https://www.scirp.org/journal/ns Natural Science, 2019, Vol. 11, (No. 12), pp: 323-335

Entropy and Temperature of Electromagnetic Radiation

Oded Kafri

Kafri Nihul Ltd., Tel Aviv, Israel

Correspondence to: Oded Kafri, kafri.entropy@gmail.com

Keywords: Thermodynamics, Statistical Physics, Optics, Coherence
Received: November 19, 2019 **Accepted:** December 16, 2019 **Published:** December 19, 2019

ABSTRACT

Electromagnetic (EM) radiation is both wave and heat. Waves are characterized by spectral distribution, spatial distribution, time coherence, spatial coherence, energy flux, and polarization. Heat, namely energy transferred from a hot body to a cold one, is characterized by its energy and entropy, and the ratio between them is the temperature. Here we calculate the entropy and temperature of a single radiation mode from the wave properties of the radiation. Using the Heisenberg uncertainty principle and Planck law, we calculate, from the optical properties of the radiation, the number of modes and their occupation number. Then we calculate the entropy and temperature of a single-mode EM radiation. It is shown that the entropy of a single-mode varies from zero for low occupation number, namely in the quantum limit to one Boltzmann constant for high occupation number, namely in the classical limit. The temperature varies from zero Kelvin in the quantum limit to infinity at high energies in the classical limit. This analysis is consistent with Fourier optics and statistical mechanics namely: Stephan-Boltzmann Law, Wein Law, Zipf law, IT File's entropy, and the canonical distribution.

1. INTRODUCTION

Thermodynamics is the science of energy flow. Most energy flow in nature is electromagnetic radiation. The electromagnetic radiation has different names for different wavelengths: The longest wavelengths (longer than one centimeter) are called radiowaves, in the millimetric range, microwaves in the microns range, infrared (that we feel as "heat") in the range between around 0.8 to 0.3 micron, visible waves or light. In the less than microns, ultraviolet or UV, the shortest wavelength is the X-ray and γ rays that can ionize molecules. The energy of the electromagnetic radiation is moving at the speed of light [1] and therefore it is, per definition, heat, namely transferred energy from the emitter to the absorber. Heat is characterized by its energy, entropy, and temperature. Hereafter we discuss in a unified way all these quantities for electromagnetic radiation and the laws that are derived from them.

Entropy is a measure of the uncertainty of a system. If a system can be found in W distinguishable configurations (microstates), its entropy is the logarithm of W multiplied by the Boltzmann constant k_B. Therefore, if we assume that each microstate has identical probability as the others, entropy is the measure of lack of knowledge in what microstate the system is. Nevertheless, entropy interpreted as uncertainty mostly in information theory, and in sparse physical systems, like gases, it is described erroneously as a disorder. The concept of uncertainty in physics is dedicated to the Heisenberg uncertainty principle that formulates the limit of the accuracy of simultaneous measurements of energy and time or momentum and space, *i.e.* if we want to measure the energy with very high accuracy, we need to give up the time resolution, etc. In fact, the Heisenberg uncertainty principle is a direct outcome of Fourier wave analysis [1] that yields, that the distribution widths multiplication of any Fourier pair is greater than 1/2π, namely,

$$\delta\nu\delta t \geq 1/2\pi \quad \text{and} \quad \delta k_x \delta x \geq 1/2\pi \tag{1}$$

where $\delta\nu$ is the spectral width, and δt is the coherence time, δk_x is the width of the wavenumber distribution and δx is the spatial coherence. This uncertainty bound 1/2π, depends on the profile of the beam, *i.e.* in a Gaussian beam the limit is 1/4π. These relations were known about 100 years before Heisenberg. Basically, Heisenberg following Planck invoked the Planck constant to connect between energy and frequency and momentum and wave number, namely $\epsilon = h\nu$ and $p_x = hk_x$ where ϵ is the energy and p_x is the momentum in the direction of the propagation axis. Therefore, for a wave described by $\Psi = A\mathrm{e}^{-i2\pi(\nu t + k_x x)}$, Equation (1) yields the famous uncertainty principle,

$$\delta\epsilon\delta t \geq h/2\pi \quad \text{and} \quad \delta p_x \delta x \geq h/2\pi \tag{2}$$

Here we address the question, what is the contribution of the Heisenberg uncertainty principle to the entropy, which is also the uncertainty of the energy emission of blackbody radiation.

2. HISTORY: BLACKBODY AND PLANCK LAW

A blackbody is any materialistic body in thermal equilibrium and therefore has a well-defined temperature *T*. The materialistic particles reach thermal equilibrium by exchanging electromagnetic radiation, namely photons, with each other. The blackbody's volume *V* contains many radiation modes, *N*, in all the frequencies and directions that are possible in its geometric boundaries. The blackbody is linked to a thermal bath of temperature *T* such that its temperature remains constant despite the electromagnetic radiation that it emits. Till Planck's publication in 1901 [2], it was assumed that each oscillator, namely a radiation mode, carries k_BT energy, where k_B is the Boltzmann constant. This value is the average energy of a classical oscillator in a bath at temperature *T* (*i.e.* each oscillation degree of freedom in ideal gas has internal energy k_BT). Therefore, to calculate the energy emitted by blackbody radiation we have to count the number of modes and multiply it by the average energy. The amount of heat *Q* emitted by the blackbody depends on the measuring time Δt. Equation (1) yields that $\delta t \geq 1/2\pi\delta\nu$. However, we can increase the measuring time without changing the spectrum as long as we wish, namely $\Delta t > \delta t = 1/2\pi\delta\nu$. The ratio $\frac{\Delta t}{\delta t} = N_l$, which is the number of the coherence length in the pulse, is called the number of longitudinal modes.

The number of the possible oscillators per unit volume *V*, in a frequency interval $\nu + \delta\nu$, is derived from the number of radiation modes *N* calculated from Maxwell equation [3] is given by,

$$N = \frac{8\pi\nu^2\delta\nu V}{c^3} \tag{3}$$

The fraction of the modes contained within a solid angle Ω will be,

$$N(\Omega) = \frac{N}{4\pi} = \frac{2\nu^2}{c^3} V\delta\nu \tag{4}$$

Now assume that a beam of duration Δt emerges from a square on the blackbody surface having a cross-section a^2. Its volume is $V = a^2 c\Delta t$ and the number of modes that participate in its emission is,

$$N(\Omega)=\frac{2v^2}{c^2}\delta va^2\Delta t=2\frac{a^2}{\lambda^2}\delta v\Delta t=2N_l\left(\frac{a^2}{\lambda^2}\right) \tag{5}$$

Now we go back to Equation (2) $\delta p_x \delta x \geq h/2\pi$ and divide both sides of Equation (2) by p_x. Since the wave vector $k_x=\frac{2\pi}{\lambda}$, where λ is the wavelength, we obtain that $\left(\frac{\delta p_x}{p_x}\right)\delta x \geq \frac{\lambda}{2\pi}$, $\frac{\delta p_x}{p_x}=\delta\Omega_x$ is the diffraction limit angular divergence in the *x* direction and δx is the size of the spot of the focused beam in the *x* direction, therefore, $\delta x > \lambda/2\pi$ and the minimum area resolution of the spot size is $\delta x \delta y > \lambda^2/(2\pi)^2$. With analogy to the definition of longitudinal modes, the ratio between the spot size resolution $a^2/(2\pi)^2$ and the diffraction limit resolution, namely $\frac{a^2}{\lambda^2}$ is called the number of spatial modes N_s. We see that a spatially coherent beam, in which $N_{sx}=1$, cannot be resolved better than $\lambda/2\pi$. Equation (5) now can be written as,

$$N=2N_lN_s \tag{6}$$

The number 2 stands for the two independent polarizations, N_l is the number of the coherence length of a beam and N_s is our ability to focus the beam as compared to the diffraction limit spot size [4].

N is a function of the temporal coherence and the spatial coherence and the polarization. The heat emitted from a blackbody is thus,

$$Q=Nk_BT \tag{7}$$

The entropy reduction of a Blackbody due to energy emission is,

$$S=\frac{Q}{T}=Nk_B \tag{8}$$

Equations (7-8) are the expressions of the heat and entropy of *N* classical modes of electromagnetic radiation. A single classical radiation mode has an energy $q=\frac{Q}{N}=k_BT$ and an entropy $s=\frac{S}{N}=k_B$. This well-known result is sometimes overlooked and it is assumed by some, that a single-mode pulse has zero entropy. The reason for this somewhat common mistake is that the expression of the entropy in statistical mechanics is proportional to the logarithm of the number of microstates. Since a single oscillator may be considered as one microstate, and since the logarithm of one is zero, one may conclude that a solitary event, well defined in space and time, has zero entropy. However, single-mode radiation, because of the Heisenberg uncertainty principle, has a built-in uncertainty (entropy) in the oscillation itself, namely the potential and the kinetic energy and the spatial resolution, therefore even a single classical oscillator has k_B entropy. Moreover, if a single oscillator would have zero entropy since entropy is extensive, it would lead to the conclusion that any combination of single oscillators has zero entropy, including the blackbody radiation that is a plurality of oscillators in different modes. This conclusion is, of course, a violation of the second law. A simple proof that electromagnetic radiation should carry entropy can be demonstrated by building a heat machine which consists of a hot bath that is a blackbody at temperature T_H and a cold bath that is also a blackbody with a temperature T_C. We replace the cylinder with a piston that contains gas by an ideal fast shutter that can transmit or reflect the electromagnetic emitted radiation. Now we open the shutter for a very short time such that the energy of the electromagnetic radiation leaves the hot blackbody but not yet be absorbed by the cold blackbody. In the short time interval that the electromagnetic radiation is between the two black bodies, the entropy of the hot blackbody decreases but the entropy of the cold blackbody is not yet increased, therefore, if electromagnetic radiation does not carry, entropy, the second law is violated. The only explanation for this "paradox" is that the electromagnetic radiation must carry entropy equal to that removed from the hot blackbody.

Equation (7) is known as Rayleigh-Jeans approximation. The problem in Equation (7) is that when λ is very small, the number of the spatial modes increases to infinity and the blackbody should then emit an infinite amount of energy which is, of course, not possible. This problem was known as the "UV catastrophe". In 1901 Planck replaced the frequency-independent average energy of a mode $q = k_B T$ in Equation (2) with the famous frequency-dependent formula,

$$q = nhv = \frac{hv}{\exp\left(\frac{hv}{k_B T}\right) - 1} \tag{9}$$

and added a new universal constant h to physics.

The famous Planck's assumption is that energy is not continuous and can be divided into quanta hv that are a linear function of their frequencies. Therefore, a spatial radiation mode having a very small wavelength has a very high frequency and therefore its quantum, the photon, may have energy that is higher than $k_B T$. Therefore, according to classical mechanics, it cannot emit energy.

In quantum mechanics, the emission of photons with energy higher than the average energy $k_B T$ is possible. The reason for this strange phenomenon is the uncertainty expressed by the entropy. The second law is about the propensity of entropy to grow. In a closed system, it will reach its maximum value where it cannot grow anymore. This point is the equilibrium point. Planck was the first to calculate the distribution of photons among the radiation modes that maximized the Boltzmann entropy $S = k_B \ln W$ under the constraint of conservation of energy [5]. His technique is called today MaxEnt or microcanonical ensemble. Planck started by calculating the number of different microstates of B indistinguishable particles (photons having a given frequency) in N distinguishable boxes (radiation modes). This yields, $W = \frac{(N+B-1)!}{(N-1)!B!}$, applying Stirling's approximation to calculate $\ln W$ he obtained that,

$$S \cong k_B N\left[(n+1)\ln(n+1) - n\ln n\right] \tag{10}$$

Here Planck replaced the total number of particles B by $n = B/N$ which is the average number of particles in a box. To maximize the entropy, he applied the conservation of energy constraint $Q(n) = Nq = Nnhv$, where, q is the energy per mode, and therefore, to find the equilibrium distribution, he used Lagrange multiplier technique, namely to find the maximum of the function $F = S(n) - \beta Q(n)$ with respect to n, where β is a Lagrange multiplier, namely $\frac{\partial F}{\partial n} = 0$, where,

$$F \cong k_B N\left[(n+1)\ln(n+1) - n\ln n - \beta(q - nhv)\right] \tag{11}$$

Since it can be shown that $\beta = \frac{1}{k_B T}$, he found out his famous result for n that maximizes the entropy namely,

$$n(v,T) = \frac{1}{e^{\frac{hv}{k_B T}} - 1} \tag{12}$$

Now the energy carried by a radiation mode is not $k_B T$ but $n(v,T)hv$ as given by Equation (9).

3. CLASSICAL LIMIT: ZIPF LAW

In the classical limit, the energy is continuous, the photon's energy is infinitely small, and therefore the number of photons is infinitely large. We rewrite Equation (12) as

$$hv = k_B T \ln\left(1 + \frac{1}{n}\right) \quad (13)$$

Since the energy of the mode is $q = nhv$, in the classical limit where $n \to \infty$

$$q = nhv = k_B T \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right)^n = k_B T$$

And the entropy of a classical mode is $s = k_B$, namely, one Boltzmann constant as was assumed in the Rayleigh-Jeans formula.

It is worth noting that both the entropy and the frequency of a classical oscillator are independent of its energy. The energy-independent frequency of the classical oscillators is used for time measurements i.e. pendulum clocks and watches. The energy-independent entropy of classical oscillators is used for digital communication [6]. Since the entropy of a sequence of N classical bits is $S = Nk_B \ln 2$ (N bits), theoretically, it is possible to recover the full content of a file, regardless of the unavoidable energy attenuation in its transmission. This explains the huge success of digital communication.

In the classical limit where the energy of the photon is very small, $e^{\frac{hv}{k_B T}} \approx 1 + \frac{hv}{k_B T}$, therefore, from Equation (12) we can see that $n \approx \frac{k_B T}{hv}$. Namely, the number of photons is the average energy of a mode divided by the energy of the photon. That means that the higher the energy of photons the higher their frequency and smaller their number, n. Namely, there are more poor-energy photons than high-energy photons.

To find the exact ratio between the frequency v and the number of photons n in N modes, we normalize the photon energy of Equation (13), by dividing it in the total sum of the energies over the modes. Namely,

$$\sum_{n=1}^{N} hv(n,T) = k_B T \sum_{n=1}^{N} \ln\left(1 + \frac{1}{n}\right) = k_B T \ln(N+1)$$

The normalized distribution is given by $\varepsilon(n,T) = hv(n,T) \big/ \sum_{n=1}^{N} hv(n,T)$ or:

$$\varepsilon(n,N) = \frac{\ln\left(1 + \frac{1}{n}\right)}{\ln(N+1)} = \log_{N+1}\left(1 + \frac{1}{n}\right) \quad (14)$$

$\varepsilon(n)$, which is a parameter-free long-tail distribution, called the Planck-Benford distribution [7-9], is the relative energy of the modes having n photons as compared to the energy of all the N modes. It is seen that the higher the photon's energy, the smaller the number of photons having this energy.

The reason for the surprising result, that the Planck-Benford distribution has no physical constants, is that in the classical limit the radiation modes are pure harmonic oscillators and therefore have the universal statistics of harmonic oscillators.

This outcome that in the energy normalization process all "physics" disappeared, namely: h, k_B and T are canceled out, makes the Planck-Benford distribution more universal than the Maxwell-Boltzmann distribution. The Planck-Benford law is also the Maximum Entropy distribution of identical balls in N distinguishable boxes, where the number of balls is greater than the number of boxes [7]. This statistics yields that there are more poor people than rich people (energy is money and people are modes), and there are more people in the big cities than there are in small villages (here people are the energy and cities are the modes). Planck-Benford law yields without any free parameter for $N = 9$ the Benford law [7]. For large N it yields Zipf law with slop 1 [8]. In the Economy, it yields correctly the wealth distribution in the OECD countries, the average Gini inequality index in the OECD countries, the Pareto 80:20 law, the vote's distribution among parties in democracies, the population of cities distribution and most of sociological statistics [9] [10].

Here we show that Planck-Benford law is the empirical Zipf law. Zipf law is a probability function found in many distributions in nature *i.e.* n people in N cities or the redundancy n of words in a text of N different words. Zipf law was found, empirically, to be, $\varphi(n,N)=1/n^s H_N$, where H_N is the harmonic number and s is some parameter (usually close to one). To show that Zipf law is Planck-Benford law with $s=1$, we assume that n is a continuous function, therefore, we can use the Riemann sum to find the area between two sequential integers n and $n+1$, namely, $\int_n^{n+1}\frac{\mathrm{d}n'}{n'}=\ln(n+1)-\ln(n)=\ln\left(1+\frac{1}{n}\right)$. Similarly, the Harmonic number defined by $H_N=\sum_{n=1}^{N}\frac{1}{n}$ becomes,

$$H_{N+1}=\int_1^{N+1}\frac{\mathrm{d}n'}{n'}=\ln(N+1).$$

Note that the integration is until $N+1$ because we start to count the N ranks from $n=1$ and not from zero (from zero to one we are in the quantum regime). Now we can write Planck-Benford law as,

$$\varepsilon(n,N)=\frac{\ln\left(1+\frac{1}{n}\right)}{\ln(N+1)}=\frac{\frac{1}{n}}{\sum_{n=1}^{N}\frac{1}{n}}=\frac{1}{nH_N} \tag{15}$$

This is the Zipf law with slope 1.

4. QUANTUM LIMIT: MAXWELL-BOLTZMANN DISTRIBUTION

In the quantum limit, the photon energy has approximately $1/n$ times more energy than the average energy k_BT. One may expect that the probability to emit photons will be also $1/n$. However, from Equation (13) we see that in the quantum limit where $n<1$, the frequency of a photon increases linearly with the temperature of the blackbody and logarithmically with the number of photons n. For very small n's, $1+\frac{1}{n}\approx\frac{1}{n}$ and $h\nu(n,T)=k_BT\ln\left(1+\frac{1}{n}\right)\approx -k_BT\ln n$. This equation usually is written as $n_i\approx\exp\left(-\frac{h\nu_i}{k_BT}\right)$. When it is normalized, by dividing it by the probability partition function, $Z_P(n,T)=\sum_i n_i$, we obtain the famous exponential "canonical energy distribution".

$$p_i=\frac{\exp\left(-\frac{h\nu_i}{k_BT}\right)}{\sum_i\exp\left(-\frac{h\nu_i}{k_BT}\right)} \tag{16}$$

The canonical distribution derivation deserves some remarks. Zipf law was obtained by energy normalization. However, the energy is inversely proportional to the number of photons in a mode that is proportional to probability. The canonical distribution was obtained directly by probability normalization in which n was changed to p. The energy emission of a single-mode in this limit is, $\varepsilon_i=h\nu_i\exp\left(-\frac{h\nu_i}{k_BT}\right)$.

5. STEPHAN-BOLTZMANN LAW

Stephan-Boltzmann law enables us to calculate the energy flux of a blackbody as a function of its temperature [11]. The general expression of the emission of given single-mode radiation, having frequency

v is obtained from Equation (12),

$$q(v,T) = nhv = \frac{hv}{e^{\frac{hv}{k_B T}} - 1} = k_B T \frac{\frac{hv}{k_B T}}{e^{\frac{hv}{k_B T}} - 1} \tag{17}$$

And the entropy of a single mode of frequency v is,

$$s(v,T) = \frac{q(v,T)}{T} = k_B \frac{\frac{hv}{k_B T}}{e^{\frac{hv}{k_B T}} - 1} \tag{18}$$

Let's designate $u = \frac{hv}{k_B T}$, therefore, the expression for the energy of a single-mode is

$q(v,T) = k_B T \frac{u}{e^u - 1}$, and its entropy is $s(v,T) = k_B \frac{u}{e^u - 1}$. See Figure 1,

To calculate the energy emission of a blackbody we have to integrate Equation (17) over all the temporal modes. Let us start with the emission of a single spatial mode. The energy flux $I(v,T)$ is $q(v,T)$, is divided by the area of the spot size of a single spatial mode λ^2 on the surface of the blackbody and multiplied by the two polarizations. Thus the energy flux is,

$$I_E(T) = \int_0^\infty \left(\frac{2q(v,T)}{\lambda^2} \right) \mathrm{d}v \tag{19}$$

Or,

$$I_E(T) = 2k_B \int_0^\infty \frac{Tu}{\lambda^2 (e^u - 1)} \mathrm{d}v$$

Since $\mathrm{d}u = \frac{h}{k_B T} \mathrm{d}v$ and $\frac{1}{\lambda^2} = \frac{v^2}{c^2}$, we obtain,

$$I_E(T) = 2k_B \int_0^\infty \frac{Tuv^2}{c^2 (e^u - 1)} \mathrm{d}v = 2k_B^4 \int_0^\infty \frac{T^4 u^3}{c^2 h^3 (e^u - 1)} \mathrm{d}u$$

The finite integral $\int_0^\infty \frac{u^3}{e^u - 1} \mathrm{d}u = \zeta(4) = \frac{\pi^4}{15}$ where ζ is Reimann zeta function, therefore, the energy flux of a single spatial mode is,

$$I_E(T) = \frac{2(\pi k_B T)^4}{15 c^2 h^3} = \sigma T^4 \tag{20}$$

where $I(T)$ has units of energy per unit area per unit time.

The entropy flux is,

$$I_s(T) = \sigma T^3 \tag{21}$$

One should remember that the area of a single-mode λ^2 that is part of the integration, vary from zero to the significant part of the blackbody surface. This point deserves careful consideration. The function $f(u) = \frac{u}{e^u - 1} = \frac{s}{k_B}$ is monotonically decreasing function as is seen in Figure 1. However, in the quantum limit $f(u \gg 1) \cong u e^{-u}$ has a maximum.

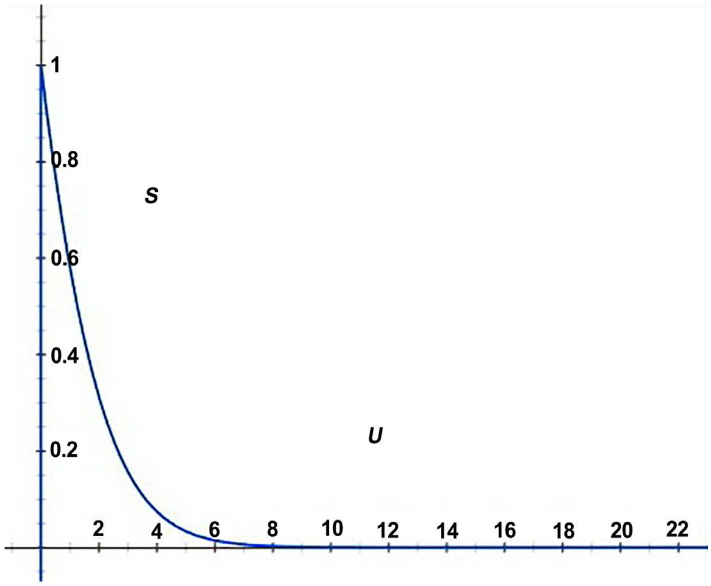


Figure 1. The entropy of a single oscillator in units of k_B as a function of u. The expression of the entropy according to Planck law is $S(u)=\frac{u}{\exp(u)-1}$ where $u=\frac{hv}{k_B T}$. The entropy of the oscillator reduces very fast from k_B in the classical limit were $u \ll 1$ to zero in the quantum limit, when $u > 1$.

The reason for this paradox is that quantum modes have very short wavelengths and the energy flux increases by the factor $1/\lambda^2$ as was derived in Equation (19). The function that obtained $f'=\frac{u^3}{e^u-1}$ has a maximum.

Stephan-Boltzmann law was derived in the Ph. D. thesis of Boltzmann under the supervision of Stephan. The derivation was done by classical thermodynamics argumentations and therefore the value of the Stephan-Boltzmann constant was not calculated. The constant that was calculated later was an additional validation of Planck's theory.

6. WEIN'S DISPLACEMENT LAW

Another quantity that derives from Planck's law is the maximum frequency of a blackbody's emission as a function of its temperature [12]. Wein's law was discovered several years before the Planck calculation in 1901. Here it is derived from the Planck law. We start with $q(v,T)=k_B T\frac{u}{e^u-1}$ remembering that $u=hv/k_B T$, the energy per mode per unit blackbody surface is

$q_s(v,T)=2q(v,T)/\lambda^2=\frac{2k_B T}{c^2}\frac{v^2 u}{e^u-1}=\frac{2h}{c^2}\frac{v^3}{e^u-1}$. To find the maximum frequency on the energy distribution we ask that $\frac{dq_s(v,T)}{dv}=0$ or,

$$\frac{dq_s(v,T)}{dv}=\left(\frac{2h}{c^2}\right)\frac{d}{dv}\frac{v^3}{e^u-1}=\frac{2h}{c^2(e^u-1)}\left(3v^2-\frac{v^2ue^u}{e^u-1}\right)=0$$

Namely,

$$\frac{ue^u}{e^u - 1} = 3 \tag{22}$$

The function $\frac{ue^u}{e^u - 1}$ has poles; nevertheless, we can calculate it in the limits. In the classical limit $u \ll 1$, so then we obtain that $e^{u_{\max}} \approx 3$ and therefore $\frac{h\nu_{\max}}{k_B T} = \ln 3$.

In the quantum limit $u \gg 1$ $e^u \approx e^u - 1$, therefore, $\frac{h\nu_{\max}}{k_B T} = 3$.

Substitute $\nu_{\max}\lambda_{\max} = c$ and obtain,

$$\lambda_{\max} = \frac{b}{T} \tag{23}$$

In the classical limit $b_{class} = \frac{hc}{k_B \ln 3}$ and in the quantum limit $b_{qunt} = \frac{hc}{3k_B}$.

Usually, Wein's law applied to Maxwell-Boltzmann distribution, which is in the quantum regime. Therefore, Equation (23) converges very fast to the area where b is constant.

7. THE TEMPERATURE OF THE LIGHT

Temperature is considered a macroscopic property. However, a microscopic entity which its energy can be measured and its entropy can be calculated from Clausius entropy definition $s = \frac{q}{T}$, has a temperature. One should ask, what is the physical meaning of it?

In the classical approximation, the answer is straightforward: energy flows from the hot to the cold. I.e. two identical classical pendulums, one hot with high amplitude and one cold with low amplitude, if they linked together, their amplitudes will eventually equate. The laser beam is a classical oscillator as $n \gg 1$, its beam temperature is the maximum temperature that it can heat a materialistic object. Consider a laser beam having power P and coherence length δx, therefore, its coherence time is $\delta t = \delta x / c$. The energy per mode is the energy within the coherence time. Therefore, $q = P\delta t = \frac{P\delta x}{c}$. The temperature of the beam is,

$$T_{class} = \frac{q}{k_B} = \frac{P\delta x}{ck_B} = \frac{nh\nu}{k_B} \tag{24}$$

I.e. HeNe laser having a power of 1 mW and coherence length 30 cm yields a temperature of about 10^{10} K and the energy of a mode is about 10^{-13} J (1 mW·0.3/c). This is a very high temperature, but not infinite as it is usually assumed. Nevertheless, for any practical reason, this temperature is equivalent to infinite temperature. The reason that the HeNe laser beam does not fit for heating is its negligible energy per mode 10^{-13} J. Namely, to heat 1 gram of water in one degree Kelvin one needs about 4×10^{13} modes, which means, of about 4 hours of heating.

Digital communication is done by a transmission of a file that is a sequence of classical oscillators from Bob to Alice. Therefore, the entropy of the file which reflects its amount of data in it is not a function of the temperature of the bits. Nevertheless, we want that it will be well above the ambient's temperature such that the thermal noise will not add noise bits the file. Therefore the power namely, the temperature of our antenna has practical importance.

The general expression of the temperature of EM radiation is given by Planck expression of Equation

(12) for the occupation number *n*, that can be rewritten as,

$$T = \frac{h\nu}{k_B \ln\left(1+\frac{1}{n}\right)} \tag{25}$$

In **Figure 2(a)** and in **Figure 2(b)** we see that the temperature at $n \ll 1$ starts from zero and very fast go to the linear zone of the classical approximation in which $T \propto n$.

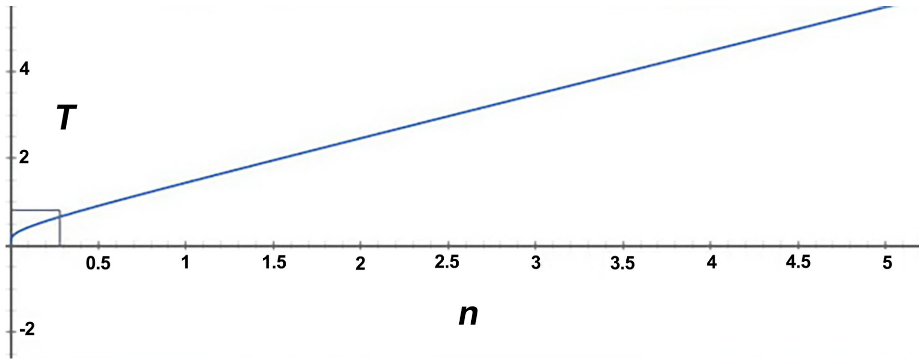
It is seen that when *n* approach to zero both the entropy and the temperature are approaching zero. Entropy is heat divided by the temperature, therefore we might expect that zero temperature will yield infinite entropy, like in a zero temperature bath. The explanation to this "abnormality", namely that $\lim_{n\to 0}\frac{q}{T}=0$, is that the energy reduces exponentially to zero, while the temperature decay linearly. From Equation (17) we obtain that in the quantum limit $q = nh\nu = \frac{h\nu}{\mathrm{e}^{\frac{h\nu}{k_B T}}-1} \approx h\nu \mathrm{e}^{-\frac{h\nu}{k_B T}}$. The temperature of Equation (25), when we multiply by *n* both nominator and the denominator, yields $T = \frac{nh\nu}{k_B \ln\left(1+\frac{1}{n}\right)^n}$. Therefore in the quantum limit $\lim_{n\to 0} T = \lim_{n\to 0}\frac{nh\nu}{k_B}$ the temperature converges to zero linearly with *n* while the heat converges exponentially so that both temperature and entropy approach to zero when *n* approach to zero.

How to calculate this temperature? Suppose that we want to measure the temperature of a single photon, a single spatial mode source. We need a sensitive detector that can detect a single photon. Suppose that we measure on the average, *P* single photons over a time period Δt. The detector is equipped with a filter that passes the frequency *ν*, and its bandwidth is $\delta\nu$. The number of modes $N = 2\pi\Delta t\delta\nu$. The occupation number is $n = \frac{B}{N}$. Photons with a narrower bandwidth $\delta\nu$ can also pass the filter, therefore, $N \le 2\pi\Delta t\delta\nu$ and $n \ge \frac{B}{2\pi\Delta t\delta\nu}$.

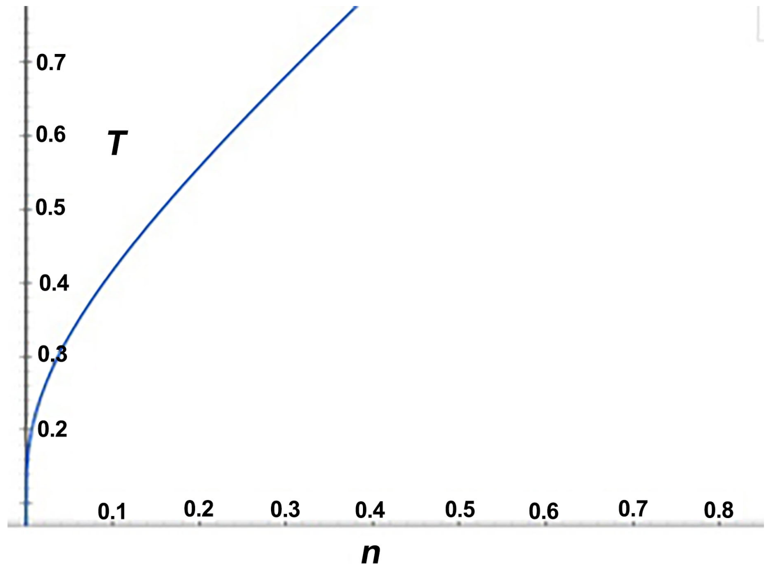
Can we transmit data with single-photon files? Suppose that we have a file composed of HeNe single-photons bits. We ask that its temperature will be at least ten times higher than that of the ambient, to reduce thermal noise, namely $T = \frac{2.28\times 10^4}{\ln\left(1+\frac{1}{n}\right)} \ge 10T_{ambient}$. At room temperature $T = 300\ \mathrm{K}$, thus we obtain $\ln\left(1+\frac{1}{n}\right) \ge 7.6$ or $n \ge \frac{1}{2000}$, that is an acceptable limit for communication.

8. SUMMARY

EM radiation is heat. In thermodynamics, heat is characterized by two independent quantities, energy and entropy. EM radiation has many more characteristics namely; frequency, spectral distribution, temporal distribution, temporal coherence, spatial coherence, beam profile, polarization, and direction. How all these properties are connected to that of heat in general? Planck's law connects between the numbers of photons in a mode (occupation number) and their frequency in a thermal bath (blackbody) of a given temperature. Here we use the spectral distribution, temporal distribution, temporal coherence, spatial



(a)



(b)

Figure 2. (a) The temperature of a mode (in units of $h\nu/k_B$) as a function of occupation number of single modes. I.e. for HeNe laser each unit is about 2.28×10^4 K. (b) The same as Figure 2(a) but for small n's (within the rectangular of Figure 2(a) in the quantum limit).

coherence, beam profile, and polarization to calculate the number of modes. From the EM beam's power, we calculate the number of photons. This enables us to calculate for EM radiation the entropy and the energy of a mode that is consistent with Planck's law.

It is shown that the EM radiation statistics vary smoothly from the classical limit in which all the radiation modes in equilibrium have an energy of $k_B T$ like in ideal gas oscillators and the quantum limit in which most of the modes are empty and some have very high energy as compared to $k_B T$. Using Planck law, we review in a consistent way the calculation of all the thermodynamical properties, namely classical energy distribution—Zipf law; quantum energy distribution—Maxwell Boltzmann law; Energy flux—Stephan-Boltzmann law, and maximum frequency—Wein displacement law.

The entropy of a single-mode as given by Equation (18) namely $s(\nu,T) = \frac{q(\nu,T)}{T} = k_B \frac{u}{\exp(u)-1}$ where $u = \frac{h\nu}{k_B T}$, monotonically increases from zero at very low occupation number in which u is very large namely, $h\nu \gg k_B T$ to k_B when the occupation number is going to infinity namely $h\nu \ll k_B T$ and

$u \to 0$. From Equation (25) it is seen that for low occupation number $n \to 0$ temperature increases monotonically from zero in low occupation number to infinity in a very large occupation number.

CONFLICTS OF INTEREST

The author declares no conflicts of interest regarding the publication of this paper.

REFERENCES

1. Eugene, H. (2005) Optics. In: Goodman, J.W., Ed., *Introduction to Fourier Optics*, 4th Edition, Pearson Education, Robert & Company.
2. Planck, M. (1901) On the Law of Distribution of Energy in the Normal Spectrum. *Annalen der Physik*, **4**, 553-562. https://doi.org/10.1002/andp.19013090310
3. Yariv, A. (1967) Quantum Electronics. Wiley, NY, 86.
4. Kafri, O. and Glatt, I. (1990) The Physics of Moire Metrology. John Wiley and Son, Hoboken, 40-51.
5. Kafri, O. and Kafri, H. (2013) Entropy-God's Dice Game. *CreateSpace*, 198-201. http://www.entropy-book.com
6. Shannon, C.E. (1949) A Mathematical Theory of Communication. University of Illinois Press, Evanston, Ill.
7. Kafri, O. (2016) A Novel Approach to Probability. *Advances in Pure Mathematics*, **6**, 201-211.
   https://arxiv.org/pdf/1606.03696
   https://doi.org/10.4236/apm.2016.64017
8. Kafri, O. (2016) Sociological Inequality and the Second Law. *Journal of Economic and Social Thought*, **3**, 476-480. arXiv preprint arXiv:0805.3206
9. Kafri, O. and Fishof, E. (2016) Economic Inequality as a Statistical Outcome. *Journal of Economics Bibliography*, **3**, 570-576.
10. Kafri, O. (2018) Money, Physics and Distributive Justice. CreateSpace-Amazon.
11. Surhone, L.M., Timpeldon, M.T. and Marseken, S.F. (2010) Stefan-Boltzmann Law. Betascript Publishing, Mauritius.
12. Grant, B.G. (2011) Rayleigh-Jeans Law and Wien Approximation in Field Guide to Radiometry. SPIE.

LIST OF SYMBOLS

δv : spectral width
δt : coherence time
δk_x : wavenumber's width
δx : spatial coherence
ϵ : energy
h: Planck constant
v: frequency
λ : wavelength
c : speed of light
T: temperature
V: volume
N: radiation modes
N_l: longitudinal modes
N_s: Spatial modes
Ω : Solid angle
a: beam waist
k_B: Boltzmann constant
Q: Heat
q: heat per mode
Δt : measuring time
S: entropy
s: entropy per mode
n: occupation number – the number of photons in a mode
B: total number of photons
W: number of microstates
β : Lagrange multiplier
$\varepsilon(n)$: normalized distribution of frequency
H_N: Harmonic number
Z_P: canonical partition function
u: the energy of a photon relative to the average energy.
I: energy flux
$v_{\max}$: The frequency of maximum emission of a blackbody.
$\lambda_{\max}$: The wavelength of maximum emission of a blackbody
b: Stephan Boltzmann constant
P: power