
400	(6, 6, 6, 1)	8 15 21 27 33 39 45 51 59 65 78 84 90 96 102 107 120 126 132 138 144 150 162 168 174 180 186 192 198 210 216 222 228 234 240 253 258 264 270 276 282 295 301 306 312 318 324 337 343 349 354 360 366 379 385 391 397

Table 2. Plane and lines of *PG*(3,7).

1	2	2	2	2	2	2	2	2	9	9	9	9	9	9	9	16	16	16	16	16	16	16	23	23	23	23	23	23	23	30	30	30	30	30	30	30
	9	58	107	156	205	254	303	352	58	65	72	79	86	93	100	58	65	72	79	86	93	100	58	65	72	79	86	93	100	58	65	72	79	86	93	100
	16	65	114	163	212	261	310	359	107	114	121	128	135	142	149	114	121	128	135	142	149	107	121	128	135	142	149	107	114	128	135	142	149	107	114	121
	23	72	121	170	219	268	317	366	156	163	170	177	184	191	198	170	177	184	191	198	156	163	184	191	198	156	163	170	177	198	156	163	170	177	184	191
	30	79	128	177	226	275	324	373	205	212	219	226	233	240	247	226	233	240	247	205	212	219	247	205	212	219	226	233	240	219	226	233	240	247	205	212
	37	86	135	184	233	282	331	380	254	261	268	275	282	289	296	282	289	296	254	261	268	275	261	268	275	282	289	296	254	289	296	254	261	268	275	282
	44	93	142	191	240	289	338	387	303	310	317	324	331	338	345	338	345	303	310	317	324	331	324	331	338	345	303	310	317	310	317	324	331	338	345	303
	51	100	149	198	247	296	345	394	352	359	366	373	380	387	394	394	352	359	366	373	380	387	387	394	352	359	366	373	380	380	387	394	352	359	366	373

37	**37**	**37**	**37**	**37**	**37**	**37**	**44**	**44**	**44**	**44**	**44**	**44**	**44**	**51**	**51**	**51**	**51**	**51**	**51**	**51**
58	65	72	79	86	93	100	58	65	72	79	86	93	100	58	65	72	79	86	93	100
135	142	149	107	114	121	128	142	149	107	114	121	128	135	149	107	114	121	128	135	142
163	170	177	184	191	198	156	177	184	191	198	156	163	170	191	198	156	163	170	177	184
240	247	205	212	219	226	233	212	219	226	233	240	247	205	233	240	247	205	212	219	226
268	275	282	289	296	254	261	296	254	261	268	275	282	289	275	282	289	296	254	261	268
345	303	310	317	324	331	338	331	338	345	303	310	317	324	317	324	331	338	345	303	310
373	380	387	394	352	359	366	366	373	380	387	394	352	359	359	366	373	380	387	394	352

	1	1	1	1	1	1	1	1	9	9	9	9	9	9	9	10	10	10	10	10	10	10	11	11	11	11	11	11	11	12	12	12	12	12	12	12
	9	58	107	156	205	254	303	352	58	59	60	61	62	63	64	58	59	60	61	62	63	64	58	59	60	61	62	63	64	58	59	60	61	62	63	64
	10	59	108	157	206	255	304	353	107	108	109	110	111	112	113	108	109	110	111	112	113	107	109	110	111	112	113	107	108	110	111	112	113	107	108	109
	11	60	109	158	207	256	305	354	156	157	158	159	160	161	162	158	159	160	161	162	156	157	160	161	162	156	157	158	159	162	156	157	158	159	160	161
2	12	61	110	159	208	257	306	355	205	206	207	208	209	210	211	208	209	210	211	205	206	207	211	205	206	207	208	209	210	207	208	209	210	211	205	206
	13	62	111	160	209	258	307	356	254	255	256	257	258	259	260	258	259	260	254	255	256	257	255	256	257	258	259	260	254	259	260	254	255	256	257	258
	14	63	112	161	210	259	308	357	303	304	305	306	307	308	309	308	309	303	304	305	306	307	306	307	308	309	303	304	305	304	305	306	307	308	309	303
	15	64	113	162	211	260	309	358	352	353	354	355	356	357	358	358	352	353	354	355	356	357	357	358	352	353	354	355	356	356	357	358	352	353	354	355

13	**13**	**13**	**13**	**13**	**13**	**13**	**14**	**14**	**14**	**14**	**14**	**14**	**14**	**15**	**15**	**15**	**15**	**15**	**15**	**15**
58	59	60	61	62	63	64	58	59	60	61	62	63	64	58	59	60	61	62	63	64
111	112	113	107	108	109	110	112	113	107	108	109	110	111	113	107	108	109	110	111	112
157	158	159	160	161	162	156	159	160	161	162	156	157	158	161	162	156	157	158	159	160
210	211	205	206	207	208	209	206	207	208	209	210	211	205	209	210	211	205	206	207	208
256	257	258	259	260	254	255	260	254	255	256	257	258	259	257	258	259	260	254	255	256
309	303	304	305	306	307	308	307	308	309	303	304	305	306	305	306	307	308	309	303	304
355	356	357	358	352	353	354	354	355	356	357	358	352	353	353	354	355	356	357	358	352

	8	8	8	8	8	8	8	8	15	15	15	15	15	15	15	21	21	21	21	21	21	21	27	27	27	27	27	27	27	33	33	33	33	33	33	33
	15	59	107	162	210	258	306	354	59	65	78	84	90	96	102	59	65	78	84	90	96	102	59	65	78	84	90	96	102	59	65	78	84	90	96	102
	21	65	120	168	216	264	312	360	107	120	126	132	138	144	150	120	126	132	138	144	150	107	126	132	138	144	150	107	120	132	138	144	150	107	120	126
400	27	78	126	174	222	270	318	366	162	168	174	180	186	192	198	174	180	186	192	198	162	168	186	192	198	162	168	174	180	198	162	168	174	180	186	192
	33	84	132	180	228	276	324	379	210	216	222	228	234	240	253	228	234	240	253	210	216	222	253	210	216	222	228	234	240	222	228	234	240	253	210	216
	39	90	138	186	234	282	337	385	258	264	270	276	282	295	301	282	295	301	258	264	270	276	264	270	276	282	295	301	258	295	301	258	264	270	276	282
	45	96	144	192	240	295	343	391	306	312	318	324	337	343	349	343	349	306	312	318	324	337	324	337	343	349	306	312	318	312	318	324	337	343	349	306
	51	102	150	198	253	301	349	397	354	360	366	379	385	391	397	397	354	360	366	379	385	391	391	397	354	360	366	379	385	385	391	397	354	360	366	379

39	**39**	**39**	**39**	**39**	**39**	**39**	**45**	**45**	**45**	**45**	**45**	**45**	**45**	**51**	**51**	**51**	**51**	**51**	**51**	**51**
59	65	78	84	90	96	102	59	65	78	84	90	96	102	59	65	78	84	90	96	102
138	144	150	107	120	126	132	144	150	107	120	126	132	138	150	107	120	126	132	138	144
168	174	180	186	192	198	162	180	186	192	198	162	168	174	192	198	162	168	174	180	186
240	253	210	216	222	228	234	216	222	228	234	240	253	210	234	240	253	210	216	222	228
270	276	282	295	301	258	264	301	258	264	270	276	282	295	276	282	295	301	258	264	270
349	306	312	318	324	337	343	337	343	349	306	312	318	324	318	324	337	343	349	306	312
379	385	391	397	354	360	366	366	379	385	391	397	354	360	360	366	379	385	391	397	354

Table 3. Spread in $PG(3,7)$.

ƫ$_i$				ʅ$_i$					(k_i,ʅ)-span
ξ	1	2	3	4	5	6	7	8	(1,ʅ)-span
ν	9	58	107	156	205	254	303	352	(2,ʅ)-span
μ	10	65	115	165	215	265	315	365	(3,ʅ)-span
θ	11	72	123	174	225	269	320	371	(4,ʅ)-span
η	12	79	131	183	228	280	325	377	(5,ʅ)-span
ζ	13	86	139	185	238	284	337	383	(6,ʅ)-span
ε	14	93	147	194	241	295	342	389	(7,ʅ)-span
δ	15	100	155	203	251	299	347	395	(8,ʅ)-span
γ	16	61	117	173	229	285	341	397	(9,ʅ)-span
β	17	68	125	182	239	289	346	354	(10,ʅ)-span

Continued

α	18	75	133	184	242	300	309	360	(11,ʅ)-span
ϑ	19	82	141	193	252	255	314	366	(12,ʅ)-span
ϐ	20	89	142	202	206	266	319	379	(13,ʅ)-span
ώ	21	96	150	162	216	270	324	385	(14,ʅ)-span
ύ	22	103	109	164	219	281	336	391	(15,ʅ)-span
ό	23	64	127	190	253	267	330	393	(16,ʅ)-span
ϋ	24	71	128	192	207	271	335	399	(17,ʅ)-span
ϊ	25	78	136	201	217	275	340	356	(18,ʅ)-span
ω	26	85	144	161	220	286	345	362	(19,ʅ)-span
ψ	27	92	152	163	230	290	308	368	(20,ʅ)-span
σ	28	99	111	172	233	301	313	374	(21,ʅ)-span
ς	29	106	119	181	243	256	318	380	(22,ʅ)-span
ϡ	30	60	130	200	221	291	312	382	(23,ʅ)-span
Ђ	31	67	138	160	231	302	317	388	(24,ʅ)-span
ϟ	32	74	146	169	234	257	329	394	(25,ʅ)-span
Ϝ	33	81	154	171	244	261	334	358	(26,ʅ)-span
Ϙ	34	88	113	180	247	272	339	364	(27,ʅ)-span
ϖ	35	95	114	189	208	276	351	370	(28,ʅ)-span
ϱ	36	102	122	191	218	287	307	376	(29,ʅ)-span
Ϭ	37	63	140	168	245	273	350	378	(30,ʅ)-span
Ϫ	38	70	148	170	248	277	306	384	(31,ʅ)-span
Ϧ	39	77	149	179	209	288	311	390	(32,ʅ)-span
Ϣ	40	84	108	188	212	292	323	396	(33,ʅ)-span
Ͼ	41	91	116	197	222	296	328	353	(34,ʅ)-span
Б	42	98	124	199	232	258	333	359	(35,ʅ)-span
Џ	43	105	132	159	235	262	338	372	(36,ʅ)-span
Ў	44	59	143	178	213	297	332	367	(37,ʅ)-span
Ѝ	45	66	151	187	223	259	344	373	(38,ʅ)-span
Ћ	46	73	110	196	226	263	349	386	(39,ʅ)-span
Њ	47	80	118	198	236	274	305	392	(40,ʅ)-span
Љ	48	87	126	158	246	278	310	398	(41,ʅ)-span
Ђ	49	94	134	167	249	282	322	355	(42,ʅ)-span
Л	50	101	135	176	210	293	327	361	(43,ʅ)-span
Щ	51	62	153	195	237	279	321	363	(44,ʅ)-span
Ю	52	69	112	204	240	283	326	369	(45,ʅ)-span
Ы	53	76	120	157	250	294	331	375	(46,ʅ)-span
ж	54	83	121	166	211	298	343	381	(47,ʅ)-span
ц	55	90	129	175	214	260	348	387	(48,ʅ)-span
Ϥ	56	97	137	177	224	264	304	400	(49,ʅ)-span
ϾϽ	57	104	145	186	227	268	316	357	(50,ʅ)-span

4.2. The (*k*,ℓ)-Span in *PG*(3,*p*)

In Table 3 any two non-intersecting lines can be taken in *PG*(3,7).

In Table 3 any elements of the set $ƫ_i = \{\xi, \nu, \mu, \cdots, GƆ\}$ except the first element can be representing by union of below set and non-intersecting of them.

Finally, the line GƆ = {57, 104, 145, 186, 227, 268, 316, 357} cannot intersect any line of the set ($ţ_i$) and (GƆ) is (50,ℓ)-span, which is the maximum (*k*,ℓ)-span of $PG(3,7)$ can be obtained. Thus GƆ is called a Spread of fifty lines of $PG(3,7)$ which partitions $PG(3,7)$; that every point of $PG(3,7)$ lies in exactly one line of $ƫ_i$. and every line are disjoint. From the above results the number of the planes in the projective space

$PG(3,7)$ are 400 planes and each plane contains 57 lines, therefore the total number of the lines in $PG(3,7)$ are 22,800. We found that the number of the lines do not intersect with some of them are fifty lines, these lines contains the whole points of the projective space $PG(3,7)$, and called him a (50,ℓ)-span, *i.e.*

$$(50,ℓ)\text{-span} = \{ℓ_1, ℓ_2, \cdots, ℓ_{50}\} = PG(3,7) = \{1, 2, 3, \cdots, 400\}$$

Moreover, we found that a (50,ℓ)-*span* is a maximum complete (*k*,ℓ)-span in $PG(3,7)$.

5. Conclusions

1) Complete Arcs and Surfaces are constructed in $PG(3,q)$ over Galois field *GF*(*q*), $q = 7$ by two methods and some theorems are proved on the (*k*,*n*)-arc of $PG(3,q)$ and on a 3-dimensional projective space.

2) We prove, the number of spread in projective space $PG(3,p)$ where *p* is prime, and $P \geq 2$ is $p^2 + 1$.

Proof

In $PG(3,p)$, there are $p^3 + p^2 + p + 1$ planes, but each line is on $p + 1$ planes; then there is exactly $\frac{p^3 + p^2 + p + 1}{p + 1} = p^2 + 1$ spread in $PG(3,p)$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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