Suzuki-Type Fixed Point Theorem in $b_2$-Metric Spaces

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Abstract
In this paper, we establish a fixed point theorem for two mappings under a contraction condition in $b_2$-metric space, and this theorem is related to a Suzuki-type of contraction.

Subject Areas
Mathematical Analysis

Keywords
Common Fixed Point, $b_2$-Metric Space, Generalized Suzuki-Type Contraction

1. Introduction
Banach [1] proved a principle, and this famous Banach contraction principle has many generalizations, see [2]-[7], and in 2008, Suzuki [8] established one of those generalizations, and this generalization is called Suzuki principle.

The aim of this paper is to prove a fixed point result generalized from the above mentioned principle in $b_2$-metric space [9].

2. Preliminaries
Before giving our results, these definitions and results as follows will be needed to present.

Definition 2.1 [9] Let $X$ be a nonempty set, $s \geq 1$ be a real number and let $d: X \times X \times X \rightarrow R$ be a map satisfying the following conditions:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

2) If at least two of three points $x, y, z$ are the same, then $d(x, y, z) = 0$.

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3) The symmetry:
\[ d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x) \quad \text{for all} \quad x, y, z \in X. \]

1) The rectangle inequality:
\[ d(x, y, z) \leq s[d(x, y, a) + d(y, z, a) + d(z, x, a)], \quad \text{for all} \quad x, y, z, a \in X. \]

Then \( d \) is called a \( b_2 \) metric on \( X \) and \( (X, d) \) is called a \( b_2 \) metric space with parameter \( s \). Obviously, for \( s = 1 \), \( b_2 \) metric reduces to 2-metric.

**Definition 2.2** [9] Let \( \{x_n\} \) be a sequence in a \( b_2 \) metric space \( (X, d) \).

1) \( \{x_n\} \) is said to be \( b_2 \)-convergent to \( x \in X \), written as
\[ \lim_{n \to \infty} x_n = x, \]
if all \( a \in X \) \( \lim_{n \to \infty} d(x_n, x, a) = 0. \)

2) \( \{x_n\} \) is Cauchy sequence if and only if \( d(x_n, x_m, a) \to 0 \), when \( n, m \to \infty \) for all \( a \in X \).

3) \( (X, d) \) is said to be complete if every \( b_2 \)-Cauchy sequence is a \( b_2 \)-convergent sequence.

**Definition 2.3** [9] Let \( (X, d) \) and \( (X', d') \) be two \( b_2 \)-metric spaces and let \( f : X \to X' \) be a mapping. Then \( f \) is said to be \( b_2 \)-continuous, at a point \( z \in X \) if for a given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( d(z, x, a) < \delta \) for all \( a \in X \) imply that \( d'(fz, fx, a) < \varepsilon. \) The mapping \( f \) is \( b_2 \)-continuous on \( X \) if it is \( b_2 \)-continuous at all \( z \in X \).

**Definition 2.4** [9] Let \( (X, d) \) and \( (X', d') \) be two \( b_2 \)-metric spaces. Then a mapping \( f : X \to X' \) is \( b_2 \)-continuous at a point \( x \in X \) if and only if it is \( b_2 \)-sequentially continuous at \( x \); that is, whenever \( \{x_n\} \) is \( b_2 \)-convergent to \( x \), \( \{fx_n\} \) is \( b_2 \)-convergent to \( f(x) \).

**Lemma 2.5** [9] Let \( (X, d) \) be a \( b_2 \)-metric space and suppose that \( \{x_n\} \) and \( \{y_n\} \) are \( b_2 \)-convergent to \( x \) and \( y \), respectively. Then we have
\[ \frac{1}{s} d(x, y, a) \leq \liminf_{n \to \infty} d(x_n, y_n, a) \leq \limsup_{n \to \infty} d(x_n, y_n, a) \leq s d(x, y, a), \quad \text{for all} \quad a \in X. \]

In particular, if \( y_n = y \) is a constant, then
\[ \frac{1}{s} d(x, y, a) \leq \liminf_{n \to \infty} d(x_n, y, a) \leq \limsup_{n \to \infty} d(x_n, y, a) \leq s d(x, y, a), \quad \text{for all} \quad a \in X. \]

**Lemma 2.6** [10] Let \( (X, d) \) be a \( b_2 \) metric space with \( s \geq 1 \) and let \( \{x_n\}_{n=0}^\infty \) be a sequence in \( X \) such that
\[ d(x_n, x_{n+1}, a) \leq \lambda d(x_{n-1}, x_n, a), \quad (2.1) \]
for all \( n \in \mathbb{N} \) and all \( a \in X \), where \( \lambda \in \left[0, \frac{1}{s}\right] \). Then \( \{x_n\} \) is a \( b_2 \)-Cauchy sequence in \( (X, d) \).

### 3. Main Results

**Theorem 3.1.** Let \( (X, d) \) be a complete \( b_2 \)-metric space. Let \( f, g : X \to X \) be two self-maps and \( \phi : [0,1) \to \left[0, \frac{1}{2} \right] \) be defined as follows
Assume there exists \( \rho \in [0,1) \) such that for every \( x,y \in X \), the following condition is satisfied

\[
\frac{1}{s} \phi(\rho) \min \{d(x,fx,a),d(fx,fy,a)\} \leq d(x,y,a)
\]

(3.2)

\[\Rightarrow \max \{d(gx,gy,a),d(gx,fy,a),d(fx,fy,a),d(gx,gy,a)\} \leq \frac{\rho}{s^2} d(x,y,a).\]

Then \( f,g \) have a unique common fixed point \( z \in X \).

Proof in (3.2), we take \( y = fx \)

\[
\frac{1}{s} \phi(\rho) \min \{d(x,fx,a),d(x,gy,a)\} \leq d(x,gy,a)
\]

(3.3)

\[\Rightarrow \max \{d(gx,g^2x,a),d(gx,fgx,a),d(fx,fgx,a),d(g^2x,fx,a)\} \leq \frac{\rho}{s^2} d(x,gy,a),\]

Therefore,

\[
d(gx,fgx,a) \leq \frac{\rho}{s^2} d(x,gy,a).
\]

(3.4)

Now we take \( y = fx \) in (3.2)

\[
\frac{1}{s} \phi(\rho) \min \{d(x,fx,a),d(x,gy,a)\} \leq d(x,fy,a)
\]

(3.5)

\[\Rightarrow \max \{d(gx,gy,a),d(gx,f^2y,a),d(fx,f^2y,a),d(gfx,fx,a)\} \leq \frac{\rho}{s^2} d(x,gy,a),\]

Therefore,

\[
d(fx,f^2x,a) \leq \frac{\rho}{s^2} d(x,fx,a).
\]

(3.6)

And

\[
d(gfx,fx,a) \leq \frac{\rho}{s^2} d(x,fx,a).
\]

(3.7)

Given an arbitrary point \( x_0 \) in \( X \) then by \( x_{2n+1} = gx_{2n} \) and \( x_{2n+1} = fx_{2n+1} \) we construct a sequence \( \{x_n\} \), for \( n \in N \).

From (3.4), we get

\[
d(x_{2n+1},x_{2n+2},a) = d(gx_{2n},fgx_{2n},a) \leq \frac{\rho}{s^2} d(x_{2n},gx_{2n},a) = \frac{\rho}{s^2} d(x_{2n},x_{2n+1},a).
\]

(3.8)
\[ d(x_{n+1}, x_n, a) = d(gf^{n-1}x_n, f_kx_{n-1}, a) \leq \frac{\rho}{s^2} d(x_{n-1}, f_kx_{n-1}, a) = \frac{\rho}{s^2} d(x_{n-1}, x_n, a), \]
that is,
\[ d(x_{n+1}, x_n, a) \leq \frac{\rho}{s^2} d(x_n, x_{n-1}, a), \]
since \( \frac{\rho}{s^2} \in [0, 1) \), by Lemma 2.6, we get \( \{x_n\} \) is a Cauchy sequence.

Since \( X \) is complete, there exists \( z \) in \( X \), such that \( \lim_{n \to \infty} x_n = z \), that is \( \lim_{n \to \infty} gx_{2n} = \lim_{n \to \infty} x_{2n+1} = z \), and \( \lim_{n \to \infty} fx_{2n+1} = \lim_{n \to \infty} x_{2n+2} = z \).

Now let us give that
\[ d(fz, z, a) \leq \rho d(x, z, a), \]
for every \( x \neq z \). For \( \{d(x_n, gx_{2n}, a)\} \) is convergent to 0, and by Lemma 2.5, we get
\[ \frac{1}{s} d(x, z, a) \leq \lim \sup_{n \to \infty} d(x_n, x, a), \]
thus we have \( \lim \sup_{n \to \infty} d(x_n, x, a) > 0 \), thus from the above relation, there exists a point \( x_{2n} \) in \( X \) such that
\[ \frac{1}{s} \phi(\rho) \min \left\{ d(x_{2n}, gx_{2n}, a), d(x_{2n}, fx_{2n}, a) \right\} \leq d(x_{2n}, x, a). \]

For such \( x_{2n} \), (3.2) implies that
\[ d(gx_{2n}, fz, a) \leq \max \left\{ d(gx_{2n}, gx, a), d(gx_{2n}, fx, a), d(fz, fx, a) \right\} \leq \frac{\rho}{s^2} d(x_{2n}, x, a), \]
therefore by Lemma 3.5,
\[ \frac{1}{s} d(fz, z, a) \leq \lim \sup_{n \to \infty} d(gx_{2n}, fx, a) \leq \frac{\rho}{s^2} \lim \sup_{n \to \infty} d(x_{2n}, x, a) \leq \frac{\rho}{s} d(x, z, a), \]
therefore we get
\[ d(fz, z, a) \leq \rho d(x, z, a), \] for each \( x \neq z \). (3.9)

Now we show that for each \( n \in N \),
\[ d(f^n z, z, a) \leq d(fz, z, a), \] (3.10)

It is obvious that the above inequality is true for \( n = 1 \), assume that the relation holds for some \( m \in N \). We get (3.10) is true when we have \( f^m z = fz \) if \( f^m z = z \), then if \( f^m z \neq z \), we get the following relation from (3.9) and induction hypothesis, and that is
\[ d(z, f^{m+1}z, a) \leq \rho d(z, f^m z, a) \leq \rho^2 d(z, f^{m-1}z, a) \leq \cdots \leq \rho^{m+1} d(z, fz, a) \leq \rho d(fz, z, a) \leq d(fz, z, a), \]
then (3.10) is proved.

Now we consider the following two possible cases in order to prove that \( f \) has a fixed point \( z \) in \( X \), and that is \( fz = z \).

Case 1 \( 0 \leq \rho < \frac{1}{\sqrt{2}} \), therefore, \( \phi(\rho) \leq \frac{1-\rho}{\rho^2} \). First, we prove the following
relation
\[ d\left( f^n z, fz, a \right) \leq \frac{\rho}{s} d\left( f z, z, a \right), \quad \text{for } n \in \mathbb{N}. \]  
(3.11)

When \( n = 1 \) it is obvious, and it follows from (3.6) when \( n = 2 \), from (3.10) and take \( a = fz \) we have
\[ d\left( f^n z, z, fz \right) \leq d\left( f z, z, fz \right) = 0, \]
then we get \( d\left( f^n z, fz, z \right) = 0 \).

Now suppose that (3.11) holds for some \( n > 2 \),
\[ d\left( f z, z, a \right) \leq s\left( d\left( z, f^n z, a \right) + d\left( f^n z, fz, a \right) + d\left( f^n z, fz, z \right) \right) \]
\[ \leq sd\left( z, f^n z, a \right) + sd\left( z, fz, a \right), \]
then we get
\[ (1 - \rho) d\left( z, fz, a \right) \leq sd\left( z, f^n z, a \right), \]
that is \( d\left( z, fz, a \right) \leq \frac{s}{1 - \rho} sd\left( z, f^n z, a \right), \) (3.11.1)
then by taking \( x = f^{n-1} z \) in (3.6)
\[ d\left( f^n z, f^{n+1} z, a \right) \leq \frac{\rho}{s} d\left( f^{n+1} z, f^n z, a \right) \leq \cdots \leq \frac{\rho^n}{s^n} d\left( z, fz, a \right), \]  
(3.11.2)
using the above two relations, (3.11.1) and (3.11.2) we have
\[ \frac{1}{s} d\left( \rho \right) \min \left\{ d\left( gf^n z, f^n z, a \right), d\left( f^n z, f^{n+1} z, a \right) \right\} \]
\[ \leq 1 - \rho \frac{d\left( f^n z, f^{n+1} z, a \right)}{s s} \leq 1 - \rho \frac{d\left( f^n z, f^{n+1} z, a \right)}{s s} \leq 1 - \rho \frac{d\left( f^n z, f^{n+1} z, a \right)}{s s} \leq 1 - \rho \frac{d\left( f^n z, f^{n+1} z, a \right)}{s s} \leq \frac{1}{s} d\left( z, f^n z, a \right) \leq \frac{1}{s} d\left( z, f^n z, a \right). \]

From (3.2) and (3.10) with \( x = f^n z \) and \( y = z \), we have
\[ \max \left\{ d\left( gf^n z, g z, a \right), d\left( f^n z, fz, a \right), d\left( f^{n+1} z, fz, a \right), d\left( g z, f^{n+1} z, a \right) \right\} \]
\[ \leq \frac{\rho}{s} d\left( z, f^n z, a \right) \leq \frac{\rho}{s} d\left( z, f^n z, a \right) \leq \frac{\rho}{s} d\left( z, fz, a \right). \]
Therefore,
\[ d\left( f^{n+1} z, z, a \right) \leq \frac{\rho}{s} d\left( f z, z, a \right). \]  
(3.12)

So by induction we prove the relation of (3.11).

Now (3.11) and \( fz \neq z \) show that for every \( n \in \mathbb{N} \) \( f^n z \neq z \), thus, (3.9) shows that
\[ d\left( z, f^{n+1} z, a \right) \leq \rho d\left( z, f^n z, a \right) \leq \rho^2 d\left( z, f^{n+1} z, a \right) \leq \cdots \leq \rho^n d\left( z, fz, a \right). \]

Therefore \( \lim_{n \to \infty} d\left( z, f^{n+1} z, a \right) = 0 \). Furthermore by using Lemma 2.5, we get
\[ \frac{1}{s} d\left( z, \liminf_{n \to \infty} f^{n+1} z, a \right) \leq \liminf_{n \to \infty} d\left( z, f^{n+1} z, a \right) = 0, \]
so
\[ d\left( z, \liminf_{n \to \infty} f^{n+1} z, a \right) = 0. \]
In the same way,
\[ d\left(z, \limsup_{n \to \infty} f^{n+1}z, a\right) = 0, \] thus we have
\[ d\left(z, \lim f^{n+1}z, a\right) = 0, \] that is \( f^{n+1}z \to z \), and by using Lemma 2.5 in (3.12), we get
\[ \frac{1}{s} d\left(z, f_z, a\right) \leq \limsup_{n \to \infty} d\left(f^{n+1}z, f_z, a\right) \leq \frac{\rho}{s} d\left(z, f_z, a\right), \] which claims that
\[ d\left(z, f_z, a\right) = 0, \] and that is a contraction.

Case 2. \( \frac{1}{\sqrt{2}} \leq \rho < 1 \), and that is when \( \phi(\rho) = \frac{1}{1 + \rho} \). We now prove that we can find a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that
\[ \frac{1}{s(1 + \rho)} \min \left\{ d\left(x_{n_k}, g_{x_{n_k}}, a\right), d\left(x_{n_k}, f_{x_{n_k}}, a\right)\right\} \leq d\left(x_{n_k}, z, a\right), \] for \( k \in N \). (3.13)

The contraries of the above relation are as follows
\[ \frac{1}{s(1 + \rho)} d\left(x_{n_k}, f_{x_{n_k}}, a\right) \geq \frac{1}{s(1 + \rho)} \min \left\{ d\left(x_{n_k}, g_{x_{n_k}}, a\right), d\left(x_{n_k}, f_{x_{n_k}}, a\right)\right\} > d\left(x_{n_k}, z, a\right), \] and
\[ \frac{1}{s(1 + \rho)} d\left(x_{n_k}, f_{x_{n_k}}, a\right) \geq \frac{1}{s(1 + \rho)} \min \left\{ d\left(x_{n_k}, g_{x_{n_k}}, a\right), d\left(x_{n_k}, f_{x_{n_k}}, a\right)\right\} > d\left(x_{n_k}, z, a\right), \] for \( n \in N \). If \( n \) is even we have
\[ \frac{1}{s(1 + \rho)} d\left(x_{2n}, g_{x_{2n}}, a\right) \]
\[ \geq \frac{1}{s(1 + \rho)} \min \left\{ d\left(x_{2n}, g_{x_{2n}}, a\right), d\left(x_{2n}, f_{x_{2n}}, a\right)\right\} \]
\[ > d\left(x_{2n}, z, a\right), \]
if \( n \) is odd then we get
\[ \frac{1}{s(1 + \rho)} d\left(x_{2n+1}, f_{x_{2n+1}}, a\right) \]
\[ \geq \frac{1}{s(1 + \rho)} \min \left\{ d\left(x_{2n+1}, g_{x_{2n+1}}, a\right), d\left(x_{2n+1}, f_{x_{2n+1}}, a\right)\right\} \]
\[ > d\left(x_{2n+1}, z, a\right), \]
for \( n \in N \). By (3.8) we have
\[ d\left(x_{2n}, x_{2n+1}, a\right) \]
\[ \leq s \left(d\left(x_{2n}, z, a\right) + d\left(x_{2n+1}, z, a\right) + d\left(x_{2n}, x_{2n+1}, z\right)\right) \]
\[ < \frac{s}{s(1 + \rho)} d\left(x_{2n}, g_{x_{2n}}, a\right) + \frac{s}{s(1 + \rho)} d\left(x_{2n+1}, f_{x_{2n+1}}, a\right) \]
\[ + \frac{s}{s(1 + \rho)} d\left(x_{2n}, g_{x_{2n}}, x_{2n+1}\right) \]
\[ = \frac{1}{1 + \rho} \left(d\left(x_{2n}, x_{2n+1}, a\right) + d\left(x_{2n+1}, x_{2n+2}, a\right) + d\left(x_{2n}, x_{2n+1}, x_{2n+1}\right)\right) \]
\[ \leq \frac{1}{1+\rho} d(x_{2n}, x_{2n+1}, a) + \frac{\rho}{s^2} d(x_{2n+1}, x_{2n}, a) \]
\[ = \frac{1}{1+\rho} d(x_{2n}, x_{2n+1}, a) + \frac{\rho}{1+\rho} d(x_{2n+1}, x_{2n}, a) \]
\[ = d(x_{2n}, x_{2n+1}, a), \]

this is impossible. Therefore, one of the following relations is true for every \( n \in N \),

\[ \frac{1}{s} \phi(\rho) \min \{d(x_{2n}, gx_{2n}, a), d(x_{2n}, fx_{2n}, a)\} \leq d(x_{2n}, z, a), \]

or

\[ \frac{1}{s} \phi(\rho) \min \{d(x_{2n+1}, gx_{2n+1}, a), d(x_{2n+1}, fx_{2n+1}, a)\} \leq d(x_{2n+1}, z, a). \]

That means there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that (3.13) is true for every \( k \in N \). Thus (3.2) shows that

\[ d(gx_{2n}, fz, a) \]
\[ \leq \max \{d(fx_{2n}, gz, a), d(fz, gx_{2n}, a), d(fx_{2n}, fz, a), d(gz, fx_{2n}, a)\} \]
\[ \leq \frac{p}{s^2} d(x_{2n}, z, a). \]

or

\[ d(fx_{2n+1}, fz, a) \]
\[ \leq \max \{d(gx_{2n+1}, gz, a), d(fz, gx_{2n+1}, a), d(fx_{2n+1}, fz, a), d(gz, fx_{2n+1}, a)\} \]
\[ \leq \frac{p}{s^2} d(x_{2n+1}, z, a). \]

From Lemma 2.5, we have

\[ \frac{1}{s} d(z, fz, a) \leq \limsup_{n \to \infty} d(gx_{2n}, fz, a) \leq \frac{p}{s^2} \limsup_{n \to \infty} d(x_{2n}, z, a) \leq \frac{p}{s} d(z, z, a) = 0, \]

or

\[ \frac{1}{s} d(z, fz, a) \leq \limsup_{n \to \infty} d(fx_{2n+1}, fz, a) \leq \frac{p}{s^2} \limsup_{n \to \infty} d(x_{2n+1}, z, a) \leq \frac{p}{s} d(z, z, a) = 0, \]

Therefore \( d(z, fz, a) \leq 0 \), which is impossible unless \( fz = z \). Hence \( z \) in \( X \) is a fixed point of \( f \). From the process of the above proof, we know \( fz = z \), then by

\[ 0 = \frac{1}{s} \phi(\rho) \min \{d(z, fz, a), d(z, gz, a)\} \leq d(z, fz, a), \]

it implies

\[ d(gz, z, a) \leq \max \{d(gz, gfz, a), d(gz, f^2z, a), d(fz, f^2z, a), d(gfz, fz, a)\} \]
\[ \leq \frac{p}{s^2} d(fz, z, a) = 0, \]

this proves that \( gz = z \). By (3.2) we can prove the uniqueness of the common fixed point \( z \).
\[ \frac{1}{s} \phi(\rho) \min \{d(z, fz, a), d(z, gz, a)\} \leq d(z, z', a), \] so (3.2) shows that
\[ d(z, z', a) = \max \{d(gz, gz', a), d(fz, fz', a), d(gz', fz, a), d(gz', fz, a)\} \]
\[ \leq \frac{\rho}{s^2} d(z, z', a), \]
which is impossible unless \( z = z' \).

\[\Box\]

**Conflicts of Interest**

The authors declare no conflicts of interest regarding the publication of this paper.

**References**


