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Bounded Turning of an *m*-th Partial Sum of Modified Caputo's Fractional Calculus Derivative Operator

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Abstract

In this article, we consider subclasses of functions with bounded turning for normalized analytic functions in the unit disk, we investigate certain conditions under which the partial sums of the modified Caputo's fractional derivative operators of analytic univalent functions of bounded turning are also of bounded turning.

Subject Areas

Mathematical Analysis

Keywords

Analytic Functions, Close-to-Convex, Bounded Turning, Univalent

1. Introduction and Definitions

Let $\mathcal{A}$ denote a class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$

Definition 1.

Let $B(\mu), 0 \le \mu < 1$ denote the class of functions of the Form (1.1) then if $\Re\{f'\} > \mu$, that is the real part of its first derivative map the unit disk onto the right half plane, then the class of functions in $B(\mu)$ are called functions of bounded turning.

By Nashiro Warschowski, see [1], it is proved that the functions in $B(\mu)$ are univalent and also close to convex in *U*. In [2], it was also shown that the partial sums of the Libera integral operator of functions of bounded turning are also of bounded turning. For more works on bounded turning see [3] [4].

Definition 2.

If $f(z)=\sum_{n=0}^{\infty}a_n z^n$ and $g(z)=\sum_{n=0}^{\infty}b_n z^n$ are analytic in *U*, then their Hadamard product $f*g$ defined by the power series is given by:

$$(f*g)(z)=\sum_{n=0}^{\infty}a_n b_n z^n. \tag{1.2}$$

Note that the convolution so defined is also analytic in *U*.

For f of the Form (1.1) several interesting derivatives operators in their different forms have been studied, here we consider (1.1) using the modified Caputo's derivative operator $J_{\eta,\lambda}f(z)$, see [5] [6], stated as follow:

$$\text{For } f\in\mathcal{A},\ J_{\eta,\lambda}f(z)=\frac{2+\eta-\lambda}{\eta-\lambda}z^{\lambda-\eta}\int_0^z\frac{\Omega^{\eta}f(\xi)}{(z-\xi)^{\lambda+1-\eta}}\mathrm{d}\xi \tag{1.3}$$

where η is a real number and $\eta-1<\lambda\le\eta<2$. Notice that (1.3) can also be express as:

$$J_{\eta,\lambda}f(z)=z+\sum_{n=2}^{\infty}\frac{(n+1)^2(2+\eta-\lambda)(2-\eta)}{(n+\eta-\lambda+1)(n-\eta+1)}a_n z^n \tag{1.4}$$

and its partial sum given as:

$$P_M(z)=z+\sum_{n=2}^{M}\frac{(n+1)^2(2+\eta-\lambda)(2-\eta)}{(n+\eta-\lambda+1)(n-\eta+1)}a_n z^n \tag{1.5}$$

We determine conditions under which the partial sums of the operator given in (1.4) are of bounded turning. We shall use the following lemmas in the sequel to establish our result.

Lemma 1. [7]

For $z\in U$, we have

$$\Re\left\{\sum_{n=1}^{\infty}\frac{z^n}{n+2}\right\}>-\frac{1}{3},(z\in U) \tag{1.6}$$

Lemma 2. [1]

Let *P*(*z*) be analytic in *U*, such that *P*(0) = 1, and $\Re(P(z))>\frac{1}{2}$ in *U*. For function Q analytic in U the convolution function $P*Q$ takes values in the convex hull of the image U under *Q*.

We shall implore lemmas 1 and 2 to show conditions under which the m-th partial sum (2.1) of the modified Caputoes derivative operator of analytic univalent functions of bounded turning is also of bounded turning.

2. Main Theorem

Let $f(z)\in\mathcal{A}$ be of the Form (1.1), if $\frac{1}{2}<\mu<1$ and $f(z)\in B(\mu)$, then

$$P_M(z) \in B\left(\frac{\left(3-(2+\eta-\lambda)(2-\eta)(1-\mu)\right)}{3}\right), \quad \eta-1<\lambda\le\eta<2.$$

Proof.

Let $f(z)$ be of the Form (1.1) and $\Re\{f'(z)\}>\mu, \frac{1}{2}<\mu<1, z\in U$. This implies that

$$\Re\left\{1+\sum_{n=2}^{\infty} n a_n z^{n-1}\right\} > \frac{\mu}{2} \tag{2.1}$$

Now for $\frac{1}{2}<\mu<1$ we have

$$\Re\left\{1+\sum_{n=2}^{\infty} a_n \frac{n}{1-\mu} z^{n-1}\right\} > \Re\left\{1+\sum_{n=2}^{\infty} n a_n z^{n-1}\right\} \tag{2.2}$$

Applying the convolution properties to $P'(z)$, where

$$P_M'(z) = 1+\sum_{n=2}^{M} \frac{n(n+1)^2(2+\eta-\lambda)(2-\eta)}{(n+\eta-\lambda+1)(n-\eta+1)} a_n z^{n-1} \tag{2.3}$$

$$\left[\left\{1+\sum_{n=2}^{\infty} a_n \frac{n}{1-\mu} z^{n-1}\right\}\right] * \left[1+\sum_{n=2}^{M} \frac{n(n+1)^2(2+\eta-\lambda)(2-\eta)}{(n+\eta-\lambda+1)(n-\eta+1)}(1-\mu) a_n z^{n-1}\right] \tag{2.4}$$
$$= P(z) * Q(z)$$

with recourse for Lemma 1 and $J=m-1$ we have

$$\Re\left\{\sum_{n=2}^{M} \frac{z^{n-1}}{n+1}\right\} > -\frac{1}{3} \tag{2.5}$$

Then for $\eta-1<\lambda\le\eta<2$

$$\Re\left\{\sum_{n=2}^{M} \frac{z^{n-1}}{\left(n(n+1)^2\right)^{-1}(2+\eta-\lambda)(2-\eta)(n+\eta-\lambda+1)(n-\eta+1)(1-\mu)a_n z^{n-1}}\right\} \tag{2.6}$$
$$\ge \Re\left\{\sum_{n=2}^{M} \frac{z^{n-1}}{n+1}\right\}$$

Hence

$$\Re\left\{\sum_{n=2}^{M} \frac{z^{n-1}}{\left(n(n+1)^2\right)^{-1}(2+\eta-\lambda)(2-\eta)(n+\eta-\lambda+1)(n-\eta+1)(1-\mu)a_n z^{n-1}}\right\} \tag{2.7}$$
$$\ge -\frac{1}{3}$$

Relating Lemma 1 and with $Q(z)$, a computation gives

$$\begin{aligned}\Re Q(z) &= \left\{1+\sum_{n=2}^{M} \frac{n(n+1)^2(2+\eta-\lambda)(2-\eta)}{(n+\eta-\lambda+1)(n-\eta+1)}(1-\mu) a_n z^{n-1}\right\} \\ &> \frac{3-(2+\eta-\lambda)(2-\eta)(1-\mu)}{3}\end{aligned} \tag{2.8}$$

Recall the power series

$$P(z)=\left\{1+\sum_{n=2}^{\infty}a_n\frac{n}{1-\mu}z^{n-1}\right\}, z\in U \tag{2.9}$$

satisfies $p(0)=1$ and $\Re(P(z))=\Re\left\{1+\sum_{n=2}^{\infty}a_n\frac{n}{1-\mu}z^{n-1}\right\}>\frac{1}{2}, z\in U$. Therefore by Lemma 2 we have

$$\Re(P'(z))>\frac{3-(2+\eta-\lambda)(2-\eta)(1-\mu)}{3}, z\in U \tag{2.10}$$

This proves our results.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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