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JGP-Ring

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Abstract

A Ring R is called right *JGP*-ring; if for every $a \in J(R)$, $r(a)$ is a left *GP*-ideal. In this paper, we first introduced and characterize *JGP*-ring, which is a proper generalization of right *GP*-ideal. Next, various properties of right *JGP*-rings are developed; many of them extend known results.

Subject Areas

Algebra

Keywords

GP-Ideal, *J*-Regular, Reduced Rings, Right Almost *J*-Injective Rings

1. Introduction

Throughout this paper, every ring is an associative ring with identity unless otherwise stated. Let R be a ring, the direct sum, the Jacobson radical, the right (left) singular, the right (left) annihilator and the set of all nilpotent elements of R are denoted by $\oplus$, $J(R)$, $Y(R)(Z(R))$, $r(a)(l(a))$ and $N(R)$, respectively.

2. Characterization of Right *JGP*-Rings

Call a right *JGP*-rings, if for every $a \in J(R)$, $r(a)$ is left *GP*-ideal. Clearly, every left *GP*-ideal [1], $r(a)$ is *GP*-ideal for every $a \in J(R)$.

2.1. Example 1

1) The ring Z of integers is right *JGP*-ring which is not every ideal of Z is *GP*-ideal.

2) Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a,b,c \in Z_2 \right\}$. Then $J(R) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$. Clearly

$r\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$ is left *GP*-ideal. Therefore R is *JGP*-ring.

2.2. Theorem 1

Let R be a right *JGP*-ring and I is pure ideal. Then R/I is *JGP*-ring.

Proof: Let $a \in J(R)$ and $a + I \in R/I$. Since R is *JGP*-ring, then $r(a)$ is left *GP*-ideal. Let $x + I \in r(a+I)$, $ax \in I$. Since I is pure ideal. Then there exists $y \in I$ such that $ax = axy, (x - xy) \in r(a)$ and $r(a)$ is *GP*-ideal. So there exist $w \in r(a)$ and a positive integer n such that

$$(x-xy)^n = w(x-xy)^n$$

$$x^n - nx^{n-1}xy + n(n-1)\frac{x^{n-2}x^2y^2}{2!} + \cdots + (xy)^n$$
$$= wx^n - nwx^{n-1}xy + \cdots + w(xy)^n$$

$$x^n - nx^n y + n\frac{(n-1)x^n y^2}{2!} + \cdots + x^n y^n = wx^n - nwx^n y + \cdots + wx^n y^n$$

$$x^n - wx^n = nx^n y - n\frac{(n-1)x^n y^2}{2!} - \cdots - x^n y^n - nwx^n y$$
$$+ n\frac{(n-1)wx^n y^2}{2!} + \cdots + wx^n y^n$$

So $(x^n - wx^n) \in I$, and $x^n + I = wx^n + I = (w+I)(x^n + I)$. Therefore $r(a+I)$ is a left *GP*-ideal. Hence R/I is *JGP*-ring.

2.3. Proposition 1

If R is right *JGP*-ring and $r(a) \subseteq J(R)$ for all $a \in J(R)$, then $r(a)$ is nil ideal.

Proof: Let R be *JGP*-ring, then $r(a)$ is *GP*-ideal. For every $b \in r(a)$ there exist a positive integer n and $x \in r(a)$ such that $b^n = xb^n$, $(1-x)b^n = 0$. Since $x \in r(a) \subseteq J(R)$, then $x \in J(R)$ implies $(1-x)$ is unit. Then there is $v \in R$ such that $v(1-x) = 1$, so $v(1-x)b^n = b^n$ then $b^n = 0$. Therefore $r(a)$ is nil ideal.

A ring R is called reversible ring [2], if for $a, b \in R$, $ab = 0$ implies $ba = 0$. A ring R is called reduced if $N(R) = 0$. Clearly, reduced rings are reversible.

2.4. Theorem 2

Let R be a reversible. Then R is right *JGP*-ring iff $r(a) + r(b^n) = R$ for all $a \in J(R)$ and $b \in r(a)$, a positive integer n.

Proof: Let R be *JGP*-ring, then $r(a)$ is *GP*-ideal. For every $b \in r(a)$ and a positive integer n, considering $r(a) + r(b^n) \neq R$. Then there is a maximal ideal M contain $r(a) + r(b^n)$. Since $r(a)$ is *GP*-ideal and $b \in r(a)$. Then there exists $c \in r(a)$ and a positive integer n such that $b^n = cb^n$, implies $(1-c) \in r(b^n) \subseteq M$.

But $c \in r(a) \subseteq M$, then $1 \in M$, this contradiction with $M \neq R$. Therefore $r(a)+r(b^n)=R$. Conversely, let $r(a)+r(b^n)=R$. For all $a \in J(R)$ and $b \in r(a)$, then $x+y=1$ when $x \in r(a)$ and $y \in r(b^n)$ multiply by b^n we get $xb^n=b^n$, $r(a)$ is *GP*-ideal. Therefore *R* is *JGP*-ring.

3. *JGP*-Rings and Other Rings

In this section we consider the connection between *JGP*-rings and *J*-regular rings.

Following [3] a ring is called NJ, if $N(R) \subseteq J(R)$.

3.1. Theorem 3

Let *R* be *JGP* and *NJ*-ring. Then *R* is reduced if, $l(a^n) \subseteq r(a)$ for every $a \in R$, and positive integer *n*.

Proof: Consider *R* not reduced ring, then there is $0 \neq a \in J(R)$ and since *R* is *JGP*-ring, then $r(a)$ is left *GP*-ideal. Implies $b \in r(a)$ and a positive integer *n* such that $a^n = ba^n$, $(1-b) \in l(a^n) \subseteq r(a)$. So $a = ab$. Since $b \in r(a)$, then $ab = 0$ implies $a = 0$ and this a contradiction. Therefore *R* is reduced.

A ring *R* is called regular if for every $x \in R, x \in xRx$ [4].

Following [5], a ring *R* is *J*-regular if for each $a \in J(R)$, there exists $x \in R$ such that $a = axa$. Every regular ring is *J*-regular ring [5].

3.2. Theorem 4

If $J(R) = N(R)$ and $l(a^n) \subseteq r(a)$ for all $a \in R$, and positive integer *n*, then *R* is *JGP*-ring iff *R* is *J*-regular ring.

Proof: Let *R* be *JGP*-ring, from Theorem 3 *R* is reduced ring implies that $N(R) = 0$. Since $J(R) = N(R)$, then $J(R) = 0$. Therefore *R* is *J*-regular.

Conversely: it is clear.

3.3. Definition 1

Let M_R be a module with $S = End(M_R)$. The module *M* is called right almost *J*-injective, if for any $a \in J(R)$, there exists an *S*-sub module X_a of *M* such that $l_M r_R(a) = Ma \oplus X_a$ as left *S*-module. If R_R is almost *J*-injective, then we call *R* is a right almost *J*-injective ring [6].

3.4. Proposition 2

If *R* is almost *J*-injective ring, then $J(R) \subseteq Y(R)$ [6].

From Proposition 2 we get:

3.5. Corollary 1

If *R* is right almost *J*-injective and *NJ*-ring, then $N(R) \subseteq Y(R)$.

An element $a \in R$ is said to be strongly regular if $a = a^2 b$ for some $b \in R$ [4].

3.6. Theorem 5

Let R be *NJ*, *JGP* and right almost J–injective ring. Then every element in $J(R)$ is strongly regular. If $l(a^n) \subseteq r(a)$ for all $a \in R$, and positive integer *n*.

Proof: For all $0 \neq a \in J(R)$, then $a^2 \in J(R)$. Since R is almost *J*-injective ring, then there exist a left ideal X in R such that

$Ra \oplus X_a = l(r(a)) = l(r(a^2)) = Ra^2 \oplus X_a$, by using Theorem 3,

$a \in l(r(a)) = l(r(a^2)) = Ra^2 \oplus X_a$. For all $b \in R$ and $x \in X$, $a = ba^2 + x$, then $a^2 = aba^2 + ax$ implies $a^2 - aba^2 = ax \in Ra \cap X_a = 0$, $a^2 = aba^2$. Therefore $(1 - ab) \in l(a^2) \subseteq r(a)$. Since R is reduced, then $a = a^2 b$. Therefore a is strongly regular element.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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