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Modeling Expected Failure Considering Repair Time and Degradation: A Rail System Case Study

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Abstract

The repeated failures of any equipment or systems are modeled as a renewal process. The management needs an assessment of the number of future failures to allocate the resources needed for fast repairs. Based on the idea of expectation by conditioning, special Volterra-type integral equations are derived for general types of repairs, considering the length of repair and reduced degradation of the idle object. In addition to minimal repair and failure replacement, partial repairs are also discussed when the repair results in reduction of the number of future failures or decreases the effective age of the object. Numerical integration-based algorithm and simulation study are performed to solve the resulting integral equation. Since the geometry degradation in different dimensions of a rail track and controlling and maintaining defects are of importance, a numerical example using the rail industry data is conducted. Expected number of failures of different failure type modes in rail track is calculated within a two-year interval. Results show that within a two-year period, anticipated occurrences of cross level failures, surface failures, and DPI failures are 2.4, 3.8, and 5.8, respectively.

Keywords

Renewal Theory, Expected Number of Failures, Partial Repair, Minimal Repair

1. Introduction

A system may need maintenance work done to either renew or restore it to a predetermined condition. In the United States (US), the expected loss is estimated at about a third of every dollar spent due to unnecessary maintenance activities [1]. In addition to the financial costs, unplanned downtime can also result in lost revenue, decreased productivity, increased maintenance and repair costs, and safety risks. Renewal theory can be used to address the issue of aging equipment by reducing its effective age and extending its useful life. By investing in regular maintenance and preemptive maintenance solutions, companies can reduce the risk of equipment failure, minimize downtime, and maximize efficiency, ultimately leading to improved profitability and competitiveness. The majority of comparative studies show when an item fails, it is replaced by a new one, meaning that it is restored to "as good as new" condition. Although this operation is ideal, it can result in waste and additional costs. The best way to deal with this issue is to perform only partial repairs when a failure occurs, allowing the machinery to be used up until a catastrophic failure [2]. A system is returned to its "as good as new condition" with perfect repair, and to its "as bad as old condition" with minimal repair. A renewal process is used to mathematically model perfect repair.

For better understanding of renewal theory, a repeated failure/repair system is shown in **Figure 1**. A process known as renewal process (RP) in which the subsequent instances of a component or system failing are considered as independently and identically distributed random variables. This is equivalent to assuming that the system is immediately repaired and returned to its "As good as new condition" state. Repairable systems can also reach "better than old but worse than new," "better than new," or "worse than old" states in addition to two previous mentioned states. These states are attempted to be included in the analysis of partial repair models [3]. In most of the studies the repair time is considered instantaneous which means failure and repair happen simultaneously.

The most important paper in this regard is by [4] that used the concept of the "effective age" to model imperfect repair using the Generalized Renewal Process (GRP). A practical method for obtaining maintenance policies for repairable systems is provided by Kijima's "effective age" models. The GRP framework is based on two major approaches for imperfect repair, namely arithmetic reduction of age (ARA) and arithmetic reduction of intensity (ARI) [5]. In the ARI approach, the change in failure intensity before and after failure is used to determine the repair effect. A decrease in the system "effective age" in ARA models represents the effect of repair as shown in **Figure 2**. Real age as compared to repair effectiveness in renewal theory. The fundamental idea behind this group of

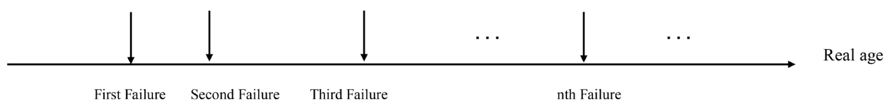

Figure 1. Repeated Failure/Repair system for a single system considering instantaneous repair.

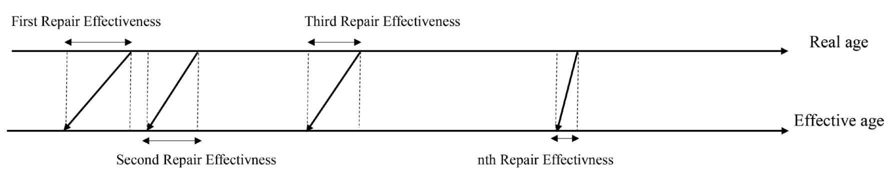

Figure 2. Real age as compared to repair effectiveness in renewal theory.

models is that system renewal occurs through repair. The "effective age' of a system is defined as a positive function of its real age, possibly depending on previous failures while the real age of a system is its functioning time *t*. Kijima's effective age models are based on the hypothesis that repair actions shorten the system's age.

We plan to study the expected number of failures of a single system using renewal theory specifically in imperfect repairs with two new assumptions such as repair time which is not noticed significantly in the literature and degradation of the system when the system is idle. These assumptions make the model more realistic and give a better understanding to develop maintenance policies needed for potential failures in the future. Maintenance mangers need more accurate estimation of failures to be able to allocate enough resources to prevent a catastrophic failure and keep the expenses low. Furthermore, if the number of failures within short period and cost of repairs increase, the managers can use the model to determine whether they need to deploy corrective maintenance rather preventive maintenance.

In this paper we use the renewal theory to calculate the expected number of failures of a single system considering that the repair time is not negligible and repair mathematically results in an increase in effective age of the system because of degradation caused by environmental conditions like temperate, humidity etc., while deploying partial repair make the state of the system better by, replacement, maintenance, and renewal which decreases the expected number of future failures. Time to first failure (TTFF) distribution is known, therefore based on the conditional expectation concept, we derive a special Volterra-type integral equations for general types of repairs. Since this type of integral has no analytical solution, we suggest a numerical integration-based algorithm to find the value of expected number of failures, then we apply a Mont Carlo simulation study to validate the model. The geometry degradation in rail track is of an importance, as a Norfolk Southern train carrying hazardous materials derailed in Ohio, USA in February 2023, highlighting the importance of proper maintenance and repair of rail systems. Renewal theory can be used to detect and estimate future failures of rail systems, considering the effects of environmental conditions on track degradation. By minimizing failure risks and improving reliability, rail system safety can be improved. Therefore, we fit the proposed model to BNSF data. The defects are categorized in three failure type modes, and we determine how many failures we might face in a two-year interval for each failure type mode.

This paper is structured as follows. The related works has been reviewed in Section 2. Section 3 introduces the general model and special cases and model variants and suggests a practical numerical method to find the value of $M(t)$ for any future time *t*. Section 4 presents an illustrative case study and shows the implementation of simulation study and, sensitivity analysis, and results. Conclusions and future research directions are outlined in Section 5.

2. Literature Review

Probability theory, as outlined in [6] and [7], is widely applicable in various fields, including engineering. In engineering, there's a strong focus on reliability and quality, involving the analysis and prediction of random failures. Renewal theory, foundational discussions found in [8] and [9], plays a key role in studying these random failures. Additional resources for reliability analysis include books by [10] [11] [12].

Since there is not a closed-form solution for the RP model [13] and [14] propose a numerical solution using the Monte Carlo (MC) simulation technique to calculate the expected number of failures. Reference [15] proposes an imperfectly repairable system with restoration levels that decrease based on previous repairs. They suggest that this model may reflect actual repair patterns better than an imperfect repair with a constant discount restoration level. The authors then use a Monte Carlo simulation based on cumulative hazard functions to estimate the number of expected failures in this new repairable system.

Scholars study a railroad track degradation analysis for three different geometry failure modes. Effective degradation factors are found using the BNSF data set, and inspection intervals are researched to lessen the impact of hidden maintenance actions [16]. A numerical method is introduced for estimating the Expected Number of Failures (ENF) and Cumulative Intensity Function (CIF). Examples using simulated data as well as a real-world example of locomotive braking grids using actual data are used to show the proposed approach's high degree of accuracy [17]. A novel approach, data and mechanics integrated approach, for analyzing the lifespan of small-radius rails considering defects and wear. The approach calculates the expected mean gross tonnage for rail renewal based on defects and wear and compares it with existing standards [18]. A valuable contribution to the field of rail renewal and maintenance planning is provided in [19], with a focus on using deep reinforcement learning to optimize the schedule and locations of renewal and maintenance activities. The neural network takes as input various factors such as track condition, traffic volume, and weather conditions, and outputs a schedule for renewal and maintenance activities.

The analysis of renewal and renewal-intensity functions for various underlying lifetime distributions, including Normal, Gamma, Uniform, and the notably important Weibull distribution is studied in [20]. The study of [21] extends a mean remaining time to renewal model for failure-prone systems with minimal repairs, accommodating changes in operating states and age, and historical repair data. It uses a bivariate approach to predict failures and quantify reliability indices, making it versatile for various models. The model facilitates the implementation of a preventive replacement policy based on renewal-reward arguments, minimizing repair costs. Numerical examples demonstrate the model and optimal solution behavior with changing parameters. Researcher in [22] propose an approach for predictive maintenance scheduling that considers the phase of the machine life cycle and its associated reliability characteristics, using the renewal theory to model the machine's life cycle and determine the optimal maintenance strategy based on the average reward criterion. New bounds for the renewal function and the variance of the renewal process is introduced in [23]. It presents a general lower bound for the renewal function, improving upon existing lower bounds.

A mathematical model is provided that considers the impact of long-term maintenance policies in addition to short-term conditions, aiming to minimize preventive and corrective maintenance costs in manufacturing systems [24]. The researchers emphasize the significance of robust estimation methods, such as M-estimators in quality control [25]. An approach is proposed for comparing numerically three maintenance strategies, involving minimal repairs at failure, replacement with complete renewal only at the first failure, and replacement with complete renewal at each failure. The approach proceeds by presenting the mathematical models at the component level and at the system level. A novel asymptotic algorithm is introduced for estimating the replacements number [2]. The methodology outlined in [26] considers two different ways of improving the state of the object by partial repairs; reduce the failure rate (or expected number of future failures) and reduce the effective age of the object. The study of [27] shows the imperfect maintenance effect in wind turbine technology by proposing a failure rate function update model considering the effective age and failure intensity update factors. Based on their proposed model, they studied a periodic dynamic imperfect preventive maintenance decision model to minimize the maintenance costs using GRP.

3. The Mathematical Method

In this research, the mathematical method is developed in several subsections. The first subsection presents a general model for the problem at hand, while the subsequent subsections consider special cases and model variants. In the final subsection, the testing phase is discussed, which involves assessing the model parameter values to ensure their validity and effectiveness.

3.1. General Model

In planning future repairs, allocating parts, tools and skilled workforce, the expected number of future failures plays a crucial role. Let $M(t)$ denote the expected number of failures in the interval $(0,t)$, then its determination depends on the types of repairs being used. The most important characteristic of any random failure is the time when it occurs; let $F(t)$ be the Cumulative Distribution Function (CDF) of the time of first' failure, $f(t)$ the probability density function (pdf), and failure rate $\rho(t) = f(t)/(1-F(t))$. The formal definition of the failure rate is as Equation (1).

$$\rho(t) = \lim_{\Delta t \to 0} \frac{P(t < X < t + \Delta t \mid t < X)}{\Delta t} \tag{1}$$

where X is the time of the first failure. The numerator is the conditional probability that the object will break down during the next Δt time periods given that it is working at time t. The list of parameters is shown in **Table 1**.

In the case of minimal repairs, it is well known that the expected number of failures is the solution of the Volterra-type integral equation:

$$M(t) = M(t)F(t) + F(t) - \int_0^t M(\tau) f(\tau) \mathrm{d}\tau \tag{2}$$

which can be shown by substitution to be:

$$M(t) = \int_0^t \rho(\tau) \mathrm{d}\tau \tag{3}$$

which is the cumulative failure rate. If the equipment is replaced after each failure, then:

$$M(t) = F(t) + \int_0^t M(t-\tau) f(\tau) \mathrm{d}\tau \tag{4}$$

Table 1. List of parameters.

Notations	Definition
$M(t)$	Expected number of failures in the interval $(0,t)$
$F(t)$	Cumulative Distribution Function (CDF) of the time of first failure
$f(t)$	probability density function (pdf)
$\rho(t)$	Failure rate
T	Repair Time
α	Quality of repair (shows partial repair)
d	Degradation rate when the item is idle during repair time
S_1	First time failure
h	Small grid point using in numerical integration-based algorithm
K, L	Positive integers using in numerical integration-based algorithm

In the case of partial repairs, the form of the integral equation depends on the types of the repairs [26]. This paper introduces a general model which generalizes several earlier ones incorporating effects which were ignored in the earlier studies.

The earlier models assumed instantaneous repairs. Here we assume that each repair takes T time periods, when the object is idle, and the degradation rate of the object is less than that of the working one. Mathematically it is assumed that a repair results in an increase of the effective age by dT ($0 < d < 1$).

Partial repairs also make the state of equipment better by decreasing the expected number of future failures by a factor α ($0 < \alpha < 1$). Parameter α depends on the actual steps of the repair process including which parts are replaced, renewed, maintained, etc, while the value of d is a function of conditions (like temperature, humidity, etc.) under which the repairment is performed. In this model CDF $F(t)$, pdf $f(t)$ and parameters α and d are assumed to be known. Let $t > 0$ be a future time. For the value of $M(t)$ we can generalize the integral Equations (2) and (4) based on the well-known idea of expectation by conditioning. Assume that the first failure occurs at time S_1, then repair is finished at time $S_1 + T$, when the effective age of the object becomes $S_1 + dT$. Therefore, the effective age at time t becomes $t - T + dT$, since during T time periods the effective age increases with dT instead of T. The assumptions are illustrated in **Figure 3**. Expected number of failures and effective age of a single system. Assuming that the value of S_1 is fixed, then:

$$E(X_t \mid S_1) = \begin{cases} 0 & \text{if } t < S_1 \\ 1 & \text{if } S_1 \le t < S_1 + T \\ 1 + \alpha\left[M(t - T + dT) - M(S_1 + dT)\right] & \text{if } t \ge S_1 + T \end{cases} \quad (5)$$

where X_t is the true number of failures in interval $(0, t)$. The first case shows that no other failure can occur before the first failure. In the second case the first failure already occurred, and repair is not finished before time $S_1 + T$. In the third case notice that in effective age interval $(S_1 + T, t - T + dT)$), the expected number of failures would be the difference $M(t - T + dT) - M(S_1 + T)$, which is multiplied by α as the result of repair. And now the expectation of this function has to be determined with respect to S_1:

$$\begin{aligned} M(t) &= \int_t^{\infty} 0 f(s)\mathrm{d}s + \int_{t-T}^{t} 1 f(s)\mathrm{d}s \\ &\quad + \int_0^{t-T} \left\{1 + \alpha\left[M(t - T + dT) - M(s + dT)\right]\right\} f(s)\mathrm{d}s \end{aligned} \quad (6)$$

which can be simplified as

$$M(t) = F(t) + \alpha M(t - T + dT) F(t - T) - \alpha \int_0^{t-T} M(s + dT) f(s)\mathrm{d}s \quad (7)$$

if $t \ge T$. Otherwise, if $t < T$, then only one failure may occur before time t because repair cannot be finished before or at time T. The number of failures in interval $(0, t)$ is either 1 or 0 with probability $F(t)$ or $1 - F(t)$, respectively. Consequently $M(t) = F(t)$ in this case. A numerical integration-based algorithm can be suggested to find the value of function M in a grid between 0 and any $t > 0$.

Trapezoidal Rule

The trapezoidal rule calculates the area beneath a curve by subdividing it into

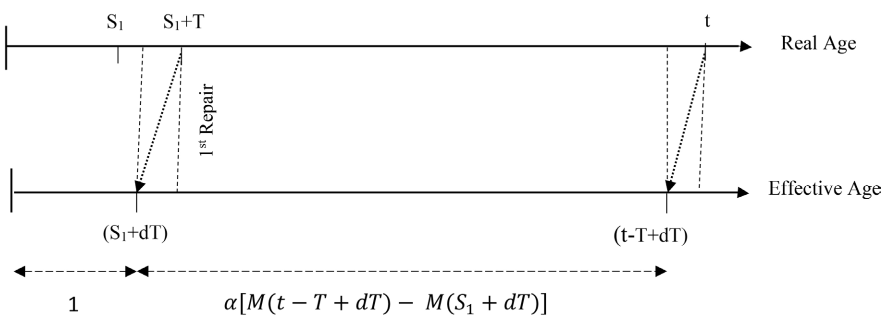

Figure 3. Expected number of failures and effective age of a single system.

multiple trapezoidal shapes and then summing the areas of these individual trapezoids. Therefore, this method is used as a numerical integration-based algorithm to find the expected number of failures.

Considering **Figure 4**. Trapezoidal Rule, $0, h, 2h, \cdots, Nh = t$ be the grid points and assume that the small value of h is selected so that $dT = Kh$ and $T = Lh$ with K and L being positive integers. The value of h can be selected so that dT is an integer multiple of h. If T is not an integer multiple of h, then select L so that $Lh \le T < (L+1)h$. For $\ell = 0,1,2,\cdots,L$ clearly $M(\ell h) = F(\ell h)$ since $\ell h \le T$. If $\ell > L$, then using the trapezoidal rule for approximating the integral in Equation (7) we get the approximation:

$$M(Nh) = F(Nh) + \alpha M((N-L+K)h) F((N-L)h) - \alpha h \left\{ \frac{1}{2} M((N-L+K)h) f((N-L)h) + \sum_{\ell=1}^{N-L-1} M((\ell+K)h) f(\ell h) \right\} \quad (8)$$

Notice that in the right-hand side the value of function M appears at earlier grid points than Nh, since $L > K$. So given the initial values of M at poins ℓh, $\ell = 0,1,2,\cdots,L$, the further values of $M(lh)$ can be obtained by using equation (8) in the order of $\ell = L+1, L+2, \cdots, N$.

Any other integral approximation can be used similarly. The most popular formulas is discussed in [28].

3.2. Special Cases and Model Variants

Assume first that there is no degradation during repair, when $d = 0$. Then Equation (7) reduces to

$$M(t) = F(t) + \alpha M(t-T) F(t-T) - \alpha \int_0^{t-T} M(s) f(s) \mathrm{d}s \quad (9)$$

and if minimal repair is assumed, then $\alpha = 1$, so in this simpler case:

$$M(t) = F(t) + M(t-T) F(t-T) - \int_0^{t-T} M(s) f(s) \mathrm{d}s \quad (10)$$

which is the same as given in [26]. If this model is further simplified by

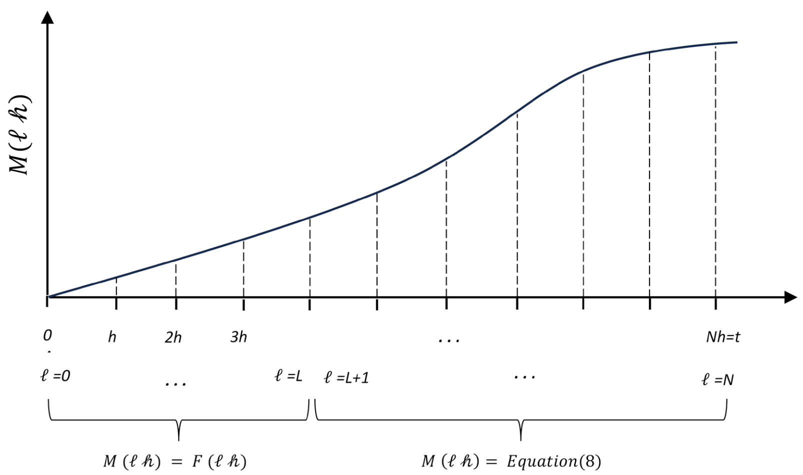

Figure 4. Trapezoidal rule.

assuming instantaneous repairs with $T=0$, then the well-known model in Equation (2) is obtained.

Assume next failure replacement. A similar model to Equation (7) can be derived similarly. Using the earlier notation:

$$E\left(X_t \mid S_1\right)=\begin{cases}0 & \text{if } t<S_1 \\ 1 & \text{if } S_1 \leq t<S_1+T \\ 1+E\left(X_{t-S_1-T}\right) & \text{if } t \geq S_1+T\end{cases} \tag{11}$$

since after repair $t-S_1-T$ periods are needed to reach *t*. So

$$\begin{aligned} M(t) &= \int_t^{\infty} 0 f(s) \mathrm{d}s+\int_{t-T}^t 1 f(s) \mathrm{d}s+\int_0^{t-T}\{1+M(t-s-T)\} f(s) \mathrm{d}s \\ & F(t)+\int_0^{t-T} M(t-s-T) f(s) \mathrm{d}s \end{aligned} \tag{12}$$

for $t \geq T$. If $t<T$,, then $M(t)=F(t)$. Assume again instantaneous repairs with $T=0$, then the equation reduces to Equation (4) as it should.

Assume next that the partial repairs decrease the effective age of the object by a fixed factor $\alpha<1$ as is shown in Figure 5. S_1 be the time of the first failure, then at time S_1 the effective age is clearly S_1, at time S_1+T it is S_1+dT and as the result of repair it decreases to $\alpha\left(S_1+dT\right)$. Without failure and repair it would be S_1+T, so the effective age is decreased by $S_1+T-\alpha\left(S_1+dT\right)$ in comparison to the no failure case.

So, the effective age at time *t* has to be

$$t-\left(S_1+T-\alpha\left(S_1+dT\right)\right)=t-(1-\alpha) S_1-T+\alpha dT \tag{13}$$

implying that

$$E\left(X_t \mid S_1\right) = \begin{cases}0 & \text{if } t<S_1 \\ 1 & \text{if } S_1 \leq t<S_1+T \\ 1+\left[M\left(t-(1-\alpha) S_1-T+\alpha dT\right)-M\left(\alpha\left(S_1+dT\right)\right)\right] & \text{if } t \geq S_1+T\end{cases} \tag{14}$$

so

$$M(t)=F(t)+\int_0^{t-T}[M(t-(1-\alpha) s-T+\alpha dT)-M(\alpha(s+dT))] f(s) \mathrm{d}s \tag{15}$$

Assuming minimal repair with $\alpha=1$, then Equation (15) becomes

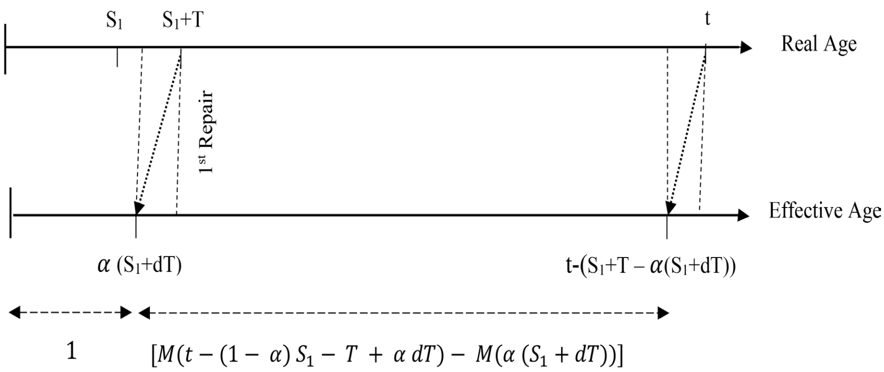

Figure 5. Expected number of failures and effective age of a single system.

$$\begin{aligned} M(t) &= F(t) + \int_0^{t-T} \left[M(t-T+dT) - M(s+dT) \right] f(s) \mathrm{d}s \\ &= F(t) + M(t-T+dT) F(t-T) - \int_0^{t-T} M(s+dT) f(s) \mathrm{d}s \end{aligned} \tag{16}$$

which is identical to Equation (7) with $\alpha = 1$.

In the further specialized case when $d = 0$ we have a simpler Equation:

$$M(t) = F(t) + M(t-T)F(t-T) - \int_0^{t-T} M(s) f(s) \mathrm{d}s \tag{17}$$

which gives back Equation (2) in the case of instantaneous repairs with $T = 0$. If $T > 0$, then the integral terms have $t - T$ as the upper bound for the interval. Therefore, a similar algorithm to Equation (8) can be developed, since in the right-hand sides only earlier values of function M are used. However, if $T = 0$, then the right-hand side also has the value of $M(t)$ so the same successive method cannot be used. Formulating the approximations of the integral equations for all $M(\ell h)$ values for $L+1 \le \ell \le N$, a system of linear algebraic equations is obtained for the unknown $M(\ell h)$ which can be then easily determined as follows. The equation for $\ell = L+1$ has only one unknown $M(L+1)$, so it can be solved for this unknown. After $M(L+1)$ is determined the equation for $M(L+2)$ gives the value of $M(L+2)$, and so on.

In very special cases, like for equation (4), Laplace transforms can be used. Let F^*, f^* and M^* denote the Laplace transforms of F, f and M, respectively, then from Equation (4),

$$M^*(s) = F^*(s) + M^*(s) f^*(s) \tag{18}$$

so $M^*(s)$ is now given and $M(t)$ is determined by using inverse Laplace transforms.

3.3. Testing Assessed Model Parameter Value

The CDF of first failures $F(t)$ can be obtained from the first failures data. After each repair the failure rate as well as the CDF changes. We can however transform the later failure times into effective ages of the object if $F(t)$ would not change. A successive algorithm can be offered to find the $(F(t)-)$ effective ages at each later failure.

Let t_1 denote the time of first failure, and before this event $F(t)$ is the CDF. Before repair the effective age is $\overline{t_1} = t_1$, which becomes $\overline{t_1} = t_1 + dT$ at calendar time $t_1 + T$.

Consider now the k^{th} failure of an object, which occurs at time t_k and $F_k(t)$ is the CDF before this failure occurs. Repair is finished at time $t_k + T$. If $\overline{t_k}$ is the $(F(t)-)$ effective age at the time of failure, then at the time $t_k + T$ it becomes $\overline{t_k} + dT$. If $\rho_k(t)$ is the failure rate before breakdown, then after repairing it becomes $\alpha\rho_k(t)$, since

$$F_k(t) = 1 - \mathrm{e}^{-\int_0^t \rho_k(\tau)\mathrm{d}\tau} \tag{19}$$

after repair the CDF becomes

$$F_{k+1}(t) = 1 - \mathrm{e}^{-\alpha\int_0^t \rho_k(\tau)\mathrm{d}\tau} \tag{20}$$

so, the corresponding reliability functions are as follows:

$$R_k(t) = \mathrm{e}^{-\int_0^t \rho(\tau)\mathrm{d}\tau}, \; R_{k+1}(t) = \mathrm{e}^{-\alpha\int_0^t \rho(\tau)\mathrm{d}\tau} = \left[R_k(t)\right]^{\alpha} \tag{21}$$

So

$$F_{k+1}(t) = 1 - R_{k+1}(t) = 1 - \left[1 - F_k(t)\right]^{\alpha} \tag{22}$$

Under $F_{k+1}(t)$ object is working from effective age $\bar{t}_k + dT$ at calendar age $t_k + T$ to calendar age t_{k+1}, so the $(F_{k+1}(t)-)$ effective age of the object is $\bar{t} + dT + t_{k+1} - t_k - T$, and if the $(F(t)-)$ effective age is $\bar{t}_{k+1}$, then

$$\bar{F}_k\left(\bar{t}_k + dT + t_{k+1} - t_k - T\right) = F\left(\bar{t}_{k+1}\right) \tag{23}$$

which always has a unique solution in $F(t)$ is strictly increasing. The solution $\bar{t}_{k+1}$ gives the $(F(t)-)$ effective age at time t_{k+1}.

The repeated application of this process gives the $(F(t)-)$ effective ages $\bar{t}_1, \bar{t}_2, \cdots$ of the subsequent failure times. If several identical objects are observed then for each of them the procedure can be repeated, so a set of failure $(F(t)-)$ effective ages become available. To test if the assessed parameter values α and d are acceptable we can use a hypothesis test to see if the obtained data set is homogeneous, or data of the first failures and that of the later failures belong to the same statistical family, that is, have the same distribution.

4. Numerical Experiment

In this section, we present the results of our numerical experiment. The experiment consists of five subsections: the first subsection concludes numerical example, the second subsection focuses on a rail application, where we evaluate the performance of our model on BNSF data set. In the third subsection, we conduct a simulation study to further validate our model's effectiveness. The fourth subsection involves a sensitivity analysis, where we investigate how the model's performance varies under different input conditions. Finally, in the last subsection, we consider multiple failure modes.

4.1. Numerical Example

Assume Weibull distribution for time to failure with parameters $\lambda = k = 2$, so

$$F(t) = 1 - \mathrm{e}^{-\left(\frac{t}{\lambda}\right)^k} = 1 - \mathrm{e}^{-\frac{t^2}{4}}, \; f(t) = \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1} \mathrm{e}^{-\left(\frac{t}{\lambda}\right)^k} = \frac{t}{2}\mathrm{e}^{-\frac{t^2}{4}} \tag{24}$$

The parameter values $\alpha = 0.9$, $d = 0.1$ and $T = 1$ are chosen. The grid step size was selected as $h = 0.01$. We considered interval (0, 10). Therefore $K = 10$, $L = 100$ and $t = Nh$ with $N = 1000$. Equation (8) was used to find the values of $M(T)$ for $t = \ell h$ for $\ell = 0, 1, 2, \cdots, 1000$. **Table 2** shows the computer results for integer values of t and **Figure 6**. The expected number of failures in the time interval (0,10) shows the expected number of failures within the time interval.

Table 2. Function values of $M(t)$.

t	0	1	2	3	4	5	6	7	8	9	10
$M(t)$	0	0.221	0.496	1.138	2.043	3.084	4.154	5.191	6.168	7.069	7.891

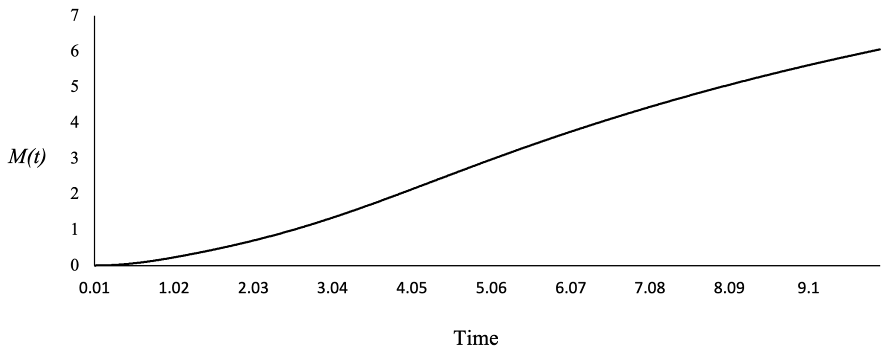

Figure 6. The expected number of failures in the time interval (0,10).

$$\begin{aligned}
&M(t=0.01)=F(0.01)\\
&M(t=0.02)=F(0.02)\\
&\vdots\\
&M(t=1)=F(1)\\
&M(t=1.1)=F(1.1)+\alpha M(0.11)F(0.01)-\alpha h\big(1/2\,M(0.11)\,f(0.01)\big)\\
&M(t=1.2)=F(1.2)+\alpha M(0.12)F(0.02)\\
&\qquad\qquad -\alpha h\big(1/2\,M(0.12)\,f(0.02)+M(0.11)\,f(0.01)\big)\\
&M(t=1.3)=F(1.3)+\alpha M(0.13)F(0.03)-\alpha h\big(1/2\,M(0.13)\,f(0.03)\\
&\qquad\qquad +M(0.11)\,f(0.01)+M(0.12)\,f(0.02)\big)\\
&\vdots\\
&M(t=10)=F(10)+\alpha M(9.11)F(9)-\alpha h\big(1/2\,M(9.11)\,f(9)\\
&\qquad\qquad +M(0.11)\,f(0.01)+\cdots+M(9.1)\,f(8.99)\big)
\end{aligned}$$

The expected time to failure is $\lambda\Gamma\left(1+\frac{1}{k}\right)=2\Gamma\left(1+\frac{1}{2}\right)=\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ which is approximated by 1.772. So, if after each repair brand new object would start working, then we might think that:

$$10/1.772\approx 5.643$$

is the expected number of failures. Since in time the object's degradation speeds up, it is clear that for larger values of t, the value of $M(t)$ becomes larger than this ratio.

4.2. Rail Application

BNSF railway is a significant North American network of freight railroads. The data set from BNFS 2007 to 2013 is used in this study. In the data set, a track can fail geometrically in one of three ways as follows [16]:

- The first is cross level failure mode, which assesses the variation in top surface elevation between two rails at any particular location along the railroad track. Since the rails can move up or down when under load, the cross-level measurement is typically done while they are in motion (**Figure 7**).

- The second is the surface failure mode, which assesses any irregularities in a rail's top surface. When there is a hump or a dip, the surface measurement can be positive or negative (**Figure 8**).
- DIP, which measures a decrease or increase in the track's centerline, is the third failure mode (**Figure 9**).

To model the time to failure for each failure mode, we use the Weibull distribution. The Weibull distribution as one of the most flexible and powerful distributions can be used to model both increasing and decreasing failure. The Weibull distribution has been used in [29] to model rail geometry defect time to failure. Equations (6) and (7) show the Weibull CDF and PDF for the failure at time t, respectively. The k and λ show shape and scale parameter of Weibull distribution parameters. For each failure mode the Weibull parameters (shape and scale) has been estimated and their standard deviation has been calculated based on the BNSF data and the result is illustrated in **Table 3**.

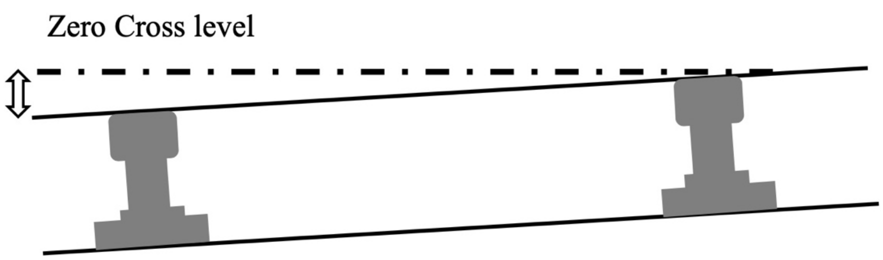

Figure 7. Graphical presentation of cross level defect.

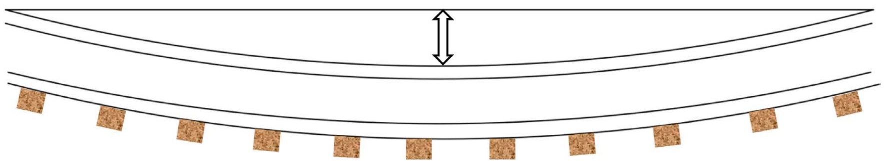

Figure 8. Graphical presentation of surface defect.

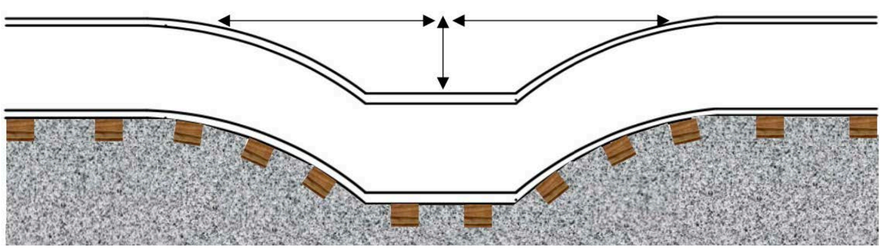

Figure 9. Graphical presentation of DIP defect.

Table 3. Estimated Weibull parameters for different modes of failure in railway [16].

Defect type	Estimate. Shape	Estimate. Scale	sd.shape	sd.scale
Cross level	1.3	238	0.085	16
DIP	1.5	146	0.12	12
Surface	1.2	212	0.131	27

parameter values $\alpha = 0.8$, $d = 0.1$ and $T = 1$ are chosen. The grid step size is selected as $h = 0.01$. We consider an interval of two years or (0, 730) days to study the number of failures within the selected interval for each of failure mode. Therefore $K = 1$, $L = 10$ and $t = Nh$ with $N = 7300$. The proposed algorithm is used to find the values of $M(T)$ for $t = \ell h$ for $\ell = 0,1,2,\cdots,7300$. Figure 10 shows the computer results for $M(t)$ for different type of failures.

4.3. Simulation Study

To evaluate our model, we conduct a Monte Carlo (MC) simulation study to examine different failure modes. We take inspiration from [14] particularly sections 5.1 and 5.3.

Step 1: Generating Random Values

In this step, we assume a Weibull distribution for inter-arrival failures. To represent $F(t_i)$, we generate random values from a uniform distribution ranging between 0 and 1.

Step 2: Accumulating Time Intervals

As part of the simulation process, we repeatedly generate random t_i values. Each generated t_i is cumulatively added to the sum of previously generated t_i values.

Step 3: Comparing with Time Interval

We compare the cumulative sum of these generated t_i values with a predefined time interval. In our study, this time interval is set at 2 years (730 days).

Step 4: Determining Simulation Outcome

If the cumulative sum of all generated t_s equal or exceed 730 days, the simulation algorithm terminates. At this point, we calculate the average number of failures generated at each step. This average represents the expected number of failures based on 10,000 simulation runs.

Step 5: Continuing Simulation

Conversely, if the cumulative sum of generated t_s is still less than the study time interval (730 days), we continue generating additional t_i values.

Step 6: Result Presentation

Finally, the result of this MC simulation is used to determine the number of failures occurring within a two-year timeframe, which is presented in **Table 4**.

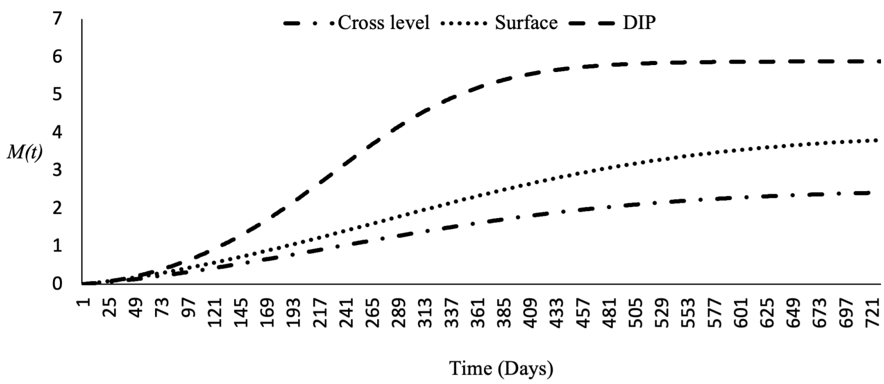

Figure 10. Expected number of failures within two years in different Failure modes.

Table 4. Number of failures within two years using simulation method.

Failure Mode	Cross level failure	Surface failure	DIP failure
$M(t)$	2.4150	3.8074	5.881

The findings indicate that, within a two-year timeframe, the anticipated occurrences of cross level failures, surface failures, and DPI failures are 2.4, 3.8, and 5.8, respectively.

4.4. Sensitivity Analysis

In this part the sensitivity of parameters regarding expected number of failures is discussed and cross level mode is selected as the failure analysis. Three parameters are considered to analyze the results. First, the shape parameter of Weibull distribution (*i.e.*, k) is selected. As it is shown in **Figure 11**, by increasing value of k in range (0.5, 4.5) and keeping other parameters constant, the expected number of failures is increased dramatically. The next two parameters have impacts on the effective age of the system. **Figure 12** shows by decreasing the value of a, the expected number of failures is dropped since the value of $\alpha = 0$ shows the quality of repair is as good as new, while $\alpha = 1$ shows the quality of repair is bad as old. Also, the effect of degradation when the system is idle (d) is studied. By increasing the value of d, the expected number of failures increased but not significantly. Therefore, we can conclude the effect of d on $M(t)$ is negligible

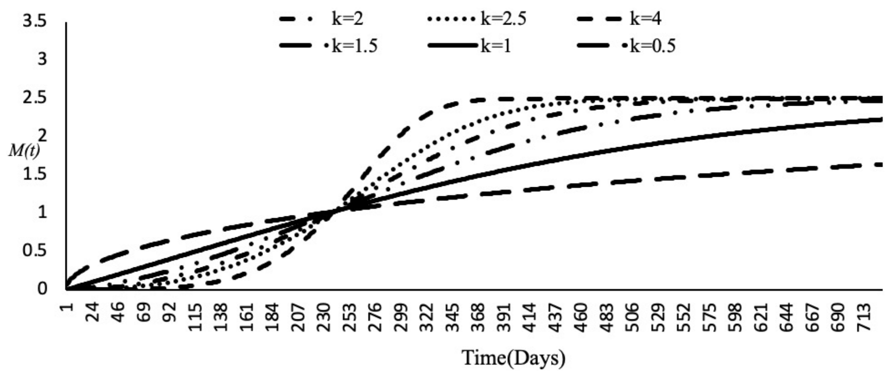

Figure 11. Sensitivity analysis on shape parameter of Weibull distribution (*i.e.*, k).

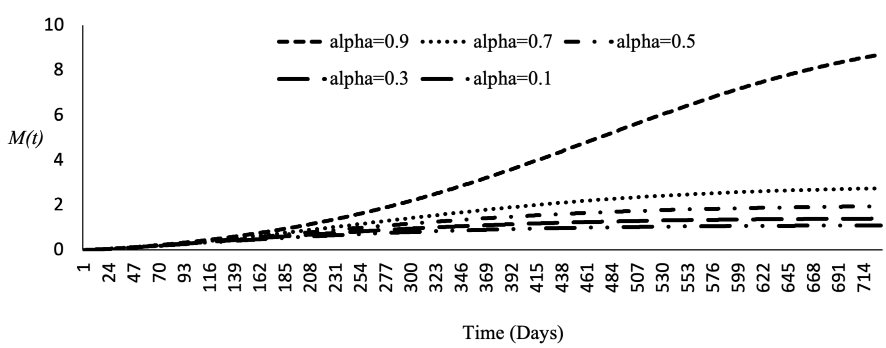

Figure 12. Sensitivity analysis on quality of repair parameter (α).

in this case. When the degradation rate increases, the effective age of the system increases and consequently, the system gets old more than the other modes and the chance of facing the failure increases.

5. Conclusions and Future Research

Any system, process, equipment and parts are subject to random failures, which have to be then repaired. The appropriate planning of repairs needs an assessment of number of future failures in any given time intervals in order to secure the needed parts, material and qualified manpower. This central problem of reliability theory is considered, and algorithms are offered for its solution. Special renewal processes are used to model the repeated failures of any equipment or system. When estimating the expected number of failures within an interval (0, *t*), it is important to consider that repair time is not negligible, despite most studies assuming that failure and repair occur simultaneously. Instead, failures and repairs occur sequentially, making it necessary to account for repair time when modeling the expected number of failures. Also, quality of repair and degradation of the single system when the system is idle is noticed. The Volterra-type integral equations are used in the modeling the mentioned problem. A numerical integration-based algorithm is also suggested to solve the general model and some special cases is discussed subsequently.

A numerical example illustrates the methodology, the trapezoidal rule is used for illustration. The distribution of time to the first failure is a Weibull distribution, which is known. So, the expected number of failures is equal to CDF of the Weibull function and for the other failures that happen after the first failure the distribution is unknown since partial repair is used. The algorithm uses a numerical integration-based method to solve the proposed model. The BNSF data for rail track defects is used for numerical experiments. A simulation study and sensitivity analysis have implemented using the proposed model. Sensitivity study shows when partial repair α rate increase, the expected number of failures increase. The degradation d rate also shows slight increase in the expected number of failures which is negligible. The result shows the expected number of future failures for each failure type mode within a two-year maintenance interval. Therefore, the expected number of future failures provides valuable insight for management to determine the most appropriate course of action, whether it be partial repairs or investing in a new system. By considering this key factor, management can make an informed decision that aligns with their organizational goals and priorities.

There are several ways to expand the research reported in this paper. Two types of partial repairs are considered: reducing the number of future failures or decreasing the effective age of the object by a factor. However other types are also discussed in the literature: decreasing the effective age by a constant term, reducing the failure rate, etc. In this study, the length of repair (parameter α) were deterministic, however in reality they are uncertain. So, stochasticity could be included in the models, as well. These models could be also embedded in an optimal resources allocation problem of using material, equipment, and manpower for repairments as effectively as possible.

Conflict of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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