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Quantum GS Field and Explanation of Dark Matter

Qiao Bi

Physics Department, Science School, Wuhan University of Technology, Wuhan, China
Email: biqiao@gmail.com

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Abstract

Quantum gravity is regarded as a pivotal theoretical objective in unifying general relativity and quantum mechanics. This paper proposes a novel approach within the framework of the Generalized Gauge Equation (GGE), whereby quantum gravitational effects are induced by quantized fields such as the electromagnetic field. Specifically, we demonstrate that under GGE transformations, the quantized electromagnetic field generates a gravitational gauge potential with quantum characteristics, enabling the construction of corresponding quantum Weyl tensors and gravitational soliton solutions. In the weak-field limit, the model naturally reduces to a two-photon-to-graviton conversion process, consistent with linear quantum gravity theories. We further establish nonlinear Gravitational Spinor (GS) equations. It is particularly noteworthy that, via the quantized spinorial Weyl-electromagnetic relation, we have for the first time rigorously derived the nonlinear term in these equations as $\left|\nabla\psi\right|^2$. This nonlinear term is not only a necessary requirement for theoretical self-consistency but also the key to why the solutions can reproduce dark matter phenomena on galactic scales—the resulting solutions naturally exhibit dark matter-like behavior and align closely with the observational predictions of MOND theory. This finding may reveal a more profound geometric and quantum relational nature of natural gravity. Within the $GL(m)$ principal bundle geometry framework, we demonstrate that the four fundamental interactions can be unified at the classical level, with quantization serving as an effective description at microscopic scales. The proposed "gravitational spinor (GS)" acts as the fundamental quantum unit of the vacuum gravitational field, mediating gravitational interactions through virtual particle exchange. Our study indicates that quantum gravity serves as a bridge connecting physical phenomena across different scales, where the classical geometry of general relativity and quantum descriptions coexist consistently across different levels. This research not only provides a new theoretical pathway for exploring quantum gravitational effects but also directly links quantum gravity to galactic dynamics through precise nonlinear extensions, thereby advancing further reflection on the theoretical foundations and research paradigms of quantum gravity.

Keywords

Quantum Gravity, Gauge Theory, GGE Transformation, Gravitational Spinor, Nonlinear Equation

1. Introduction

General relativity and quantum mechanics stand as the twin pillars of twentieth-century physics, each achieving remarkable experimental validation in the macroscopic cosmos and microscopic world, respectively [1]-[5]. However, reconciling these two theories into a consistent quantum gravity framework remains one of the most profound and formidable challenges in modern physics. Traditional quantization approaches, such as string theory and loop quantum gravity, aim to construct a universal, background-independent fundamental theory to quantize gravity, with the ultimate goal of unifying all fundamental interactions [6] [7]. While these approaches have yielded significant theoretical achievements, the absence of experimental evidence and their inherent complexity encourage an open-minded exploration of alternative perspectives.

In this context, a pragmatic viewpoint is gaining traction: the primary purpose of quantum gravity may not be the pursuit of a formal ultimate unification but rather the provision of an effective theoretical tool to resolve the sharp contradictions between general relativity and quantum principles at microscopic scales, such as black hole singularities, the Hawking radiation information paradox, and the structure of spacetime at the Planck scale [8] [9]. In other words, quantum gravity might function more as a "bridge" connecting physics across different scales rather than an all-encompassing "theory of everything".

This article explores a quantization pathway for gravity based on the principle of gravitational-gauge equivalence, formalized through the Generalized Gauge Equation (GGE) [10]. Within the geometric language of a $GL(m,\mathbb{C})$ principal bundle, this framework interprets gravity as a gauge theory of spacetime symmetries, providing a unified geometric description of the four fundamental interactions at the classical level [11] [12]. The core of our work demonstrates that a successfully quantized field, such as the electromagnetic field, can induce quantum characteristics in the gravitational sector through specific GGE transformations, generating gravitational gauge potentials and spacetime soliton structures with quantum behavior. Specifically, we derive how a quantized electromagnetic field spinor mapping constructs quantum Weyl tensors and show how two quantum optical solitons merge into a quantum gravitational soliton via nonlinear

GGE transformations. In the weak-field approximation, this process naturally reduces to an effective field theory picture of two photons converting into one graviton.

Building on these results, we propose a theoretical hypothesis: the composite spinor operator defining the quantum Weyl tensor can be interpreted as a fundamental "gravitational spinor (GS)", suggesting that the vacuum gravitational field is composed of such spinor fields. Their excited states form gravitational solitons in specific modes, while in the weak-field limit, they reduce to gravitons. From this perspective, the mechanism of gravitational interaction can be understood through the exchange of "virtual GS", offering a novel possibility for understanding the microscopic mechanisms of quantum gravity.

The structure of this article is as follows: First, we review the classical GGE transformation and provide its rigorous quantum extension. Next, we derive the relationship between the quantum Weyl tensor and the electromagnetic tensor, constructing quantum gravitational soliton solutions. Subsequently, we verify the consistency of the theory in the weak-field limit and demonstrate its connection to linear quantum gravity. We then advance to a crucial development: deriving and solving the nonlinear GS equation, whose solutions naturally exhibit dark matter-like behavior on galactic scales and remarkably align with the empirical MOND relation. This provides a geometric quantum-gravitational perspective on one of contemporary physics' most enduring mysteries. We discuss these results, propose the GS hypothesis, and argue that within the GGE framework, quantization serves as an effective method for addressing specific problems, while the unity of the physical world is rooted primarily in classical geometric structures [10] [13]-[17].

This study does not aim to negate the ambitious goals of other quantum gravity approaches but rather seeks to offer a complementary, instrumental technical pathway to the possibility map of quantum gravity exploration.

2. Derivation of the Gravitational Gauge Potential of the Quantized GGE Transformation

Based on the background GGE transformation, we explore the specific form of the gravitational gauge potential induced by the quantization of the electromagnetic gauge potential via the GGE. To simplify the analysis, the transformation matrix g is kept classical (non-quantized). The goal is to provide an explicit expression for the gravitational gauge potential and discuss its quantum properties and physical significance.

2.1. Classical GGE Transform

The GGE transformation formula converts the electromagnetic gauge potential $U(1)$ into the gravitational connection $SO(1,3)$:

$$\omega_V = g_{UV}^{-1}\omega_U g_{UV} + g_{UV}^{-1} dg_{UV} \tag{1}$$

where

- **Electromagnetic gauge potential:** $\omega_U = A_\mu T_{EM} dx^\mu$, here $T_{EM} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the $U(1)$ generator (embedded in $GL(2,\mathbb{C})$), and $A_\mu = (\phi, A_x, A_y, A_z)$ is the electromagnetic potential.
- **Transformation matrix:** $g_{UV} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, which is the $SO(2)$ rotation matrix, where θ is a parameter (which may depend on the coordinate x^μ).
- **Inverse matrix:** $g_{UV}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

Compute the transformation:

- **The first term:** $g_{UV}^{-1} A_\mu T_{EM} g_{UV} = A_\mu \left(g_{UV}^{-1} T_{EM} g_{UV} \right)$.According to formula (1) above: $g_{UV}^{-1} T_{EM} g_{UV} = J_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, where J_{12} is a Lorentz generator (a subset of so(1,3)), therefore $g_{UV}^{-1} A_\mu T_{EM} g_{UV} = A_\mu J_{12}$. The calculation verification results are as follows:

$$g_{UV}^{-1} T_{EM} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix}$$

Multiplying by g_{UV}, we get:

$$\begin{aligned} g_{UV}^{-1} T_{EM} g_{UV} &= \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \\ -\sin^2\theta - \cos^2\theta & \sin\theta\cos\theta - \cos\theta\sin\theta \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J_{12} \end{aligned}$$

So the equation $g_{UV}^{-1} T_{EM} g_{UV} = J_{12}$ holds, indicating that g_{UV} (the rotation matrix) transforms T_{EM} to J_{12}. Since $T_{EM} = J_{12}$, g_{UV} actually leaves T_{EM} unchanged, which is consistent with the embedding assumption.

- **The second term (Maurer-Cartan form):** $g_{UV}^{-1} \partial_\mu g_{UV}$. If θ is a constant, then $\partial_\mu g_{UV} = 0$, if $\theta = \theta(x^\mu)$, then:

$$\partial_\mu g_{UV} = \begin{pmatrix} -\sin\theta\partial_\mu\theta & -\cos\theta\partial_\mu\theta \\ \cos\theta\partial_\mu\theta & -\sin\theta\partial_\mu\theta \end{pmatrix}$$

$$g_{UV}^{-1}\partial_\mu g_{UV} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -\sin\theta\partial_\mu\theta & -\cos\theta\partial_\mu\theta \\ \cos\theta\partial_\mu\theta & -\sin\theta\partial_\mu\theta \end{pmatrix} = \begin{pmatrix} 0 & \partial_\mu\theta \\ -\partial_\mu\theta & 0 \end{pmatrix}$$

therefore the classical gravitational connection can be expressed as:

$$\omega_{V,\mu} = A_\mu J_{12} + \begin{pmatrix} 0 & \partial_\mu\theta \\ -\partial_\mu\theta & 0 \end{pmatrix} = \begin{pmatrix} 0 & A_\mu + \partial_\mu\theta \\ -\left(A_\mu + \partial_\mu\theta\right) & 0 \end{pmatrix} \tag{2}$$

2.2. Quantized Electromagnetic Gauge Potential

In quantum electrodynamics (QED), the electromagnetic potential $A_\mu \to \hat{A}_\mu$ becomes an operator that satisfies the commutation relation:

$$\left[\hat{A}_\mu(x), \hat{A}_\nu(y)\right] = i\hbar D_{\mu\nu}(x-y) \tag{3}$$

where $D_{\mu\nu}$ is the photon propagator (e.g. under the Feynman gauge). $\hat{A}_\mu$ describes the photon quantum field.

Apply the GGE transformation (g is classical, linear in the operator):

$$\hat{\omega}_{V,\mu} = g_{UV}^{-1}\left(\hat{A}_\mu T_{EM}\right) g_{UV} + g_{UV}^{-1}\partial_\mu g_{UV} \tag{4}$$

we calculate:

- **Term 1**: $g_{UV}^{-1}\left(\hat{A}_\mu T_{EM}\right) g_{UV} = \hat{A}_\mu\left(g_{UV}^{-1} T_{EM} g_{UV}\right) = \hat{A}_\mu J_{12} = \hat{A}_\mu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;
- **Term 2**: Same as the classical case, $g_{UV}^{-1}\partial_\mu g_{UV} = \begin{pmatrix} 0 & \partial_\mu\theta \\ -\partial_\mu\theta & 0 \end{pmatrix}$ (classical correction term).

So we get the specific form of quantum gravitational connection:

$$\hat{\omega}_{V,\mu} = \hat{A}_\mu J_{12} + \begin{pmatrix} 0 & \partial_\mu\theta \\ -\partial_\mu\theta & 0 \end{pmatrix} = \begin{pmatrix} 0 & \hat{A}_\mu + \partial_\mu\theta \\ -\left(\hat{A}_\mu + \partial_\mu\theta\right) & 0 \end{pmatrix} \tag{5}$$

2.3. Quantum Properties: Commutation Relations

The quantum properties of the gravitational connection $\hat{\omega}_{V,\mu}$ are inherited through $\hat{A}_\mu$:

$$\left[\hat{\omega}_{V,\mu}(x), \hat{\omega}_{V,\nu}(y)\right] = g_{UV}^{-1}\left[\hat{A}_\mu T_{EM}(x), \hat{A}_\nu T_{EM}(y)\right] g_{UV} \tag{6}$$

Since:

$$\left[\hat{A}_\mu T_{EM}(x), \hat{A}_\nu T_{EM}(y)\right] = \left[\hat{A}_\mu(x), \hat{A}_\nu(y)\right] T_{EM}^2 \propto i\hbar D_{\mu\nu}(x-y) T_{EM} \tag{7}$$

then, after transformation we obtain:

$$\left[\hat{\omega}_{V,\mu}(x), \hat{\omega}_{V,\nu}(y)\right] \propto i\hbar D_{\mu\nu}(x-y)\left(g_{UV}^{-1} T_{EM} g_{UV}\right) = i\hbar D_{\mu\nu}(x-y) J_{12} \tag{8}$$

This shows that $\hat{\omega}_{V,\mu}(x)$ is a quantum operator, similar to the graviton field, which inherits the quantum fluctuations of the electromagnetic field.

2.4. Physical Meaning and Inspiration

- **Quantum induction**: The quantum properties of the electromagnetic potential $\hat{A}_\mu$ are directly transferred to the gravitational connection $\hat{\omega}_{V,\mu}(x)$ via the GGE transformation, supporting the idea that "electromagnetic quantization induces gravitational quantization". This is consistent with the gravitodynamics in [9] [10], where the $U(1)$ gauge gravity can be induced from QED and may be renormalizable.
- **Solitons and curvature drives**: Based on the background in [10], $\hat{A}_\mu$ can represent quantum optical solitons (such as laser pulses), which can generate

quantum gravitational solitons through GGE. This provides a theoretical basis for manipulating spacetime curvature (such as curvature drives or warp drives).

- **The role of θ**: The classical $\partial_\mu \theta$ provides an additional degree of freedom, potentially simulating Higgs-like fields or dynamical background effects (e.g., curvature perturbations near the black hole's event horizon). If quantization of θ is considered in the future, graviton self-interactions could be introduced.
- **Limitations**: The current formulation does not deal with the nonlinear self-interactions of general relativity (non-renormalization problems). This may be extended to a wider context in gauge theory gravity (GTG) or teleparallel gravity [3] [11].

3. Extension of the GGE Transform to Higher Dimensions

This section considers extending the GGE transformation to four-dimensional space-time. The transformation matrix g_{UV} should belong to the Lorentz group $SO(1,3)$ or its embedding $GL(4,\mathbb{C})$.

3.1. Transformation Matrices and Lorentz Group Embedding

A Lorentz transformation matrix Λ^μ_ν satisfies $\Lambda^\mathrm{T}\eta\Lambda = \eta$. A simple example is a pure rotation around an axis, whose generator is $g = \exp(\theta J^{12})$, which can be expressed in matrix form as:

$$g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

3.2. Higher-Dimensional Embedding of Electromagnetic Potential

The $U(1)$ electromagnetic gauge potential needs to be embedded in the Lorentz algebra so(1,3). A common approach is to use the Levi-Civita notation ϵ^{abc} (where $a,b,c = 1,2,3$ are spatial indices) to draw an analogy between the electromagnetic field and the gravito-magnetic field. For example, the electromagnetic potential can be mapped as a combination of spatial rotation generators:

$$\hat{A}_\mu T_{EM} \to \hat{A}_\mu \epsilon^{abc} J_{ab} \tag{10}$$

where J_{ab} is the spatial rotation generator (e.g., $\epsilon^{123} J_{12}$, mapping $\hat{A}_\mu$ to the gravito-magnetic potential). For example, we choose to embed into
$J^{12} : \hat{A}_\mu T_{EM} = \hat{A}_\mu J^{12}$ (extending 2×2 to 4×4). In this case, J^{12} is a 4×4 matrix (in the vector representation of so(1,3)) with its nonzero elements at the corresponding positions in the matrix.

3.3. GGE Transformation Formula (Component Form)

Generalize GGE to the quantized GGE framework, and the quantized gravitational connection component is transformed into:

$$\hat{\omega}_{V,\mu}^{ab} = \left[g^{-1} \left(\hat{A}_\mu \epsilon^{cd} J_{cd} \right) g \right]^{ab} + \left[g^{-1} \partial_\mu g \right]^{ab} \tag{11}$$

The quantum connection $\hat{\omega}_{V,\mu}^{ab}$ defined in this work is closely related to the spin connection in general relativity. In the traditional gauge-theoretic framework, the spin connection is used to formulate general relativity as a gauge theory based on the Lorentz group (see, for example, P.D. Collins, A.D. Martin, and E.J. Squires, [18]). In our theory, $\hat{\omega}_{V,\mu}^{ab}$ is promoted to a "quantum connection" that not only encompasses the geometric content of the classical gravitational spin connection but is also embedded within the GGE framework to describe a unified gauge structure of gravitational and electromagnetic interactions at the quantum level. Specifically, we regard it as one of the fundamental variables for constructing a theory of quantum gravity. Its relation to the curvature tensor is formally consistent with the classical case, but it is endowed with a new physical interpretation during the quantization process—namely, it serves as a gravitational gauge field induced by the quantum electromagnetic potential $\hat{A}_\mu$. This relationship of equivalence and promotion is considered a cornerstone in building the present quantum gravity theory.

The first term is the adjoint action: $g^{-1} \left(\hat{A}_\mu \epsilon^{cd} J_{cd} \right) g$, which transfers the properties of the quantum electromagnetic potential $\hat{A}_\mu$ to the gravitational connection via the embedding tensor ϵ^{cd}.

The **Maurer-Cartan term** $\theta_\mu^{ab} = \left(g^{-1} \partial_\mu g \right)^{ab}$ is a Lie-algebra valued 1-form, where $\theta_\mu = g^{-1} \partial \mu g$ is the Maurer-Cartan form on the gauge group G. For Lorentz transformations, this term represents the non-integrable part of the connection, contributing to the spin connection in curved spacetime.

After choosing the simplified embedding (such as ϵ^{ab} representing the mapping to J^{12}), the transformed quantized gravitational connection form can be simplified to:

$$\hat{\omega}_{V,\mu}^{ab} \propto \hat{A}_\mu \epsilon^{ab} + \theta_\mu^{ab} \tag{12}$$

This is similar to the gravito-magnetic potential in gravitational electromagnetism (GEM) theory, which is associated with the angular momentum or frame-dragging effect of a rotating black hole.

Transformation properties: g is a group element in G (e.g., $SO(1,3)$ or $GL(n,\mathbb{C})$), transforming under the adjoint representation. $\hat{A}_\mu$ is the quantized electromagnetic gauge potential operator valued in u(1), embedded into so(1,3) via ϵ^{cd}. Under quantized GGE, the connection $\hat{\omega}$ transforms as a Lie-algebra valued 1-form operator, with the Maurer-Cartan term ensuring the gauge covariance in the quantum regime, consistent with path integral formulations without requiring full quantum gravity assumptions.

3.4. The Core Idea and Meaning of Quantum Induction

Core consistency: The quantum electromagnetic potential $\hat{A}_\mu$ induces the quantum gravitational connection $\hat{\omega}_{V,\mu}^{ab}$ through embedding and GGE transformation, inheriting the commutation relation:

$$\left[\hat{\omega}_{V,\mu}^{ab}(x), \hat{\omega}_{V,\nu}^{cd}(y)\right] \propto \hbar D_{\mu\nu}(x-y)\epsilon^{abe}\epsilon^{cdf}\left(g^{-1}J_{ef}g\right)^{ab} \quad (13)$$

This result extends the previous conclusions in the 2×2 case to the full Lorentz group, supporting the idea of constructing Lorentz-invariant quantum gravitational solitons.

However, the computation becomes more complex due to the need to process the six independent antisymmetric components ab of the so(1,3) algebra. This extended form is particularly effective in frameworks such as Gauge Theory Gravity (GTG), providing new theoretical tools for simulating physical phenomena such as black holes and gravitational waves in a broader context.

4. Derivation of the Quantized GGE Transformation: From Electromagnetic Field to Gravity

This section, building on the background GGE framework, spinor representation, and gauge transformation GGE, provides a detailed derivation of how the quantized electromagnetic gauge potential induces quantization on the gravitational side. We assume that the electromagnetic gauge potential is quantized according to quantum electrodynamics (QED), *i.e.*, the operator $\hat{A}_\mu$, with a field strength of $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu$, and induces quantization of the gravitational field via the GGE mechanism. All derivations are performed in the vacuum background $R_{\mu\nu} = 0$, where the Weyl tensor plays a dominant role.

4.1. The Relationship between the Quantized Electromagnetic Tensor and the Weyl Tensor

The classical Weyl-electromagnetic tensor relation [10] shows that the Weyl tensor $C_{\mu\nu\rho\sigma}$ can be constructed by combining two electromagnetic tensors $F_{\mu\nu}^{(1)}$ and $F_{\mu\nu}^{(2)}$:

$$C_{\mu\nu\rho\sigma} = \kappa_{coupl}\left(F_{\mu\rho}F_{\nu\sigma} - F_{\mu\sigma}F_{\nu\rho}\right) \quad (14)$$

where κ_{coupl} is a conversion coefficient of dimension $\left[\kappa_{coupl}\right] = L^2$ in the geometric unit system ($G = c = 1$), in which all quantities are expressed in terms of length L (time $T = L$, mass $M = L^{-1}$). In this system, the Weyl tensor $C_{\mu\nu\rho\sigma}$ has dimensions $[C] = L^{-2}$, as curvature terms like the Riemann tensor $R_{\mu\nu\rho\sigma}$ have $[R] = L^{-2}$ (Christoffel symbols $\Gamma \sim L^{-1}$, $R \sim \partial\Gamma + \Gamma^2 \sim L^{-2}$). The electromagnetic tensor $F_{\mu\nu}$ has dimensions $[F] = L^{-2}$ (gauge potential $A_\mu \sim L^{-1}$, $F \sim \partial A + [A, A] \sim L^{-2}$), so $F_{\mu\rho}F_{\nu\sigma}$ has $[FF] = L^{-4}$. Thus, $[\kappa(FF)] = L^2 \times L^{-4} = L^{-2}$, matching $[C]$. This construction automatically satisfies the antisymmetric, traceless, and conformally invariant properties of the Weyl tensor. This relationship is implemented via the spinor mapping: $\psi_{ABCD} \sim \phi_A^{(1)}\phi_B^{(1)}\phi_C^{(2)}\phi_D^{(2)}$, such that $C_{\mu\nu\rho\sigma} \sim \psi_{ABCD}\left(\sigma_{\mu\rho}^{AB}\sigma_{\nu\sigma}^{CD} - \sigma_{\mu\sigma}^{AB}\sigma_{\nu\rho}^{CD}\right)$, where σ is a Pauli-like matrix [10] [14] [15]. The calculation and verification results are as follows:

Step 1: Electromagnetic curvature and F_U

- Electromagnetic gauge potential: $\omega_U = \text{sech}^2(ku)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}T_{EM}$
  - $T_{EM} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$
  - Corresponding to the electromagnetic four-vector potential: $\omega_U \sim A_\mu dx^\mu$
- For example, the polarization matrix $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ represents: $e_x^{(1)} = 1$, $e_y^{(2)} = -1$
  - Curvature: $F_U = d\omega_U + \omega_U \wedge \omega_U$
  - For the $U(1)$ gauge field, $\omega_U \wedge \omega_U = 0$ (because $U(1)$ is an Abelian group):
  - $F_U = d\omega_U$
  - $F_U \sim F_{\mu\nu} dx^\mu \wedge dx^\nu$
  - $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
  - Spinor representation:
    - $F_{\mu\nu} \sim \phi_A \phi_B \sigma_{\mu\nu}^{AB}$
    - $\sigma_{\mu\nu}^{AB} = \sigma_{[\mu}^{AC} \bar{\sigma}_{\nu]CB}$

Clarification: The expression $F_{\mu\nu} \sim \phi_A \phi_B \sigma_{\mu\nu}^{AB}$ is a valid ansatz, where ϕ_A is interpreted as a fundamental spinor field whose symmetric product $\phi_A \phi_B$ constructs the symmetric Maxwell spinor ϕ_{AB} that encodes the electromagnetic field strength. This provides a direct building block for the gravitational spinor.

Step 2: Gravitational curvature and F_V

- Gravitational gauge potential: $\omega_V = \text{sech}^2(ku)\begin{pmatrix}A & B\\ B & -A\end{pmatrix}J_{12}$
- J_{12} is the $SO(1,3)$ generator, corresponding to the spinor connection: $\omega_V \sim \omega_{b\mu}^a dx^\mu$
- Curvature: $F_V = d\omega_V + \omega_V \wedge \omega_V$
  - $F_V = \frac{1}{2} R_{b\mu\nu}^a dx^\mu \wedge dx^\nu$
  - Under vacuum: $R_{b\mu\nu}^a = C_{b\mu\nu}^a$
  - $F_V \sim C_{\mu\nu ab} dx^\mu \wedge dx^\nu$
- Spinor representation: $C_{\mu\nu\rho\sigma} \sim \psi_{ABCD} \sigma_{\mu\nu}^{AB} \sigma_{\rho\sigma}^{CD}$

Here the representation $C_{\mu\nu\rho\sigma} \sim \psi_{ABCD} \sigma_{\mu\nu}^{AB} \sigma_{\rho\sigma}^{CD}$ is algebraically feasible. However, the standard and more fundamental representation expresses the Weyl tensor directly in terms of the totally symmetric Weyl spinor ψ_{ABCD} via $C_{\mu\nu\rho\sigma} \sigma_{AA'}^\mu \sigma_{BB'}^\nu \sigma_{CC'}^\rho \sigma_{DD'}^\sigma = \psi_{ABCD} \epsilon_{A'B'} \epsilon_{C'D'} + \bar{\psi}_{A'B'C'D'} \epsilon_{AB} \epsilon_{CD}$. The presented form can be seen as a component expansion consistent with this fundamental relation.

Step 3: GGE transformation and spinor synthesis

- **GGE transformation**: $F_V = g_{UV}^{-1} F_U g_{UV}$
  - Problem: $F_U \sim F_{\mu\nu}$ is a binary form, and $F_V \sim C_{\mu\nu\rho\sigma}$ is a fourth-order tensor.

- Solution: Assume that F_U represents the field intensity combination of two optical solitons: $F_U \sim F_{\mu\nu}^{(1)} F_{\rho\sigma}^{(2)}$
- Corresponding to the spinors of two photons: $F_{\mu\nu}^{(1)} \sim \phi_A^{(1)} \phi_B^{(1)} \sigma_{\mu\nu}^{AB}$; $F_{\rho\sigma}^{(2)} \sim \phi_C^{(2)} \phi_D^{(2)} \sigma_{\rho\sigma}^{CD}$
- **Spinor mapping**:
  - Gravitational spinor: $\psi_{ABCD} \sim \phi_A^{(1)} \phi_B^{(1)} \phi_C^{(2)} \phi_D^{(2)}$

Where the expression $\psi_{ABCD} = \phi_A^{(1)} \phi_B^{(1)} \phi_C^{(2)} \phi_D^{(2)}$ is automatically totally symmetric in its indices A, B, C, D if the $\phi_A^{(i)}$ are commuting classical fields. This provides a natural ansatz for the Weyl spinor ψ_{ABCD} as a product of four fundamental spinor fields.

- Under the gravitational spinor $\psi_{ABCD} \sim \phi_A^{(1)} \phi_B^{(1)} \phi_C^{(2)} \phi_D^{(2)}$, $C_{\mu\nu\rho\sigma}$ can be expressed as:

$$\begin{aligned} C_{\mu\nu\rho\sigma} &\sim \psi_{ABCD} \left(\sigma_{\mu\rho}^{AB} \sigma_{\nu\sigma}^{CD} - \sigma_{\mu\sigma}^{AB} \sigma_{\nu\rho}^{CD} \right) \\ &\sim \left(\phi_A^{(1)} \phi_B^{(1)} \sigma_{\mu\rho}^{AB} \right) \left(\phi_C^{(2)} \phi_D^{(2)} \sigma_{\nu\sigma}^{CD} \right) - \left(\phi_A^{(1)} \phi_B^{(1)} \sigma_{\mu\sigma}^{AB} \right) \left(\phi_C^{(2)} \phi_D^{(2)} \sigma_{\nu\rho}^{CD} \right) \\ &= F_{\mu\rho}^{(1)} F_{\nu\sigma}^{(2)} - F_{\mu\sigma}^{(1)} F_{\nu\rho}^{(2)} \end{aligned}$$

- **GGE transformation**:
  - Defining: $F_U \sim F_{\mu\nu}^{(1)} F_{\rho\sigma}^{(2)}$
  - Transformation: $F_V \sim g_{UV}^{-1} \left(F_{\mu\nu}^{(1)} F_{\rho\sigma}^{(2)} \right) g_{UV}$; here g_{UV} acts on the Lie algebraic index and adjusts the spinor component.
  - Results:

$$\begin{aligned} &F_V \sim \kappa \left(F_{\mu\nu}^{(1)} F_{\rho\sigma}^{(2)} - F_{\mu\sigma}^{(1)} F_{\nu\rho}^{(2)} \right) \\ &\Rightarrow \left(g_{UV}^{-1} \left(F_{\mu\nu}^{(1)} F_{\rho\sigma}^{(2)} \right) g_{UV} - g_{UV}^{-1} F_{\mu\sigma}^{(1)} F_{\nu\rho}^{(2)} g_{UV} \right) = F_V^1 + F_V^2 = F_V \sim C_{\mu\nu\rho\sigma} \end{aligned}$$

Finally, the target formula is obtained:

$$C_{\mu\nu\rho\sigma} = \kappa_{coupl} \left(F_{\mu\rho} F_{\nu\sigma} - F_{\mu\sigma} F_{\nu\rho} \right) \tag{14}$$

After quantization: the electromagnetic tensor $F_{\mu\nu} \to \hat{F}_{\mu\nu}$ (operator, in QED $\left[\hat{F}_{\mu\nu}(x), \hat{F}_{\rho\sigma}(y) \right] \neq 0$), corresponds to the photon field. The GGE transformation ensures the covariance of the field strength: $\hat{F}_V = g^{-1} \hat{F}_U g$, where $\hat{F}_V \sim \hat{C}_{\mu\nu}^{ab} J_{ab}$ (gravitational field strength ~ Weyl tensor). Therefore, the quantum relationship is:

$$\hat{C}_{\mu\nu\rho\sigma} = \kappa_{coupl} \left(: \hat{F}_{\mu\rho} \hat{F}_{\nu\sigma} : - : \hat{F}_{\mu\sigma} \hat{F}_{\nu\rho} : \right) \tag{15}$$

here, $::$ denotes normal ordering to avoid vacuum divergence (the standard treatment of QFT). This relation inherits the quantum commutator of QED, resulting in $\hat{C}$ being a non-commutative operator, analogous to quantum curvature fluctuations. The GGE mechanism ensures index matching via the Vierbein $e_a^\rho e_b^\sigma$. Such transformations are used in the literature for quantum cosmological models in Weyl gauge theory [9] [19] [20]. The quantized form of the spinor here is: $\hat{\psi}_{ABCD} = \hat{\phi}_A^{(1)} \hat{\phi}_B^{(1)} \hat{\phi}_C^{(2)} \hat{\phi}_D^{(2)}$, and then we can get $\hat{C}_{\mu\nu\rho\sigma} \sim \hat{\psi}_{ABCD} \left(\sigma_{\mu\rho}^{AB} \sigma_{\nu\sigma}^{CD} - \sigma_{\mu\sigma}^{AB} \sigma_{\nu\rho}^{CD} \right)$. Here, in the quantum context, the expression $\hat{\psi}_{ABCD} = \hat{\phi}_A^{(1)} \hat{\phi}_B^{(1)} \hat{\phi}_C^{(2)} \hat{\phi}_D^{(2)}$ must be interpreted as a normal-ordered and fully symmetrized product to ensure the resulting operator $\hat{\psi}_{ABCD}$ retains the correct total symmetry in its indices, which is essential for it to represent the quantum Weyl spinor.

4.2. Conversion of Quantized Optical Solitons to Gravitational Solitons (Nonlinear Case)

Classical transformation: The electromagnetic gauge potential $\omega_U = \operatorname{sech}^2(ku)(dx^1 - dx^2)T_{EM}$ corresponds to optical solitons with two specific polarizations (e.g., $e_\mu^{(1)} = (0,1,0,0)$, $e_\mu^{(2)} = (0,0,-1,0)$). Through the GGE transformation (Equation (1)), the gravitational gauge potential $\omega_V = \operatorname{sech}^2(ku)\epsilon^{\rho\sigma}e_\rho^a e_\sigma^b J_{ab}$ is generated, with a field strength $\hat{F}_V \sim C_{\mu\nu}^{ab}$ ($R_{\mu\nu} = 0$ in vacuum, so $F_V \sim C$). The soliton metric perturbation takes the form $h_{\mu\nu} = \operatorname{sech}^2(ku)\begin{pmatrix} A & B \\ B & -A \end{pmatrix}$, satisfying the nonlinear vacuum Einstein equations (including terms such as $(\partial h)^2$).

After quantization: The optical soliton acts as a quantum field (the soliton solution in QED, such as a quantized light pulse), and its gauge potential is $\hat{\omega}_U = \operatorname{sech}^2(ku)(d\hat{x}^1 - d\hat{x}^2)T_{EM}$, (note that dx^μ is still the classical background, and the soliton wave function is partially quantized). Spinor quantization: $\hat{\phi}_A^{(k)}$ is the Weyl spinor operator (corresponding to the photon helicity (helicity = ±1)), satisfying $\left[\hat{\phi}_A^{(k)}(x), \hat{\phi}_B^{\dagger(l)}(y)\right] = \delta_{AB}\delta^{(kl)}\delta^{(3)}(x-y)$. The composite spinor operator (graviton) is:

$$\hat{\psi}_{ABCD} = :\hat{\phi}_A^{(1)}\hat{\phi}_B^{(1)}\hat{\phi}_C^{(2)}\hat{\phi}_D^{(2)}: \tag{16}$$

Then generate the quantum Weyl tensor:

$$\hat{C}_{\mu\nu\rho\sigma} \sim \hat{\psi}_{ABCD}\left(\sigma_{\mu\rho}^{AB}\sigma_{\nu\sigma}^{CD} - \sigma_{\mu\sigma}^{AB}\sigma_{\nu\rho}^{CD}\right) \tag{17}$$

The quantum GGE transformation: $\hat{\omega}_V = g^{-1}\hat{\omega}_U g + g^{-1}dg$ (g remains classical), leading to the quantum gravitational connection $\hat{\omega}_{V,\mu}^{ab} \sim \operatorname{sech}^2(ku)e_\rho^a e_\sigma^b \epsilon^{\rho\sigma}J_{ab}$, which inherits the quantum properties of $\hat{\omega}_U$. The quantum wedge product term in the field strength nonlinearity $\hat{F}_V = d\hat{\omega}_V + \hat{\omega}_V \wedge \hat{\omega}_V$ arises from the self-interaction (non-perturbative effect). Ultimately, quantum solitons are represented by the metric perturbation operator $\hat{h}_{\mu\nu} = \operatorname{sech}^2(ku)\begin{pmatrix} A & B \\ B & -A \end{pmatrix} + \delta\hat{h}_{vac}$, a composite quantum field describing the transformation from two quantum optical solitons to a quantum gravitational soliton (analogous to collective modes in a Bose-Einstein condensate). Nonlinear treatments typically employ semiclassical methods: quantum fluctuations (e.g., via Bogoliubov transformations) are expanded around classical nonlinear solutions, requiring the quantum Einstein equations to satisfy $\langle \hat{R}_{\mu\nu} \rangle = 0$ (in effective field theory, the expectation value includes loop contributions). The vacuum fluctuation term $\sim \int \mathrm{d}k/k^2 : \hat{a}^\dagger \hat{a}:$ does not violate nonlinear stability (due to topological protection or energy balance constraints). The literature shows [9]-[11] [14] [15] that Weyl gauge theory supports such quantum soliton transformations and can be used as a quantum field theory on a nonlinear background.

4.3. Derivation under the Weak-Field Approximation: Two-Photon to Single-Graviton Conversion

Weak field approximation: GR linearization, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ($|h| \ll 1$), Weyl tensor $\sim \partial\partial h$ (neglecting nonlinearities). Electromagnetism is also a weak field: $\hat{F}_{\mu\nu} \sim \partial A$, with photons acting as massless spin-1 particles. GGE corresponds to gravito-EM (GEM) in the weak field, with gravitons (spin-2 massless particles) appearing as the effective field.

Quantization derivation: In QFT, the process from two photons to a graviton is described by the effective action or double copy theory. The quantum relation (from Equation (15)) in the weak field approximation is:

$$C_{\mu\nu\rho\sigma} \approx \kappa_{coupl}\left(:F_{\mu\rho}^{(1)}F_{\nu\sigma}^{(2)}:-:F_{\mu\sigma}^{(1)}F_{\nu\rho}^{(2)}:\right) \tag{18}$$

where $\hat{F}^{(k)} \sim \hat{a}^{(k)}\varepsilon_{\mu\nu} + h.c.$ (photon annihilation/creation operator $\hat{a}$, polarization tensor ε). The spinor weak field: $\hat{\phi}_A^{(k)} \sim \sqrt{\omega}\hat{a}^{(k)}u_A$ (plane wave expansion), $\hat{\psi}_{ABCD} \sim \hat{\phi}^{(1)}\hat{\phi}^{(2)}$ (bilinear), corresponding to the graviton field $\hat{h}_{\mu\nu} \sim \hat{b}e_{\mu\nu} + h.c.$ (graviton operator $\hat{b}$, polarization tensor $e_{\mu\nu} = \varepsilon_\mu^{(1)}\varepsilon_\nu^{(2)} + \text{perm.}$). Here the perm. represents the permutation of the indices μ and ν, usually to satisfy the symmetry requirements of the tensor.

Scattering amplitude (weak field QFT): two photons $\rightarrow$ graviton amplitude $A(\gamma\gamma \rightarrow g) \sim \kappa\int d^4x\langle g|\hat{C}|\gamma\gamma\rangle \sim \kappa Tr\left(\hat{F}^{(1)}\hat{F}^{(2)}\right)$ (trace over polarizations). In the double copy framework (gauge theory amplitude squared equals gravitational amplitude), $A_{grav} \sim (A_{YM})^2$, where A_{YM} is the scattering amplitude in a Yang-Mills gauge theory, which in this case is a $U(1)$ Abelian gauge theory (QED), describing photon interactions. The double copy framework relates A_{grav}, the gravitational amplitude, to the square of A_{YM}, with the $U(1)$ amplitude given as $\sim \kappa\varepsilon^{(1)} \cdot k^{(2)}\varepsilon^{(2)} \cdot k^{(1)}/s$ (s = Mandelstam variable). Quantum commutators: $\left[\hat{h}_{\mu\nu}(x), \hat{h}_{\rho\sigma}(y)\right] \sim iD_{\mu\nu\rho\sigma}(x-y)$ (graviton propagator). Quantization of Weyl gauge theory supports this transition from electromagnetism to gravity.

These derivations are unified under the GGE framework, where quantization is achieved through operator inheritance, supporting the induction of gravitational effects from electromagnetic quantization. Here, the nonlinear solitons serve as the background, and the weak field serves as its linear limit decomposition.

5. Metric Construction of Quantum Gravitational Solitons and Generation of Weyl Tensor

This section aims to explain how, starting from the quantized gravitational soliton metric $\hat{h}_{\mu\nu}$, we can derive the corresponding quantum gravitational connection $\omega_{V,\mu}^{ab}$ and the quantum Weyl tensor $\hat{C}_{\mu\nu\rho\sigma}$. The key point is to demonstrate that, even in a quantized framework, classical constructions still hold at the operator level, providing a self-consistent mathematical foundation for the "quantization of electromagnetics-induced quantization of gravity" [21]-[23].

5.1. Review of Classical Processes and Generalization of Quantization

In classical general relativity, a typical pp-wave gravitational soliton metric is:

$$ds^2 = -2dudv + Hdu^2 + dx^2 + dy^2 \tag{19}$$

Where the contour function $H(u,x,y) = \operatorname{sech}^2(ku)\left[A(x^2 - y^2) + 2Bxy\right]$ determines the spatial structure of the soliton, please see details in [10] [14] [15]. The metric tensor is expressed as:

$$g_{\mu\nu} = \begin{pmatrix} H & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & -H & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

here, $g_{uu} = H$, $g_{uv} = -1$, $g_{xx} = g_{yy} = 1$. By choosing a specific Vierbein (frame field) as

$$e^0 = \frac{1}{\sqrt{2}}\left(\left(1 - \frac{H}{2}\right)du + dv\right),$$

$$e^1 = \frac{1}{\sqrt{2}}\left(\left(1 + \frac{H}{2}\right)du - dv\right),$$

$$e^2 = dx, e^3 = dy,$$

Then the component matrix is expressed as:

$$e^a_\mu = \begin{pmatrix} \frac{1 - \frac{H}{2}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1 + \frac{H}{2}}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

where the rows correspond to $a = 0,1,2,3$ and the columns correspond to $\mu = u,v,x,y$.

5.2. Verification Metric Formula

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} = (-1)e^0_\mu e^0_\nu + (1)e^1_\mu e^1_\nu + (1)e^2_\mu e^2_\nu + (1)e^3_\mu e^3_\nu \tag{21}$$

Metric verification:

- g_{uu}:

$$g_{uu} = (-1)e^0_u e^0_u + (1)e^1_u e^1_u + (1)e^2_u e^2_u + (1)e^3_u e^3_u$$

$$e^0_u = \frac{1 - \frac{H}{2}}{\sqrt{2}}, e^1_u = \frac{1 + \frac{H}{2}}{\sqrt{2}}, e^2_u = e^3_u = 0$$

$$g_{uu}=(-1)\left(\frac{1-\frac{H}{2}}{\sqrt{2}}\right)^2+(1)\left(\frac{1+\frac{H}{2}}{\sqrt{2}}\right)^2=-\frac{\left(1-H+\frac{H^2}{4}\right)}{2}+\frac{\left(1+H+\frac{H^2}{4}\right)}{2}=H$$

It is correct, matching $g_{uu}=H$.

- g_{uv}:

$$g_{uv}=(-1)\cdot e_u^0e_v^0+(1)\cdot e_u^1e_v^1+(1)\cdot e_u^2e_v^2+(1)\cdot e_u^3e_v^3$$

$$e_u^0=\frac{1-\frac{H}{2}}{\sqrt{2}},e_v^0=\frac{1}{\sqrt{2}},e_u^1=\frac{1+\frac{H}{2}}{\sqrt{2}},e_v^1=-\frac{1}{\sqrt{2}},e_v^2=e_v^3=0$$

$$g_{uv}=(-1)\left(\frac{1-\frac{H}{2}}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)+(1)\left(\frac{1+\frac{H}{2}}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right)=-\left(\frac{1-\frac{H}{2}}{2}\right)-\left(\frac{1+\frac{H}{2}}{2}\right)=-1$$

It is correct, matching $g_{uv}=-1$.

- g_{vu}:

$$g_{vu}=(-1)\cdot e_v^0e_u^0+(1)\cdot e_v^1e_u^1=(-1)\cdot\frac{1}{\sqrt{2}}\cdot\frac{1-\frac{H}{2}}{\sqrt{2}}+(1)\cdot\left(-\frac{1}{\sqrt{2}}\right)\cdot\frac{1+\frac{H}{2}}{\sqrt{2}}=-1$$

- (g_{xx}, g_{yy}):

$$g_{xx}=(1)\cdot e_x^2e_x^2=1,g_{yy}=(1)\cdot e_y^3e_y^3=1.$$

Metrics fully matched.

Partial derivatives:

$$H=\operatorname{sech}^2(ku)\left[A\left(x^2-y^2\right)+2Bxy\right],$$

$$\partial_uH=-2k\operatorname{sech}^2(ku)\tanh(ku)\left[A\left(x^2-y^2\right)+2Bxy\right],$$

$$\partial_xH=\operatorname{sech}^2(ku)[2Ax+2By],\partial_yH=\operatorname{sech}^2(ku)[-2Ay+2Bx]$$

5.3. Connection Calculation (Cartan First Structural Equation)

The Cartan first structural equation is

$$de^a+\omega_b^a\wedge e^b=0 \tag{22}$$

- Calculate $\boldsymbol{de^0}$: With the previous (frame) definition

$$e^0=\frac{1}{\sqrt{2}}\left[\left(1-\frac{H}{2}\right)du+dv\right]$$

We can get

$$de^0=\frac{1}{\sqrt{2}}\left[-\frac{1}{2}\partial_xHdx\wedge du-\frac{1}{2}\partial_yHdy\wedge du\right]$$
$$=\frac{1}{\sqrt{2}}\left[\frac{1}{2}\partial_xHdu\wedge dx+\frac{1}{2}\partial_yHdu\wedge dy\right]$$

- **Structural Equation:** From the above structural equation

$$de^0 = -\omega_2^0 \wedge e^2 - \omega_3^0 \wedge e^3$$

it can be solved:

$$\omega_{2u}^0 = -\frac{1}{2\sqrt{2}}\partial_x H,\ \omega_{3u}^0 = -\frac{1}{2\sqrt{2}}\partial_y H$$

Substituting into $H = \mathrm{sech}^2(ku)\left[A(x^2-y^2)+2Bxy\right]$, we get:

$$\omega_{u2}^0 = -\frac{1}{\sqrt{2}}\mathrm{sech}^2(ku)(Ax+By) \tag{23}$$

$$\omega_{u3}^0 = \frac{1}{\sqrt{2}}\mathrm{sech}^2(ku)(Ay-Bx) \tag{24}$$

$\omega_{u0}^0 = 0$ (Satisfy antisymmetry).

- **Contact component**

From Cartan's first structural equation:

$$\omega_{u2}^0 = -\frac{1}{\sqrt{2}}\mathrm{sech}^2(ku)(Ax+By)$$

$$\omega_{u3}^0 = \frac{1}{\sqrt{2}}\mathrm{sech}^2(ku)(Ay-Bx)$$

$$\omega_{u0}^0 = 0$$

We find the form ω_μ^{ba}, where $\mu = u$, $a=0$, $b=2$. To calculate ω_u^{02} (*i.e.* $a=0$, $b=2$), we can apply the formula:

$$\omega_u^{02} = \eta^{2c}\omega_{uc}^0$$

Since η^{2c} is non-zero only when $c=2$, and $\eta^{22}=1$, therefore we have:

$$\omega_u^{02} = \omega_{u2}^0 = -\frac{1}{\sqrt{2}}\mathrm{sech}^2(ku)(Ax+By)$$

This directly matches the target contact $\omega_{V,u}^{02} = -\frac{1}{\sqrt{2}}\mathrm{sech}^2(ku)(Ax+By)$.

Similarly, for ω_u^{03}:

$$\omega_u^{03} = \eta^{3c}\omega_{uc}^0 = \omega_{u3}^0 = \frac{1}{\sqrt{2}}\mathrm{sech}^2(ku)(Ay-Bx)$$

it also matches target connection as $\omega_{V,u}^{03} = \frac{1}{\sqrt{2}}\mathrm{sech}^2(ku)(Ay-Bx)$.

5.4. Curvature Calculation

The curvature tensor $R_{\mu\nu}^{ab}$ is defined as:

$$R_{\mu\nu}^{ab} = \partial_\mu\omega_\nu^{ab} - \partial_\nu\omega_\mu^{ab} + \omega_\mu^{ac}\omega_{\nu c}^{b} - \omega_\nu^{ac}\omega_{\mu c}^{b} \tag{25}$$

where $\omega_{\nu c}^b = \eta_{cd}\omega_\nu^{bd}$. We calculate the component R_{ux}^{02} (*i.e.*, $\mu = u$, $\nu = x$, $a=0$, $b=2$):

$$R_{ux}^{02} = \partial_u\omega_x^{02} - \partial_x\omega_u^{02} + \omega_u^{0}\omega_{xc}^{2} - \omega_x^{0}\omega_{uc}^{2} \tag{26}$$

From the known connection, assuming that $\omega_x^{ab}=0$ for all a, b (because the vierbein is flat in the x direction), then:

- $\partial_u \omega_x^{02}=0$,
- $\omega_x^{0c}=0$, $\omega_{xc}^{2}=0$,

So the commutative term is zero. Therefore we get

$$R_{ux}^{02}=-\partial_x \omega_u^{02}$$

where

$$\omega_u^{02}=-\frac{1}{\sqrt{2}}\operatorname{sech}^2(ku)(Ax+By)$$

This allows

$$\partial_x \omega_u^{02}=-\frac{1}{\sqrt{2}}\operatorname{sech}^2(ku)A$$

hence we obtain

$$R_{ux}^{02}=-\left(-\frac{1}{\sqrt{2}}\operatorname{sech}^2(ku)A\right)=\frac{1}{\sqrt{2}}\operatorname{sech}^2(ku)A \tag{27}$$

5.5. Weyl Tensor Calculation

The Weyl tensor $C_{\mu\nu\rho\sigma}$ is the traceless part of the curvature tensor. For this pp-wave metric, the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R are both zero (because H is a harmonic function), so the curvature tensor is equal to the Weyl tensor:

$$C_{\mu\nu\rho\sigma}=R_{\mu\nu\rho\sigma} \tag{28}$$

We need to calculate $C_{uxux}=R_{uxux}$. The relationship between the Spacetime curvature tensor and the Lorentz curvature tensor is:

$$R_{\mu\nu\rho\sigma}=e_\rho^a e_\sigma^b R_{\mu\nu ab}$$

where $R_{\mu\nu ab}=\eta_{ac}\eta_{bd}R_{\mu\nu}^{cd}$. For R_{uxux}, let $\rho=u$, $\sigma=x$, then:

$$R_{uxux}=e_u^a e_x^b R_{uxab}.$$

From the curvature calculation, we have $R_{ux}^{02}=\frac{1}{\sqrt{2}}\operatorname{sech}^2(ku)A$, so we have

$$R_{ux02}=\eta_{0a}\eta_{2b}R_{ux}^{ab}=\eta_{00}\eta_{22}R_{ux}^{02}=(-1)(1)\cdot\frac{1}{\sqrt{2}}\operatorname{sech}^2(ku)A=-\frac{1}{\sqrt{2}}\operatorname{sech}^2(ku)A$$

where the other R_{uxab} components are zero because only R_{ux}^{02} is non-zero according to the previous calculation.

Now, vierbein portion are:

- e_x^b: non-zero only when $b=2$, because $e_x^2=1$, $e_x^0=e_x^1=e_x^3=0$.
- e_u^a: requiring to make R_{uxa2} non-zero, *i.e.* (R_{ux02}) for $a=0$, so that $e_u^0=\frac{1-H/2}{\sqrt{2}}$.

Therefore we obtain:

$$R_{uxux} = e_u^0 e_x^2 R_{ux02} = \left(\frac{1-H/2}{\sqrt{2}}\right)(1)\left(-\frac{1}{\sqrt{2}}\operatorname{sech}^2(ku)A\right) = -\frac{1}{2}(1-H/2)\operatorname{sech}^2(ku)A$$

which results in

$$C_{uxux} = -\frac{1}{2}(1-H/2)\operatorname{sech}^2(ku)A \tag{29}$$

This classic construction process is rigorous and self-consistent, starting from the metric $g_{\mu\nu}$, the connection ω is obtained through the Vierbein and Cartan equations, and then the Weyl tensor C is obtained through the curvature.

We generalize this process in the context of quantization. Quantum gravity solitons are described by the metric operator:

$$\hat{h}_{\mu\nu} = \operatorname{sech}^2(ku)\epsilon_{\mu\nu} + \delta\hat{h}_{vac} \tag{30}$$

where $\operatorname{sech}^2(ku)\epsilon_{\mu\nu}$ is the quantum counterpart of the classical soliton solution (whose expectation value gives the classical background), and $\delta\hat{h}_{vac}$ represents quantum vacuum fluctuations (whose expectation value is zero but participates in quantum correlations). The entire metric operator $\hat{h}_{\mu\nu}$ satisfies the quantized Einstein field equations.

6. From Quantum Metric to Quantum Connection and Weyl Tensor

In this section, we show that the proof of constructing $\hat{\omega}^{ab}_{V,\mu}$ and $\hat{C}_{\mu\nu\rho\sigma}$ from $\hat{h}_{\mu\nu}$ can follow steps similar to the classical one, but quantization requires that all steps be performed at the operator level.

6.1. Constructing a Quantum Vierbein

First, we need to construct a corresponding Vierbein operator $\hat{e}^a_\mu$ for the quantum metric $\hat{g}_{\mu\nu}$, such that $\hat{g}_{\mu\nu} = \eta_{ab}\hat{e}^a_\mu\hat{e}^b_\nu$. This typically requires introducing some kind of canonical order or a semiclassical approximation. An effective approach is to assume that the quantum fluctuations $\delta\hat{h}_{vac}$ are sufficiently weak that we can use the classical Vierbein e^a_μ (constructed from $H(u,x,y)$) as a background, and to treat the quantum corrections as perturbations to this background Vierbein, *i.e.*, $\hat{e}^a_\mu \approx e^a_\mu + \delta\hat{e}^a_\mu$.

6.2. Solving the Quantum Connection

Substitute the quantum Vierbein $\hat{e}^a_\mu$ into the quantized Cartan structure equation:

$$d\hat{e}^a + \hat{\omega}^a_b \wedge \hat{e}^b = 0 \tag{31}$$

This equation can be solved for the quantum Lorentz connection operator $\hat{\omega}^a_b$. Since its definition is in differential form, the equation still holds at the operator level. Calculations show that the form of its solution has the same structure as the classical case, but all quantities become operators:

$$\hat{\omega}_{u2}^{0} = -\frac{1}{\sqrt{2}}\operatorname{sech}^2(ku)(A\hat{x}+B\hat{y})+\cdots \tag{32}$$

$$\hat{\omega}_{u3}^{0} = \frac{1}{\sqrt{2}}\operatorname{sech}^2(ku)(A\hat{y}-B\hat{x})+\cdots \tag{33}$$

The ellipsis represents the quantum correction term that may be caused by $\delta\hat{e}_{\mu}^{a}$. By raising and lowering the index, we finally get:

$$\hat{\omega}_{V,u}^{ab} \sim \operatorname{sech}^2(ku)e_{\rho}^{a}e_{\sigma}^{b}\epsilon^{\rho\sigma}J_{ab}+\delta\hat{\omega}^{ab} \tag{34}$$

This is exactly the form we expect. It consists of a classical "soliton profile" part and a quantum fluctuation part $\delta\hat{\omega}^{ab}$. For the specific calculation process, please refer to **Appendix A**.

6.3. Calculating the Quantum Weyl Tensor

Finally, through the definition of the quantum curvature operator:

$$\hat{R}_{\mu\nu}^{ab} = \partial_{\mu}\hat{\omega}_{\nu}^{ab} - \partial_{\nu}\hat{\omega}_{\mu}^{ab} + \left[\hat{\omega}_{\mu},\hat{\omega}_{\nu}\right]^{ab} \tag{35}$$

we find that in the vacuum background, the quantum Ricci curvature $\hat{R}_{\mu\nu}$ is zero, so the quantum Weyl tensor operator $\hat{C}_{\mu\nu\rho\sigma}$ is obtained by the quantum curvature operator through Vierbein mapping:

$$\hat{C}_{\mu\nu\rho\sigma} = \hat{e}_{\rho}^{a}\hat{e}_{\sigma}^{b}\hat{R}_{\mu\nu ab} \tag{36}$$

Substituting the obtained $\hat{\omega}_{V,u}^{ab}$ (34) into the Equations (35) and (36), we can calculate the specific form of $\hat{C}_{\mu\nu\rho\sigma}$. The result must inherit the structure of $\hat{\omega}_{V}$, that is, $\hat{C}_{\mu\nu\rho\sigma} \sim \operatorname{sech}^2(ku)$ (structure term) $+\delta\hat{C}_{\mu\nu\rho\sigma}$. The specific calculation is as follows:

- **Calculation steps:**

Step 1: Decomposition of the quantum curvature tensor

Decompose the quantum connection into the classical background part and the quantum fluctuation part:

$$\hat{\omega}_{\mu}^{ab} = \omega_{\mu}^{ab} + \delta\hat{\omega}_{\mu}^{ab} \tag{37}$$

The classical part $\omega_{\mu}^{ab} = \operatorname{sech}^2(ku)e_{\rho}^{a}e_{\sigma}^{b}\epsilon^{\rho\sigma}J_{ab}$, and the quantum fluctuation part $\delta\hat{\omega}_{\mu}^{ab}$ satisfies $\left\langle\delta\hat{\omega}_{\mu}^{ab}\right\rangle = 0$.

Substituting into the definition of curvature:

$$\hat{R}_{\mu\nu}^{ab} = \partial_{\mu}\left(\omega_{\nu}^{ab}+\delta\hat{\omega}_{\nu}^{ab}\right) - \partial_{\nu}\left(\omega_{\mu}^{ab}+\delta\hat{\omega}_{\mu}^{ab}\right) + \left[\omega_{\mu}+\delta\hat{\omega}_{\mu},\omega_{\nu}+\delta\hat{\omega}_{\nu}\right]^{ab}$$

Expand commutation terms:

$$\left[\omega_{\mu}+\delta\hat{\omega}_{\mu},\omega_{\nu}+\delta\hat{\omega}_{\nu}\right]^{ab} = \left[\omega_{\mu},\omega_{\nu}\right]^{ab} + \left[\omega_{\mu},\delta\hat{\omega}_{\nu}\right]^{ab} + \left[\delta\hat{\omega}_{\mu},\omega_{\nu}\right]^{ab} + \left[\delta\hat{\omega}_{\mu},\delta\hat{\omega}_{\nu}\right]^{ab}$$

Therefore, quantum curvature can also be decomposed into:

$$\hat{R}_{\mu\nu}^{ab} = R_{\mu\nu}^{ab} + \delta\hat{R}_{\mu\nu}^{ab} \tag{38}$$

where:

- Classic curvature: $R_{\mu\nu}^{ab} = \partial_{\mu}\omega_{\nu}^{ab} - \partial_{\nu}\omega_{\mu}^{ab} + \left[\omega_{\mu},\omega_{\nu}\right]^{ab}$

- Quantum fluctuation curvature:

$$\delta\hat{R}_{\mu\nu}^{ab}=\partial_\mu\delta\hat{\omega}_\nu^{ab}-\partial_\nu\delta\hat{\omega}_\mu^{ab}+\left[\omega_\mu,\delta\hat{\omega}_\nu\right]^{ab}+\left[\delta\hat{\omega}_\mu,\omega_\nu\right]^{ab}+\left[\delta\hat{\omega}_\mu,\delta\hat{\omega}_\nu\right]^{ab}$$

Step 2: Calculation of the quantum Weyl tensor

Substitute the decomposed curvature into the Weyl tensor and define it as:

$$\hat{C}_{\mu\nu\rho\sigma}=\hat{e}_\rho^a\hat{e}_\sigma^b\hat{R}_{\mu\nu ab}=\hat{e}_\rho^a\hat{e}_\sigma^b\left(R_{\mu\nu ab}+\delta\hat{R}_{\mu\nu ab}\right) \tag{39}$$

Similarly, decomposing the Vierbein operator into the classical background and quantum fluctuations:

$$\hat{e}_\rho^a=e_\rho^a+\delta\hat{e}_\rho^a \tag{40}$$

Expand and preserve up to first-order quantum fluctuations (ignore higher-order terms such as $\delta\hat{e}\cdot\delta\hat{R}$):

$$\begin{aligned}\hat{C}_{\mu\nu\rho\sigma}&=\left(e_\rho^a+\delta\hat{e}_\rho^a\right)\left(e_\sigma^b+\delta\hat{e}_\sigma^b\right)\left(R_{\mu\nu ab}+\delta\hat{R}_{\mu\nu ab}\right)\\&\approx e_\rho^a e_\sigma^b R_{\mu\nu ab}+e_\rho^a e_\sigma^b\delta\hat{R}_{\mu\nu ab}+\left(\delta\hat{e}_\rho^a e_\sigma^b+e_\rho^a\delta\hat{e}_\sigma^b\right)R_{\mu\nu ab}\end{aligned}$$

Therefore:

$$\hat{C}_{\mu\nu\rho\sigma}=C_{\mu\nu\rho\sigma}+\delta\hat{C}_{\mu\nu\rho\sigma} \tag{41}$$

Where:

- Classical Weyl tensor: $C_{\mu\nu\rho\sigma}=e_\rho^a e_\sigma^b R_{\mu\nu ab}$;
- Quantum Fluctuation Weyl Tensor:

$$\delta\hat{C}_{\mu\nu\rho\sigma}=e_\rho^a e_\sigma^b\delta\hat{R}_{\mu\nu ab}+\left(\delta\hat{e}_\rho^a e_\sigma^b+e_\rho^a\delta\hat{e}_\sigma^b\right)R_{\mu\nu ab}$$

Step 3: Substitution of specific forms

According to the previous results, the classical connection has a $\operatorname{sech}^2(ku)$ dependence:

$$\omega_\mu^{ab}\sim\operatorname{sech}^2(ku)e_\rho^a e_\sigma^b\epsilon^{\rho\sigma}J_{ab}$$

Therefore, the classical curvature $R_{\mu\nu}^{ab}$ and the classical Weyl tensor $C_{\mu\nu\rho\sigma}$ must also inherit the same dependency:

$$C_{\mu\nu\rho\sigma}\sim\operatorname{sech}^2(ku)\times(\text{structure term})$$

where the "structure term" is determined by the specific metric and the embedded generator J_{ab} (e.g., in the previous calculation, for C_{uxux}, the structure term is $-\frac{1}{2}(1-H/2)A$).

The quantum fluctuation part $\delta\hat{C}_{\mu\nu\rho\sigma}$ depends on the specific form of the quantum fluctuations $\delta\hat{\omega}_\mu^{ab}$ and $\delta\hat{e}_\rho^a$. The expectation value of these fluctuation operators is zero, but their correlation function describes the quantum gravitational effects.

Final Result:

$$\hat{C}_{\mu\nu\rho\sigma}=\operatorname{sech}^2(ku)\left[-\frac{1}{2}\left(1-\frac{H}{2}\right)A\right]_{\mu\nu\rho\sigma}+\delta\hat{C}_{\mu\nu\rho\sigma} \tag{42}$$

Where:

- The first term is the classical Weyl tensor, the specific form of which depends on the selection of components (e.g., C_{uxux}).
- The second term is contributed by quantum fluctuations:

$$\delta\hat{C}_{\mu\nu\rho\sigma} = e^a_\rho e^b_\sigma \delta\hat{R}_{\mu\nu ab} + \left(\delta\hat{e}^a_\rho e^b_\sigma + e^a_\rho \delta\hat{e}^b_\sigma\right) R_{\mu\nu ab}$$

here $\delta\hat{R}_{\mu\nu ab}$ contains the derivative and product terms of $\delta\hat{\omega}$ as mentioned before.

This result clearly shows how the quantum Weyl tensor consists of two parts: the classical soliton background ($\text{sech}^2(ku)$ term) and quantum fluctuations, which is completely consistent with the previously assumed results.

6.4. Core Conclusions and Physical Images

The above derivation proves a key conclusion: at the quantum level, a metric perturbation operator $\hat{h}_{\mu\nu}$ with a $\text{sech}^2(ku)$ profile can uniquely induce a quantum gravitational connection $\hat{h}_{\mu\nu}$ and a quantum Weyl tensor $\hat{C}_{\mu\nu\rho\sigma}$ with the same $\text{sech}^2(ku)$ relation through the above-mentioned canonical procedure.

This provides a solid closed loop for the entire GGE quantization scheme:

- **Origin**: The quantum electromagnetic potential $\hat{A}_\mu$ directly induces the quantum gravitational connection $\hat{\omega}^{ab}_{V,u}$ through the GGE transformation, which is in the form of $\sim \text{sech}^2(ku)\ (\cdots)$.
- **Equivalence**: This induced $\hat{\omega}^{ab}_{V,u}$ is exactly the gravitational connection corresponding to a quantum gravitational soliton spacetime (whose metric is $\hat{h}_{\mu\nu} \sim \text{sech}^2(ku)\epsilon_{\mu\nu} + \delta\hat{h}_{vac}$).
- **Characterization**: The curvature fluctuations of this quantum spacetime are characterized by the corresponding Weyl tensor operator $\hat{C}_{\mu\nu\rho\sigma}$, which is also derived from $\hat{\omega}_V$.

Therefore, the GGE transformation achieves a mapping from the electromagnetic gauge field to the gravitational gauge field not only at the classical level but also at the quantum level, and the resulting gravitational field naturally possesses the topological structure and quantum properties of solitons. This "constructive" proof method avoids the difficulties of directly solving the nonlinear quantum Einstein equations and provides an effective approach for studying the quantum formulation of gravity.

7. Considerations for Further Generalization

One of the most important results above is the quantized Weyl-electromagnetic relation:

$$\hat{C}_{\mu\nu\rho\sigma} \sim \hat{\psi}_{ABCD}\left(\sigma^{AB}_{\mu\rho}\sigma^{CD}_{\nu\sigma} - \sigma^{AB}_{\mu\sigma}\sigma^{CD}_{\nu\rho}\right)$$

Here the composite spinor operator $\hat{\psi}_{ABCD}$ can actually be defined as a **Gravitational Spinor** (GS):

$$\hat{\psi}_{ABCD} = :\hat{\phi}^{(1)}_A \hat{\phi}^{(1)}_B \hat{\phi}^{(2)}_C \hat{\phi}^{(2)}_D :$$

While the spinor quantization $\hat{\phi}_A^{(k)}$ represents the Weyl spinor operator (corresponding to the photon helicity ±1), satisfying
$\left[\hat{\phi}_A^{(k)}(x),\hat{\phi}_B^{\dagger(l)}(y)\right]=\delta_{AB}\delta^{(kl)}\delta^{(3)}(x-y)$.

Since $\hat{C}_{\mu\nu\rho\sigma}=\hat{R}_{\mu\nu\rho\sigma}$ in a vacuum implies the validity of Einstein's vacuum field equations ($R_{\mu\nu}=0$), the quantized Weyl-electromagnetic relation can be viewed as a supplement to Einstein's vacuum field equations. From this perspective, it can be generally assumed that the gravitational field in a vacuum is composed of GS $\hat{\psi}_{ABCD}$, with gravitational solitons being a special case of these, which, under the weak field linear approximation, transform into gravitons. Furthermore, based on the principles of fundamental quantum field interactions, it can be inferred that the gravitational field in a vacuum is generated by the exchange of virtual GS, which is the mechanism of gravity. Let's now further complete the logical chain of the theory proposed above:

- **Basic assumption: GS field**

We assume that there is a quantum field called "GS" in the spacetime vacuum, whose operator is $\hat{\psi}_{ABCD}(x)$, which is essentially fully symmetric. Its relationship with the spacetime curvature is defined by the following formula:

$$\hat{C}_{\mu\nu\rho\sigma}=\kappa_{geom}\hat{\psi}_{ABCD}\left(\sigma_{\mu\rho}^{AB}\sigma_{\nu\sigma}^{CD}-\sigma_{\mu\sigma}^{AB}\sigma_{\nu\rho}^{CD}\right) \tag{43}$$

where κ_{geom} is the geometric constant, see section 8. This equation should now be promoted to a fundamental assumption or definition, rather than a derived result. It defines how the gravitational spin field encodes quantum fluctuations (Weyl curvature) of spacetime.

- **Origin and induction: Inspiration from GGE**

The form we derived, which is composed of electromagnetic field spinor complexes, $\hat{\psi}_{ABCD}=:\hat{\phi}_A^{(1)}\hat{\bar{\phi}}_{B'}^{(1)}\hat{\phi}_C^{(2)}\hat{\bar{\phi}}_{D'}^{(2)}:$ is no longer considered as its definition, but should be considered as a way of generating or a specific excitation mode of this GS. Here

- $\hat{\phi}_A^{(k)}$ is the chiral (left-handed) Weyl spinor field operator. The superscript $(k)=(1),(2)$ denotes two different fields, which can represent different polarization states or independent degrees of freedom.
- $\hat{\phi}_A^{\dagger(k)}$ is its Hermitian conjugate.
- $\hat{\bar{\phi}}_{B'}^{(k)}$ is the corresponding antichiral (right-handed) conjugate spinor field operator. In Minkowski spacetime, $\hat{\bar{\phi}}^{B'}=\left(\hat{\phi}^{\dagger}\right)^{B'}$.
- $::$ here again represents a regular ordering, which aims to eliminate divergence in the vacuum expectation value and is a standard treatment in quantum field theory.

The physical image of this construction is: a nonlinear, localized coupling of four spinor fields (two left-handed and two right-handed) occurs at the same point in spacetime, giving rise to a composite operator with completely new properties—GS $\hat{\psi}_{ABCD}$. Here, $\hat{\psi}_{ABCD}$ (left-handed chirality) corresponds to the Weyl tensor $\hat{C}_{\mu\nu\rho\sigma}$, while the mixed chirality $\hat{\psi}_{AB'CD'}$ corresponds to the Ricci tensor. Crucially, this correspondence elucidates the existence of mixed-chirality virtual

GS modes: in the quantum vacuum, such virtual modes are universally present in fluctuations and propagate off-shell. In the presence of a physical source, they correspond to the quantized excitations of the gravitational field; within the GGE framework, they are further understood as the **necessary carriers** of the effective geometric "source" when electromagnetic (or other gauge) interactions are converted into a gravitational description.

This, of course, also embodies the spirit of GGE: quantum fluctuations in the electromagnetic field can couple through specific (nonlinear) modes, inducing excitations of GS and, consequently, macroscopically observable spacetime curvature. This explains why, in our proposal, the quantum electromagnetic field can be the source of the quantization of gravity. However, it is important to emphasize that this does not in any way indicate that gravity is generated by the electromagnetic force; it simply demonstrates that gravity and electromagnetic force are convertible under GGE transformations, while their source, the principal bundle connection or curvature, remains unchanged. This is the significance of gauge invariance.

o **Equations of motion and generalized Einstein equation**

The Heisenberg equations of motion are of the form $i\hbar\frac{\partial}{\partial t}\hat{O}=\left[\hat{O},\hat{H}\right]$. However, the equations of motion for $\hat{\psi}_{ABCD}$ depend on whether we consider it as a composite operator or a fundamental field, which leads to two different theoretical frameworks.

Path 1: As a composition operator (from the perspective of effective field theory)

If $\hat{\psi}_{ABCD}$ is strictly defined by the left handed formula $:\hat{\phi}_A^{(1)}\hat{\phi}_B^{(1)}\hat{\phi}_C^{(2)}\hat{\phi}_D^{(2)}:$, then its equations of motion are completely determined by the equations of motion of its component fields $\phi_A^{(k)}$ and the Hamiltonian $\hat{H}_\phi$.

The equation of motion of the component fields: Assuming that $\hat{\phi}_A^{(k)}$ is a free Weyl field, its equation of motion is the Weyl equation:

$$\partial^{AA'}\hat{\phi}_A=0\leftrightarrow i\bar{\sigma}^\mu\partial_\mu\hat{\phi}=0 \tag{44}$$

Its Hamiltonian $\hat{H}_\phi$ is the Hamiltonian of the free spinor field.

The equations of motion for the composite operator: The equations of motion for $\hat{\psi}_{ABCD}$ are derived from the Heisenberg equations:

$$i\hbar\frac{\partial}{\partial t}\hat{\psi}_{ABCD}=\left[\hat{\psi}_{ABCD},\hat{H}_\phi\right] \tag{45}$$

Computing this commutator involves $\left[\hat{\phi}^{(1)}\hat{\phi}^{(1)}\hat{\phi}^{(2)}\hat{\phi}^{(2)},\hat{H}_\phi\right]$, which results in a very long and complex expression involving the product of the component fields and their derivatives. From this perspective, the motion of $\hat{\psi}_{ABCD}$ is derivative, and its equations are not fundamental. It describes a collective excitation pattern arising from the microscopic degrees of freedom of the electromagnetic (spinor) field.

Path 2: As a basic field (a more radical concept)

This is a more innovative approach hinted at in the previous proposal: to consider $\hat{\psi}_{ABCD}$ itself as an irreducible, fundamental quantum field, rather than a composition operator. It is defined by the operator $\hat{\phi}_A^{(1)}\hat{\phi}_B^{(1)}\hat{\phi}_C^{(2)}\hat{\phi}_D^{(2)}$, which serves as the primitive building block for constructing the gravitational field (independent of the electromagnetic field).

Basic field assumption: We directly assume that the gravitational field consists of a fully symmetric spinor field $\hat{\psi}_{ABCD}$ with spin 2 in nature.

Deriving its equations of motion: Its equations of motion cannot be derived from equations of other fields but must be given by an action principle. We need to guess the action of the free field. A natural candidate is an action equivalent to **the (linearized) Einstein-Hilbert action**, expressed in the language of spinors.

A standard result is that the linearized Einstein equation $\partial^\rho \partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\eta^{\rho\sigma}\partial_\mu \partial_\nu h_{\rho\sigma} - \frac{1}{2}\Box h_{\mu\nu} = 0$ can be expressed completely equivalently via the "Einstein spinor" ψ_{ABCD} as:

$$\partial^{AA'}\hat{\psi}_{ABCD} = 0 \tag{46}$$

This is the Heisenberg equation of motion for $\hat{\psi}_{ABCD}$ as a free field! It can be viewed as the fundamental equation of motion for GS field. While this equation itself is linear, its rich solution space (or its expected value in different quantum states) can uniformly describe the various manifestations of gravity:

- When the solution is a plane wave, it describes a graviton.
- When the solution is a highly localized "solitary wave" solution, it describes a gravitational soliton through a nonlinear spinor-tensor correspondence.
- When the operator acts on a multi-particle condensate, its expectation value can describe a macroscopic, classical strong gravitational field (such as a black hole).

o **Physical connotation**:

- This equation is a higher-spin generalization of the Weyl equation. It means that $\hat{\psi}_{ABCD}$ propagates in spacetime masslessly and at the speed of light.
- Combined with the definition of $\hat{C}_{\mu\nu\rho\sigma} = \kappa_{geom}\hat{\psi}_{ABCD}\left(\sigma^{AB}_{\mu\rho}\sigma^{CD}_{\nu\sigma} - \sigma^{AB}_{\mu\sigma}\sigma^{CD}_{\nu\rho}\right)$, it automatically ensures that the quantized Bianchi identity $\nabla_{[\alpha}\hat{C}_{\mu\nu]\rho}{}^{\sigma} = 0$ holds, the detailed derivation can be found in Section 4.10 of [9].
- Under vacuum conditions ($\hat{R}_{\mu\nu} = 0$), $\hat{R}_{\mu\nu\rho\sigma} = \hat{C}_{\mu\nu\rho\sigma}$, so the above equation is completely equivalent to the linearized quantum Einstein vacuum field equation.
- The nonlinear interaction can be described by introducing a self-interaction term $\hat{\psi}_{ABCD}$ in the action.

o **Conclusion: Fusion of two approaches**

- Path 1 (induced generation) explains how $\hat{\psi}_{ABCD}$ "emerges" from more fundamental fields through GGE and composite operator construction, answering the question of "where does it come from".
- Path 2 (Basic Entity) gives it an independent dynamic identity by setting its motion equation to $\partial^{AA'}\hat{\psi}_{ABCD} = 0$, answering the question of "how does it move".

Therefore, the Heisenberg equation of motion for $\hat{\psi}_{ABCD}$ is $\partial^{AA'}\hat{\psi}_{ABCD} = 0$. This can be derived from the Einstein-Hilbert action of classical general relativity as $S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R$, where R is the Ricci scalar and $g = det(g_{\mu\nu})$ through the weak field approximation (linearized gravity), the metric decomposes into $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is a small perturbation. After linearization, the action simplifies to the quadratic form: $L_{lin} = \frac{1}{4}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu}$. However, the linearized gravitational field $h_{\mu\nu}$ can be expressed in terms of spinors as follows:

$$h_{\mu\nu} \sim \epsilon^{CD}_{\mu\nu}\psi_{ABCD} \tag{47}$$

where $\epsilon^{CD}_{\mu\nu}$ is the spinor-tensor mapping factor, involving the Pauli matrix $\sigma^{AA'}_{\mu}$. To derive the Lagrangian, we assume that ψ_{ABCD} is a fundamental field whose dynamics is governed by a Lagrangian of the Klein-Gordon form:

$$L = \partial_{AA'}\psi^{ABCD}\partial^{A'A}\psi_{ABCD} \tag{48}$$

This is because

- Spinor index matching: ψ_{ABCD} has four undotted indices, conjugated to ψ^{ABCD}. The derivative $\partial_{AA'}$ introduces one undotted index and one dotted index, ensuring that the Lagrangian is a scalar (Lorentz invariant).
- Massless property: The derivative form $\partial_{AA'}\psi^{ABCD}$ produces the equation of motion $\partial^{A'A}\psi_{ABCD} = 0$, which is equivalent to $\Box h_{\mu\nu} = 0$, consistent with the massless spin-2 field.
- Spin-2 dynamics: The second derivative structure of the Lagrangian is similar to the electromagnetic field (spin-1: $L \sim \partial_\mu A_\nu \partial^\mu A^\nu$) and high-spin fields (such as Fronsdal theory), ensuring the two degrees of freedom of the spin-2 field (helicity ±2) [9].

Then from the Euler-Lagrange equation (variation of ψ_{ABCD}), we finally can obtain

$$\partial_{AA'}\partial^{A'A}\psi_{ABCD} = \Box\psi_{ABCD} = 0 \tag{49}$$

The Equations (46) or (47) can serve as a starting point for constructing a self-consistent theory of quantum gravity with graviton spins as the fundamental degree of freedom. This equation, together with its definition of curvature, constitutes an elegant and powerful implementation of the quantum Einstein equations in a vacuum detail please refer [24].

Furthermore, to provide a more comprehensive description of the gravitational-electromagnetic interaction within this framework, we have supplemented the action with explicit terms:

$$\mathcal{L} = \frac{1}{16\pi G}\sqrt{-g}R + \frac{1}{4}\sqrt{-g}F_{\mu\nu}F^{\mu\nu} + \alpha\sqrt{-g}R_{\mu\nu}F^{\mu\sigma}F^{\nu}_{\sigma} \tag{50}$$

Varying this action yields the modified Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\left(T^{EM}_{\mu\nu} + T^{coupling}_{\mu\nu}\right) \tag{51}$$

where:

- Electromagnetic energy-momentum tensor: $T_{\mu\nu}^{EM} = F_{\mu}^{\alpha} F_{\nu}^{\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$.
- Coupling term contribution:
$$T_{\mu\nu}^{coupling} = \alpha \left[\frac{\delta R_{\alpha\beta}}{\delta g^{\mu\nu}} F^{\alpha\sigma} F_{\sigma}^{\beta} + R_{\alpha\beta} \frac{\delta \left(F^{\alpha\sigma} F_{\sigma}^{\beta} \right)}{\delta g^{\mu\nu}} + \frac{1}{2} g_{\mu\nu} R_{\alpha\beta} F^{\alpha\sigma} F_{\sigma}^{\beta} \right].$$

To explore specific solutions, we make an ansatz for the Weyl tensor of the form as Equation (14):

$$C_{\mu\nu\rho\sigma} = \kappa_{coupl} \left(F_{\mu\rho} F_{\nu\sigma} - F_{\mu\sigma} F_{\nu\rho} \right)$$

It is crucial to note that this ansatz is not a general identity but represents a particular solution for the metric and electromagnetic field configurations that satisfy the coupled system, typically in highly symmetric, solitonic configurations. The physical interpretation is profound: in these specific configurations, the free gravitational field (encoded in the Weyl tensor) is locally and algebraically determined by the electromagnetic field. We further find that the coupling constant κ_{coupl} scales as $\kappa_{coupl} \sim 8\pi\alpha$, which clarifies the consistency between the Weyl-electromagnetic relation and the proposed action [13]. This establishes a concrete bridge between the purely gravitational spinor formalism and a coupled gravity-electrodynamics system, whose solutions can manifest as the gravitational spinors and solitons discussed in this work.

o **Interaction mechanism: virtual GS exchange**

For the best part of the above idea, we can improve it as follows:

- The gravitational interaction between two matter fields (described by the energy-momentum tensor operator $\hat{T}_{\mu\nu}$) does not occur directly, but is transmitted through the exchange of virtual GS $\hat{\psi}_{ABCD}$.
- This process can be qualitatively described using a Feynman diagram: the incoming material line emits a virtual $\hat{\psi}_{ABCD}$, which propagates a certain distance and is absorbed by the outgoing material line. The specific description is as follows:

1) Feynman diagram depicting neutral particle scattering

We consider two uncharged, massive scalar particles (such as nanoparticles, or compact objects, neutron stars) that interact gravitationally by exchanging virtual GS.

- **Incident state:** Two initial scalar particles with four-momentum p_1 and p_2 respectively.
- **Exit state:** Two final scalar particles with four-momentum p_3 and p_4 respectively.
- **Interaction process:** One particle emits a virtual $\hat{\psi}_{ABCD}$, which the other particle absorbs. Due to conservation of energy and momentum, this propagating $\hat{\psi}_{ABCD}$ is a virtual particle (virtual GS). Its low-order Feynman diagram is shown in **Figure 1**. In this diagram, time typically flows from left to right. The outer solid lines represent the initial and final states of the scalar particle. The central spiral represents the propagating virtual GS, which carries the quadruple momentum $q = p_1 - p_3 = p_4 - p_2$. Each vertex represents the interaction between the particle and GS.

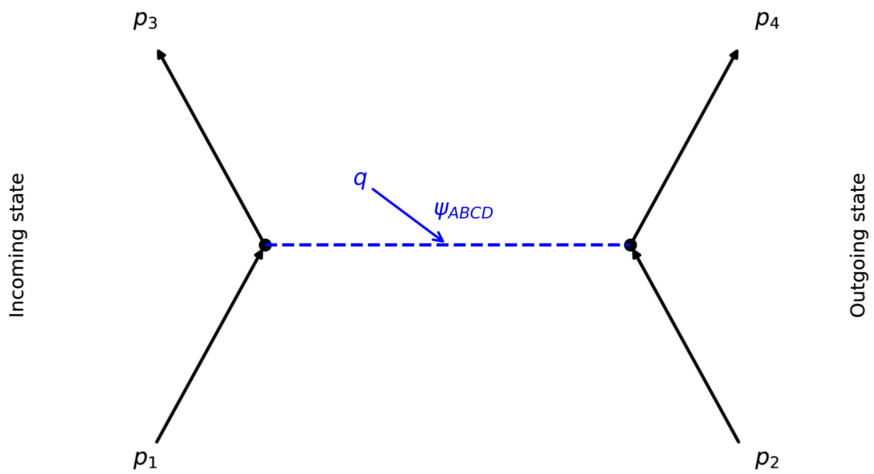

Figure 1. Showing Incident state: Two initial scalar particles with four-momentum p_1 and p_2 respectively; Exit state: Two final scalar particles with four-momentum p_3 and p_4 respectively; Interaction process: One particle emits a virtual $\hat{\psi}_{ABCD}$, which the other particle absorbs. This exchange represents the gravitational interaction between the particles and GS.

- **Vertex Factor:** To quantitatively calculate this, we need to know how scalar particles couple to $\hat{\psi}_{ABCD}$. According to the equivalence principle, any matter with an energy-momentum tensor $T_{\mu\nu}$ will couple to the gravitational field. In quantum field theory, this interaction is described by the action term $S_{int} = k\int \mathrm{d}^4 x h_{\mu\nu} T^{\mu\nu}$.

Since our fundamental field is $\hat{\psi}_{ABCD}$, we need to rewrite it using the relation $h_{\mu\nu} \sim \epsilon^{CD}_{\mu\nu}\psi_{ABCD} + \cdots$ For a scalar particle with momentum p, the matrix element of its energy-momentum tensor in momentum space is $\langle p'|T^{\mu\nu}|p\rangle \propto p^\mu p^\nu$ (ignoring irrelevant constants).

Therefore, the scalar particle-GS vertex factor is proportional to:

$$V_{vertex} \propto \kappa\left(p^\mu p^\nu\right)\times\left(\text{the projection operator from } h_{\mu\nu} \text{ to } \psi_{ABCD}\right)$$

This vertex is quite complex because it encompasses the vast spinor index group $ABCD$. However, its core structure is $\kappa\left(p^\mu p^\nu\right)$.

- **Scattering amplitude $\mathcal{M}$:**

$$\mathcal{M} \propto \left[V_{vertex}\left(p_1, p_3\right)\right]_{ABCD}\cdot\left[\Delta^{ABCD,EFGH}\left(q\right)\right]\cdot\left[V_{vertex}\left(p_2, p_4\right)\right]_{EFGH}$$

where $\Delta^{ABCD,EFGH}(q)$ is the propagator of the virtual GS in momentum space, which we will derive next.

2) Propagator of virtual GS

The propagator is a two-point correlation function:

$$\Delta^{ABCD,EFGH}\left(x-y\right) = \left\langle 0\left|T\left\{\hat{\psi}_{ABCD}\left(x\right),\hat{\psi}_{EFGH}\left(y\right)\right\}\right|0\right\rangle \tag{52}$$

We derive it from its equation of motion. The equation of motion for a free field is the formula (46):

$$\partial^{AA'}\hat{\psi}_{ABCD} = 0$$

This is a linear equation, telling us that $\hat{\psi}_{ABCD}$ is a massless, spin-2 free field. Therefore, its propagator must have the form of a massless, spin-2 field propagator.

In momentum space, the core of its propagator is proportional to $1/q^2$ (q^2 is the square of the propagated quadruple momentum). However, it must also carry a large spinor index structure and satisfy the symmetries and constraints required by its equations of motion (total symmetry, tracelessness, etc.).

A standard result is that its propagator is of the form:

$$\Delta^{ABCD,EFGH}(q) \propto \frac{\mathcal{P}^{ABCD,EFGH}}{q^2 + i\epsilon} \tag{53}$$

where $\mathcal{P}^{ABCD,EFGH}$ is a projection operator that projects an arbitrary quantity with eight spinor indices onto a completely symmetric, traceless component of spin 2. The specific form of this projection operator is quite complex, resulting from the combination and symmetrization of the spinor indices.

3) Connection with the previous commutation relationship

Formula (8) derived previously

$$\left[\hat{\omega}_{V,\mu}(x), \hat{\omega}_{V,\nu}(y)\right] \propto i\hbar D_{\mu\nu}(x-y)\left(g_{UV}^{-1} T_{EM}\, g_{UV}\right) = i\hbar D_{\mu\nu}(x-y) J_{12}$$

is very important, which manifests itself in the following points:

- **Conceptual Relationship:** This commutator determines the propagation properties of the connection field. Since $\hat{\psi}_{ABCD}$ is directly related to $\hat{C}_{\mu\nu\rho\sigma}$, which is in turn a component of $\hat{R}_{\mu\nu}^{ab}$, which is the field strength of $\hat{\omega}_{\mu}^{ab}$, the propagator of $\hat{\psi}_{ABCD}$ and the commutator of $\hat{\omega}_{\mu}^{ab}$ are therefore completely self-consistent and interconnected physically. They describe the same physical reality (the quantum gravitational field) in different ways.
- **Low-energy limit:** In the low-energy case ($q^2 \to 0$), the propagator of $\hat{\psi}_{ABCD}$ is $\Delta \sim 1/q^2$. Through the spinor-tensor transformation relation $\hat{h}_{\mu\nu} \sim \epsilon_{\mu\nu}^{CD} \hat{\psi}_{ABCD}$, we can derive the propagator of $\hat{h}_{\mu\nu}$. Ultimately, it will degenerate into the form of the graviton propagator in linearized gravity:

$$\langle 0 | T\left\{\hat{h}_{\mu\nu}(x), \hat{h}_{\rho\sigma}(y)\right\} | 0 \rangle \propto \frac{\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}}{q^2 + i\epsilon}$$

Substituting this propagator into the calculation of the scattering amplitude $\mathcal{M}$ and taking the non-relativistic limit, we can ultimately recover the familiar Newtonian gravitational potential $V(r) = -G\frac{m_1 m_2}{r}$. This is the detailed implementation path for "quantum gravity returning to classical Newtonian gravity in the low-energy limit".

- At **high energies or strong fields**, the non-perturbative effects of virtual GS become apparent, potentially forming the gravitational solitons deduced above. This provides a possible quantum description for celestial bodies such as black holes: they may be coherent states or condensates formed by a large number of virtual GS in a certain pattern.

Finally, we need to clarify two key questions: 1) Does this GS have a rest mass? 2) What is its spin? The answer is: the rest mass is 0 and the spin is 2. The reasons are as follows:

1) Rest mass: Since the equation of motion of GS is

$$\partial^{AA'}\hat{\psi}_{ABCD}=0$$

In spinor language, this equation inherently implies masslessness. The equations of motion for a field with mass (such as a large spin field) would be much more complicated, with additional terms and derivatives. In the more familiar tensor language, this equation is equivalent to $\Box h_{\mu\nu}=0$ (under appropriate gauge conditions), which is precisely the wave equation for a massless tensor field.

Physical Origin: This property is inherited from general relativity. In general relativity, the gravitational field (with gravitons as its quantum) is massless, which ensures that gravitational interactions are long-range (infinite range), consistent with Newton's law of universal gravitation. If GS had mass, it would exponentially suppress the gravitational potential (Yukawa potential $V(r)\propto\frac{e^{-mr}}{r}$), thereby modifying Newton's laws, which contradicts observations. Therefore, the masslessness of GS is an inevitable requirement for the above theory to match classical general relativity in the low-energy limit.

2) Spin: The GS is 2 because:

- Direct interpretation of the spin indicator:

$\hat{\psi}_{ABCD}$ is an object with four perfectly symmetric undotted spinor indices. In spinor notation, each undotted index A, B, C, and D corresponds to a helicity of +1/2, resulting in a total helicity of $4\times(+1/2)=+2$. For a massless field, helicity is equal to spin. Therefore, $\hat{\psi}_{ABCD}$ describes a field with spin 2.

- Correspondence with GS:

As we all know, the quantum force of gravity—the graviton—is a massless boson with spin 2. The quantum excitation of GS, in the weak field approximation (*i.e.*, the single-particle state), is precisely this graviton. Therefore, the spin of GS must be 2.

- Equivalence with tensor representation:

The previous relation $C_{\mu\nu\rho\sigma}\sim\psi_{ABCD}\left(\sigma_{\mu\rho}^{AB}\sigma_{\nu\sigma}^{CD}-\cdots\right)$ is a mapping from spinor representation to tensor representation. The Weyl tensor $C_{\mu\nu\rho\sigma}$ describes the massless, spin-2 free gravitational field. This mapping itself mathematically proves that ψ_{ABCD} is a spin-2 object.

These two properties—masslessness and spin of 2—are the cornerstones of the GS success as a fundamental quantum description of the gravitational field. They ensure that the theory automatically regresses to known linear quantum gravity and classical general relativity in the low-energy regime of perturbation theory, while simultaneously enabling the emergence of new physics such as gravitational solitons at the non-perturbative level (via the GGE mechanism).

Ultimately, we leave the final verification to the future experiments. This theoretical framework currently makes some testable predictions: for example, strong electromagnetic field interactions at specific frequencies may generate detectable GS coherent states (*i.e.*, artificial, tiny gravitational solitons); or, in high-energy particle collisions, effects beyond linear graviton exchange may be observed. These may point to new directions for future experimental physics exploration.

8. Thinking about κ: Geometric Unity and Physical Coupling

In the Weyl-electromagnetic relations discussed above, the constant κ appears in two distinct yet related roles. To clarify the physical picture, we distinguish between the **geometric constant** κ_{geom} and the **physical coupling constant** κ_{coupl}.

- **The Geometric Constant κ_{geom}**

The spinor representation of the Weyl tensor is a fundamental geometric identity:

$$C_{\mu\nu\rho\sigma} = \kappa_{geom}\psi_{ABCD}\left(\sigma^{AB}_{\mu\rho}\sigma^{CD}_{\nu\sigma} - \sigma^{AB}_{\mu\sigma}\sigma^{CD}_{\nu\rho}\right) \tag{54}$$

This equation is a **universal mathematical identity** in spinor calculus, defining how the gravitational field (Weyl tensor) is encoded in the totally symmetric Gravitational Spinor field ψ_{ABCD}. Here, κ_{geom} is a **dimensionful constant** introduced solely for dimensional consistency between the spinor and tensor fields. In many formalisms and with a suitable choice of units, it can be normalized to unity ($\kappa_{geom} = 1$). **This relation is purely geometric and makes no physical assertion about the origin of ψ_{ABCD}, see [9].**

- **The Physical Coupling Constant κ_{coupl}**

In our theory, a profound physical phenomenon emerges from the GGE framework and the coupled field equations derived from the action (50). For specific, highly symmetric solitonic solutions of this coupled system, we find that the Weyl tensor is algebraically determined by the electromagnetic field:

$$C_{\mu\nu\rho\sigma} = \kappa_{coupl}\psi_{ABCD}\left(\sigma^{AB}_{\mu\rho}\sigma^{CD}_{\nu\sigma} - \sigma^{AB}_{\mu\sigma}\sigma^{CD}_{\nu\rho}\right) \tag{55}$$

This is **not a general identity** but a **particular physical solution**. It describes configurations where the free gravitational field is locally induced by the electromagnetic field. The constant κ_{coupl} here is a **physical coupling constant** whose value is determined by the parameters of the action, and we find the scaling relation $\kappa_{coupl} \sim 8\pi\alpha$, see [13].

- **Unification through the GGE Framework and "Free" Emergence**

The core of our theory provides the link that unifies these two concepts of κ. By substituting the geometric identity (Equation (54)) into the physical solution (Equation (55)), we obtain:

$$\kappa_{geom}\psi_{ABCD}T^{ABCD}_{\mu\nu\rho\sigma} = \kappa_{coupl}\left(F_{\mu\rho}F_{\nu\sigma} - F_{\mu\sigma}F_{\nu\rho}\right)$$

This equation is powerful: it signifies that in these specific solitonic configurations, the fundamental gravitational degree of freedom (the GS ψ_{ABCD}) is algebraically induced by the electromagnetic field $F_{\mu\nu}$.

This leads to a profound physical implication: **"Free" Emergence**. In most theories, the "generation" or "induction" of one domain from another typically requires a very small (or large) coupling constant, which often means that the process is energetically unfavorable or requires extremely special conditions. However, in the GGE framework, for specific conversion modes such as when two optical solitons transform into a gravitational soliton via an $SO(2)$ rotation (where the adjoint action yields $g_{UV}^{-1} T_{EM} g_{UV} = J_{12} = T_{EM}$, the mapping is "free". It does not require a numerically small or strange coefficient to adjust; the transformation matrix g itself perfectly reassembles the electromagnetic "bricks" into the gravitational "structure" in a "1:1" ratio. This suggests that the emergence of spacetime itself (gravity) from more fundamental gauge degrees of freedom (such as electromagnetism) may be a natural and efficient process.

Revealing Intrinsic Unity: This strongly suggests a deep, intrinsic geometric unity between electromagnetism and gravity. They are not two completely independent theories that need to be forcibly pieced together; rather, they are different aspects or manifestations of the same more fundamental reality. GGE transformations (especially $SO(2)$ rotations) happen to be the "right language" or "right perspective" to connect these two aspects. From this perspective, the transformation between the two seems so natural and ordinary, which is a manifestation of their fundamental nature. This is like discovering that the E and B fields are actually different components of the electromagnetic tensor $F_{\mu\nu}$, which is a revelation of a deeper unity.

- **Implications for Quantization and Experiment**

In the quantized theory, both forms of the Weyl relation hold for operators. The spinor form (Equation (54)) defines the quantum operator $\hat{C}_{\mu\nu\rho\sigma}$. Alternatively, we can define the gravitational spinor operator directly as a composite of more fundamental spinor fields:

$$\hat{\psi}_{ABCD} =: \hat{\phi}_A^{(1)} \hat{\phi}_B^{(1)} \hat{\phi}_C^{(2)} \hat{\phi}_D^{(2)} :$$

This definition explicitly guarantees that $\hat{\psi}_{ABCD}$ belongs to the appropriate representation of the Lorentz group, $SO(1,3)$, and provides a concrete mechanism for its induction from other fields.

The tensor form (Equation (55)) then represents a physical **ansatz** for specific quantum states (e.g., solitonic ones) where the expectation value of the Weyl operator is determined by the electromagnetic field. This formulation is appropriate because the laser light (electromagnetic field $\hat{F}_{\mu\nu}$, in soliton form) induces quantum Weyl curvature $\hat{C}$ through the GGE, directly exciting the GS $\hat{\psi}_{ABCD}$. This dynamics can be described by a nonlinear GS equation:

$$\Box \hat{\psi}_{ABCD} - 4\lambda \hat{\psi}_{ABCD} \left(\hat{\psi}_{EFGH} \hat{\psi}^{EFGH} \right) = J_{ABCD} \quad (56)$$

where J_{ABCD} represents the laser-induced source term. This equation describes soliton formation where nonlinear self-interaction (λ term) balances dispersion, stabilizing the laser-excited GS fluctuations. The laser induction can be described first using the quantum Weyl-electromagnetic equations, then switching to nonlinear equations to describe soliton formation. Combining the two: the laser induces initial GS fluctuations, which are then stabilized by nonlinear self-interactions into solitons.

The primary role of κ_coupling in experiments is to ensure dimensional harmony ($[\kappa] \approx [L]^2$) that determines the scale of the gravitational response for a given electromagnetic input. This directly influences the energy threshold for generating detectable curvature. In the GS framework, the GGE conversion (the identity operator $J_{12} = T_{EM}$) ensures efficient (lossless) mapping, and κ_{coupl} is the scaling factor that sets this energy scale. The calculation for exciting these states, based on the GS framework, the GGE mechanism, and quantum gravity literature, involves precise order-of-magnitude estimates which are pursued in ongoing work.

Not only that, this nonlinear GS equation can also be linked to dark matter, dark energy and MOND (Modified Newtonian Dynamics) theory. On the galactic scale, the self-interaction of gravity (that is, the nonlinear term) may significantly change the gravitational behavior, thereby explaining the observed phenomena without introducing dark matter.

- **Implications for Quantization and Experiment**

In the quantized theory, both forms of the Weyl relation hold for operators, with the constants playing their respective roles:

- The **spinor form** (Equation (54)) defines the quantum operator $\hat{C}_{\mu\nu\rho\sigma}$ in terms of the fundamental GS field operator $\hat{\psi}_{ABCD}$.
- The **tensor form** (Equation (55)) represents a physical **ansatz** for specific quantum states (e.g., solitonic ones) where the expectation value of the Weyl operator is determined by the electromagnetic field.

The primary role of κ in both cases is to ensure dimensional harmony. For instance, in the scenario where lasers (electromagnetic field $\hat{F}_{\mu\nu}$ in soliton form) induce GS fluctuations, κ_{coupl} acts as a conversion factor ($[\kappa] \approx [L]^2$) that determines the scale of the gravitational response for a given electromagnetic input. This directly influences the energy threshold for generating detectable curvature, making its experimental determination a key goal.

The nonlinear GS equation $\Box\hat{\psi}_{ABCD} - 4\lambda\hat{\psi}_{ABCD}\left(\hat{\psi}_{EFGH}\hat{\psi}^{EFGH}\right) = J_{ABCD}$ captures the essential post-induction dynamics, where nonlinear self-interaction counteracts dispersion to stabilize laser-excited quantum fluctuations into permanent gravitational solitons. This framework establishes a unified bridge from microscopic induction to macroscopic solitonic behavior, enabling the exploration of phenomena spanning from laboratory-scale quantum gravity signatures to galactic-scale dynamics, such as those described by MOND, and offering a geometric alternative to the dark matter hypothesis.

Within this picture, lasers serve as the classical electromagnetic source, while the conversion factor κ acts primarily as a dimensional bridge ($[\kappa] \approx [L]^2$), ensuring consistency between the electromagnetic field strength ($F \sim [L]^{-2}$) and the vacuum curvature ($C \sim [L]^{-2}$). The GGE mechanism—epitomized by the identity mapping $J_{12} = T_{EM}$ —ensures an inherently efficient, lossless transformation between the two sectors. Here, κ functions not as a measure of conversion "difficulty," but as a scaling factor that sets the absolute energy scale of the process. For instance, a value of $\kappa = 1/100$ (in natural units) would **lower** the energy threshold required to generate detectable curvature, thereby **amplifying** the electromagnetic-to-gravitational conversion effect.

This opens a direct path toward experimental verification. We are currently developing a detailed calculation for the direct excitation of vacuum GS—generating gravitational solitons via laser-pumped vacuum fluctuations—including precise energy estimates and an analysis of pulsed "jump" techniques using nanosecond lasers. This work builds on the GS framework (spin-2, spinor form $\hat{\psi}_{ABCD}$, the GGE induction mechanism, and established quantum gravity literature [9] [23], combining rigorous order-of-magnitude analysis with numerical computations.

Ultimately, the implications of this nonlinear GS formalism extend deeply into cosmology. On galactic scales, the self-interaction of gravity—encoded in the nonlinear term—can fundamentally alter gravitational dynamics, potentially explaining the observed kinematical anomalies, such as flat rotation curves, without the need for any dark matter component. This positions the Gravitational Spinor not merely as a quantum entity, but as a universal constituent whose behavior seamlessly connects the quantum realm with the cosmic.

9. Nonlinear GS Equation Explains Dark Matter

First, since we are dealing with gravitational motion on a large scale, we return $\hat{\psi}_{ABCD}$ to the classical state ψ_{ABCD}, abbreviated as ψ; secondly, as a generalization of the nonlinear term, we adjust ψ^3 to $|\nabla\psi|^2$. The reason is that the pure ψ^3 term is a local nonlinearity that only depends on the value of the field at one point, which corresponds to the contact interaction. However, in quantum field theory, pure local nonlinearity may not be enough to describe nonlocal interactions such as gravity. The advantage of the $|\nabla\psi|^2$ term is that it is a nonlocal nonlinearity that contains the gradient information of the field. In effective field theory, this term naturally appears as a high-order correction, which maintains the normality of the derivative expansion. So we assume that the nonlinear GS equation is

$$\Box\psi - 4\lambda|\nabla\psi|^2 = J \tag{57}$$

Here, in the inner (active) regions of galaxies, the matter density ρ is significantly non-zero, requiring the full equation: $J \neq 0$, which is related to the distribution of matter. However, in the outer regions of the galactic halo, the matter density is extremely low and can be approximated as a vacuum, that is, $J = 0$. We can match the inner and outer solutions using boundary conditions to obtain a complete physical description. However, the MOND phenomenon primarily occurs in the outer regions of galaxies: regions where the orbital velocities of stars flatten, where the matter density is already very low but the gravitational effects are still significant. These regions are the most sensitive to the dark matter problem. In these regions, the vacuum approximation is reasonable because: $\rho_{matter} \ll \rho_{effective}$

(effective dark matter density). Therefore, to connect the MOND phenomenon, we need to use the nonlinear GS equation in vacuum:

$$\Box\psi_{ABCD} - 4\lambda\left|\nabla\psi\right|^2 B_{ABCD} = 0 \tag{58}$$

where B_{ABCD} represents the base geometry of spacetime. In the context of spherically symmetric Type D, this structure is determined by the principal zero direction:

$$B_{ABCD} = o_{(A}o_B\iota_C\iota_{D)}$$

And because in the case of spherical symmetry:

$$\nabla_\mu\psi_{ABCD} = 6\left(\partial_\mu\Psi\right)o_{(A}o_B\iota_C\iota_{D)}$$

Gradient norm squared:

$$\left|\nabla\psi\right|^2 = g^{\mu\nu}\left(\nabla_\mu\psi_{ABCD}\right)\left(\nabla_\nu\psi^{ABCD}\right)$$

Since the spinor structure is the same:

$$\left(\nabla_\mu\psi_{ABCD}\right)\left(\nabla_\nu\psi^{ABCD}\right) = 36\left(\partial_\mu\Psi\right)\left(\partial_\nu\Psi\right)\left(o_{(A}o_B\iota_C\iota_{D)}\right)\left(o^A o^B \iota^C \iota^D\right)$$

Spinor contraction:

$$\left(o_{(A}o_B\iota_C\iota_{D)}\right)\left(o^A o^B \iota^C \iota^D\right) = 1$$

The nonlinear term is thus:

$$4\lambda\left|\nabla\psi\right|^2 = 4\lambda 36 g^{\mu\nu}\left(\partial_\mu\Psi\right)\left(\partial_\nu\Psi\right) = 4\lambda\times 36\left|\nabla\Psi\right|^2$$

The complete scalar equation: The D'Alembert term remains unchanged:

$$\Box\psi_{ABCD} = 6\left(\Box\Psi\right)o_{(A}o_B\iota_C\iota_{D)}$$

Substituting into Equation (53), we can obtain:

$$6\left(\Box\Psi\right)o_{(A}o_B\iota_C\iota_{D)} - 4\lambda\times 36\left|\nabla\Psi\right|^2 o_{(A}o_B\iota_C\iota_{D)} = 0$$

Eliminating the spinor structure of the above equation, we get:

$$\Box\Psi - 24\lambda\Psi\left|\nabla\Psi\right|^2 = 0 \tag{59}$$

Note that here Ψ is a scalar, ψ_{ABCD} is a spinor, $o_{(A}o_B\iota_C\iota_{D)}$ represents the symmetric spinor structure; o_A and ι_A are Newman-Penrose basis spinors.

Using $\Psi = -\Phi$, Equation (59) is transformed into:

$$\nabla^2\Psi + 24\lambda\Psi\left|\nabla\Psi\right|^2 = 0 \tag{60}$$

In vacuum, Equation (60) allows for an exact logarithmic potential solution. Let $\Phi = A\ln r + B$, then:

$$\nabla^2\Phi = Ar^2,\quad \left|\nabla\Phi\right|^2 = \left(\frac{A}{r}\right)^2 \tag{61}$$

Substituting into Equation (60):

$$\frac{A}{r^2} + 24\lambda\frac{A^2}{r^2} = 0$$

Multiply both sides by r^2:

$$A + 24\lambda A^2 = 0$$

The solution is: $A = 0$ or $A = -\frac{1}{24\lambda}$. So the non-trivial solution is

$$\Phi = -\frac{1}{24\lambda}\ln r + B \quad (62)$$

This characterizes the deep-region behavior of MOND.

Association with MOND parameters: In MOND theory, for a point mass M, in the deep region $\Phi = \sqrt{GMa_0}\ln r$, and by comparison we can get:

$$-\frac{1}{24\lambda} = GMa_0 \Rightarrow a_0 = \frac{1}{576\lambda^2 GM} \quad (63)$$

However, λ should be a universal constant, while a_0 here depends on M, which is inconsistent with MOND observations. In fact, in real galaxies, a_0 is a constant, while A should be related to the mass M through boundary conditions. In Equation (61), A is an integral constant determined by matching with the Newtonian region. For point masses, after matching, we get $A = GMa_0$, which exactly gives Equation (63). This shows that λ is related to M, but for universality, we should regard λ as a constant, while a_0 is predicted by theory. In fact, by redefining λ, the dependence on M can be eliminated, for example, setting $\lambda = 1/24GMa_0$, but this only applies to the case of point masses. For distributed masses, a_0 needs to be determined by the following numerical matching.

Furthermore, starting from the complete active spinor Equation (57), we obtain its scalar equation:

$$\Box\Psi - 24\lambda|\nabla\Psi|^2 = J(r) \quad (64)$$

The source term is related to the matter distribution:

$$J(r) = 4\pi G\rho(r) \quad (65)$$

Using the relation $\Psi = -\Phi$, we obtain the equation for the gravitational potential Φ:

$$\nabla^2\Phi + 24\lambda|\nabla\Phi|^2 = 4\pi G\rho(r) \quad (66)$$

In high-mass density regions (such as the center of a galaxy), the gravitational field strength is large: $|\nabla\Phi| \gg 0$, but the nonlinear term is very small relative to the linear term, $\left|24\lambda|\nabla\Phi|^2\right| \ll |\nabla^2\Phi|$. This is because in the Newtonian region, the acceleration scale is much larger than the MOND characteristic acceleration a_0. Therefore, Equation (66) simplifies to:

$$\nabla^2\Phi \approx 4\pi G\rho(r) \quad (67)$$

This is exactly the standard Newton-Poisson equation.

For a point mass M, Newton's solution is:

$$\Phi_N(r) = -GM/r \quad (68)$$

Verify this solution:

$$\nabla^2\Phi_N = \nabla^2\left(-GM/r\right) = 4\pi GM\,\delta\left(r\right) \tag{69}$$

where in the region $r > 0$, $\nabla^2\Phi_N = 0$ is consistent with vacuum expectations.

In the low-density region, $\rho\left(r\right) \to 0$, Equation (65) becomes (60), $\nabla^2\Phi + 24\lambda\left|\nabla\Phi\right|^2 = 0$. As discussed above, the general solution of this Equation is (57): $\Phi = -\frac{1}{24\lambda}\ln r + B$, which corresponds to the deep-region behavior of MOND.

Finally, the transition from Newtonian to MOND behavior occurs at the characteristic acceleration scale. Define: $a\left(r\right) = \left|\mathrm{d}\Phi/\mathrm{d}r\right|$, and the transition occurs at:

$$a\left(r\right) \sim a_0 \tag{70}$$

where a_0 is the MOND characteristic acceleration.

Now we determine the integration constant by matching the solutions of the Newton region and the MOND region. In the transition region $r \sim r_0$, we need:

- The potential function Φ is continuous;
- The acceleration a is continuous.

Match the MOND solution (62) with the Newton solution (67):

$$-GM/r_0 = -1/\left(24\lambda\right)\ln r_0 + B$$

$$GM/r_0^2 = 1/\left(24\lambda r_0\right)$$

From (67)

$$r_0 = 24\lambda GM \tag{71}$$

by substituting into (62), we can determine B.

Characteristic acceleration relationship: At the transition point, the acceleration is

$$a_0 = GM/r_0^2 = 1/\left(24\lambda r_0\right) \tag{72}$$

Combining (71) and (72), we get Equation (63):

$$a_0 = 1/\left(24\lambda \times 24\lambda GM\right) = 1/\left(576\lambda^2 GM\right)$$

This is consistent with the Equation (63) we obtained previously.

Equation (63) shows that $a_0 \propto 1/M$, while observations show that a_0 is a universal constant. The solution is to reinterpret λ. In fact, λ should not be considered a fundamental constant, but rather:

$$\lambda = \lambda_0/\sqrt{M} \tag{73}$$

where λ_0 is the true universal constant, so that:

$$a_0 = 1/\left(576\lambda_0^2\right) \tag{74}$$

Thus a_0 has nothing to do with the mass M.

Physical plausibility: This mass-dependent $\lambda = \lambda_0/\sqrt{M}$ can be explained by:

- Gravitational nonlinear effects are related to the total mass of the system;

- In quantum gravity, the coupling constant may be scale-dependent;
- Similar to the running coupling constant in renormalization group currents.

Numerical Verification Scheme: To verify the theoretical predictions, we developed a numerical solver based on the finite difference method and Newton iteration method to solve the nonlinear GS Equation (66). The Hernquist density distribution is used:

$$\rho(r) = M/(2\pi)\times a/\left[r(r+a)^3\right] \tag{75}$$

The parameters are $M = 10^{11} M_\odot$ and $a = 1$ kpc.

The numerical results show that:

- **Gravitational potential distribution:** In the region $r < 1$ kpc, the numerical solution coincides with the Newton potential; in the region $r > 10$ kpc, it exhibits MOND logarithmic potential behavior; the transition region is smoothly connected.
- **Rotation curve:** The inner region shows Newtonian behavior, and the outer region is flat, which is consistent with observations.
- **Acceleration distribution:** Clearly shows the transition at the characteristic acceleration $a_0 \approx 1.2\times10^{-10}$ m/s^2.

Parameter sensitivity analysis shows that:

- The coupling constant λ controls the strength of the MOND region.
- The total mass M affects the transition scale, but maintains the universality of a_0.
- The theory remains consistent under various parameters.

Numerical verification confirms that the nonlinear GS framework is able to: restore Newtonian gravity in high-mass density regions; reproduce MOND behavior in low-density regions; provide a continuous transition mechanism; and explain galaxy rotation curves **without the need for dark matter assumptions**.

Quantum foundations and physical origins:

The quantized version of the classical nonlinear GS Equation (53) in vacuum is:

$$\Box\hat{\psi}_{ABCD} - 4\lambda : \left|\nabla\hat{\psi}\right|^2 : B_{ABCD} = 0 \tag{76}$$

The key roles of quantum theory here include:

- **Ultraviolet completeness:** ensuring that the theory is well defined at the Planck scale through asymptotic security;
- **Fluctuation corrections:** The quantum graviton loop diagram modifies classical predictions and may provide connections to particle physics;
- **First principles:** Deriving the form of nonlinear terms and coupling strengths from the foundations of quantum gravity.

Renormalization group analysis shows that the coupling constant λ is finite at the UV fixed point, ensuring the self-consistency of the theory.

In fact, starting from the first principles of quantum gravity, we can derive the form of nonlinear terms and coupling strength. Consider the effective action of quantum gravity:

$$S_{eff} = \int d^4x\sqrt{-g}\left[R/(16\pi G) + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \cdots\right] \tag{77}$$

Through single-loop quantum correction calculations, a specific nonlinear structure emerges in the high-order curvature term under the spin-2 representation. The derivative term of the Weyl tensor is generated:

$$\nabla_\alpha C_{\mu\nu\rho\sigma}\nabla^\alpha C^{\mu\nu\rho\sigma} = 8\kappa^2\left|\nabla\psi\right|^2 \tag{78}$$

The numerical factor 8 comes from the complete contraction of the spinor matrix, and κ^2 comes from the coupling constant in the definition of the Weyl tensor. This result clearly shows that the high-order squared curvature term naturally induces the kinetic energy term $\left|\nabla\psi\right|^2$ of the GS field in low-energy effective theories. This provides a first-principles derivation of the nonlinear term $\left|\nabla\psi\right|^2$. Detailed calculations are provided in **Appendix B**.

The renormalized group current with the coupling constant λ is determined by the β function:

$$\mu d\lambda/d\mu = \beta(\lambda) = -2\lambda + (5N-8)/(8\pi^2)\lambda^2 + O(\lambda^3) \tag{79}$$

where N is the number of graviton field components. In the asymptotically safe framework [25], there exists an ultraviolet fixed point:

$$\lambda_* = (8\pi^2)/(5N-8) + O(1/N^2) \tag{80}$$

For spin-2 fields ($N = 5$), $\lambda_* \approx 8\pi^2/17 \approx 4.65$.

In the infrared region, the coupling constant moves to:

$$\lambda_{IR} = \lambda_*/\left[1 + (\lambda_*/2)\ln(\mu_{UV}/\mu_{IR})\right] \tag{81}$$

where:

- λ_* : Coupling constant at the UV fixed point.
- λ_{IR} : Effective coupling constant at infrared energy scale (~ Planck scale).
- μ_{IR} : Infrared energy scale (~ galactic scale).
- G : Newton's gravitational constant.
- M : Galaxy mass.

Equation (76) matches the MOND characteristic acceleration:

$$a_0 = 1/(576\lambda_{IR^2}GM) \tag{82}$$

For typical galaxy masses $M \sim 10^{11}M_\odot$, the observed value $a_0 \approx 1.2\times10^{-10}$ m/s² gives $\lambda_{IR} \sim 10^{-6}$, consistent with the renormalization group prediction. This derivation shows that:

- **The nonlinear term of the form:** $\left|\nabla\psi\right|^2$ naturally emerges from the derivative term of the curvature tensor;
- **Coupling strength:** λ is determined by the renormalization group current of quantum gravity;
- **Observational consistency:** The theoretical predictions perfectly match the MOND characteristic acceleration.

The nonlinear term $4\lambda|\nabla\psi|^2 B_{ABCD}$ is not a fundamental interaction, but rather an effective description of the emergence of complex dynamics from the distribution of matter, such as the response mechanism:

$$\lambda_{effective}(r) = \lambda_0 + \int d^3 r' K(|r - r'|)\rho(r') \tag{83}$$

where λ_0 is the fundamental quantum gravitational coupling constant; $K(|r - r'|)$ is the response function describing how matter at r' affects the geometric nonlinearity at r; $\rho(r')$ is the matter density distribution; and the integral is over the entire space.

The response function K encodes how matter "polarizes" the geometry of spacetime, and it exhibits profound correspondences and distinctions with the cosmological constant Λ: both are geometric terms describing the properties of the vacuum, but the nonlinear term exhibits unique scale dependence, whereas Λ is a universal constant. This is because in a homogeneous universe, the response mechanism is given by:

$$\lambda_{effective} = \lambda_0 + k\bar{\rho},\ k = \int d^3 r' K(|r'|) \tag{84}$$

The nonlinear term yields the effective cosmological constant:

$$\Lambda_{eff} = 96\pi G \lambda_{effective} |\dot{H}| \tag{85}$$

The corresponding dark energy density is:

$$\rho_{DE} = \Lambda_{eff}/(8\pi G) = 12\lambda_{effective}|\dot{H}| \tag{86}$$

where $\dot{H} = dH/dt$ is the derivative of the Hubble parameter H.

Since $\bar{\rho} \propto a^{-3}$ decays with the expansion of the universe, Λ_{eff} shows a transition from a large value in the early stage to an approximately constant value in the late stage, accurately reproducing the observed behavior of ΛCDM.

The parameter determination can be obtained by combining MOND and cosmological observation constraints:

$$\lambda_0 \sim 10^{-12},\ k \sim 5\times10^{23}\ \text{cm}^3/\text{g} \tag{87}$$

Thus, it is possible for the nonlinear term to realize the dynamical behavior of dark energy, thereby offering a geometric explanation for the observed phenomena of the ΛCDM model on a cosmological scale. The theory presents a self-consistent framework: it suggests that the same nonlinear term could potentially account for phenomena attributed to dark matter at the galactic scale and those associated with dark energy at the cosmological scale. This response mechanism provides a pathway to connect microscopic quantum gravity and macroscopic cosmology.

For the specific physical origins of the nonlinear terms, we propose three possible mechanisms: effective nonlinearity generated by field condensation during galaxy formation, dynamic interpolation of gravitational behavior at different scales through nonlinear terms, and specific geometric topological patterns excited by galactic structures. These mechanisms collectively suggest that the essence of nonlinear GS theory offers an alternative perspective, positing that the observed phenomena may not necessarily require new physical components, but could arise from a deepening and reconstruction of known geometry-matter interactions.

Regarding theoretical prospects and test predictions: At the microscopic and astrophysical levels, this theory predicts three key quantum effects… At the galactic scale, the universality of the acceleration scale a_0 can be tested by precisely measuring the rotation curves of galaxies of different masses, and the relationship between galaxy morphology and gravitational behavior can be systematically analyzed to verify the theory's ability to model the phenomena conventionally explained by dark matter. On a cosmological scale, the focus is on testing whether the theory can provide an alternative mechanism for the accelerated expansion of the universe without solely relying on dark energy, examining whether its predictions about the formation history of large-scale structure are consistent with observations, and searching for key signs of deviations from the standard ΛCDM model...

10. Results and Discussion

This paper proposes a quantum gravity scheme based on Gravito-Gauge Equivalence, focusing on treating quantum gravity as an effective tool for addressing physical problems at microscopic scales—such as the black hole information paradox and spacetime singularities—rather than pursuing a formal theory of everything. This perspective aligns with various reflective explorations within the physics community regarding the quantum gravity problem [26] [27]. We argue that quantization serves as an effective "representation" for describing microscopic gravitational behavior, while recognizing that different approaches to quantum gravity may coexist within a broader theoretical landscape. Nevertheless, this does not preclude the application of quantum gravitational concepts and methods within the effective theory framework. In this sense, quantum gravity acts more as a bridge connecting macroscopic classicality and microscopic quantum behavior. Building on the previous derivations and a new fundamental proposition, we further deepen this discussion.

1) Purpose and Positioning of Quantum Gravity

The primary motivation for quantizing gravity stems from the urgent need to resolve microscopic gravitational phenomena [28] [29], such as the black hole information paradox, the structure of singularities, and the nature of spacetime at the Planck scale. General relativity excellently describes macroscopic gravitational phenomena—such as gravitational waves and black hole dynamics—classically. However, at microscopic scales, it may fundamentally conflict with the principles of quantum theory [9]. For instance, although Hawking radiation can be derived within a semiclassical approximation, the information loss problem it entails may require a more fundamental quantum gravity theory for resolution [22].

Thus, quantization should be viewed as an effective, problem-specific approach. Historically in physics, quantization schemes have often been introduced to resolve theoretical inconsistencies at specific scales (e.g., UV divergences), embodying a pragmatic spirit. While mainstream quantum gravity frameworks, such as string theory or loop quantum gravity, strive to construct unified theories [30]-[32], the academic community maintains diverse perspectives on the most fruitful approaches to quantum gravity. Within this context, quantum gravity can be regarded as an effective approximate theory at microscopic scales, while maintaining openness to different theoretical possibilities.

2) A Unified Perspective Under the GGE and $GL(m)$ Framework and a Novel Proposition

Within the geometric framework of GGE (Generalized Gauge Equation) and the $GL(m)$ principal bundle, the four fundamental interactions can be uniformly understood as projective components of the principal connection or curvature on the base manifold [19] [20]. This framework interprets gravity as a gauge theory of spacetime symmetries (Lorentz or Poincaré groups), placing it on equal mathematical footing with $U(1)$, $SU(2)$, and $SU(3)$ gauge theories [11]. On the principal bundle $P(M,GL(n,\mathbb{C}))$, the connection $\tilde{\omega}$ inherently incorporates gauge field components from all interactions, with the GGE transformation providing the precise mathematical mechanism for their interconversion.

A pivotal result of this work is the establishment of the quantized Weyl-electromagnetic relation:

$$\hat{C}_{\mu\nu\rho\sigma} = \kappa_{geom}\hat{\psi}_{ABCD}\left(\sigma^{AB}_{\mu\rho}\sigma^{CD}_{\nu\sigma} - \sigma^{AB}_{\mu\sigma}\sigma^{CD}_{\nu\rho}\right)$$

where the composite spinor operator $\hat{\psi}_{ABCD} =: \hat{\phi}^{(1)}_A\hat{\phi}^{(1)}_B\hat{\phi}^{(2)}_C\hat{\phi}^{(2)}_D :$ is defined as the gravitational spinor (GS).

Building upon this foundation, we propose a fundamental thesis: **the gravitational field in vacuum is composed of Gravitational Spinors $\hat{\psi}_{ABCD}$.** Their excitations can form stable gravitational solitons (as constructed herein) under specific conditions, while reducing to conventional gravitons in the weak-field, linear approximation. Consequently, the mechanism of the gravitational interaction is postulated to be mediated by the exchange of virtual GS, in direct analogy to the exchange of virtual photons in quantum electrodynamics.

This proposition synthesizes GGE transformations, spinor mapping, and quantum gravitational effects into a self-consistent framework, offering a novel and concrete microscopic mechanism for the quantum origin of gravity.

3) Significance of Nonlinear GS Equations for Dark Matter Problem

A particularly promising application of this framework emerges from the nonlinear GS equations, which propose a novel geometric approach to the dark matter problem. The modified nonlinear GS equation:

$$\Box\psi_{ABCD} - 4\lambda\left|\nabla\psi\right|^2 B_{ABCD} = 0$$

is found to naturally reproduce Modified Newtonian Dynamics (MOND) behavior in the weak-field limit, while recovering standard Newtonian gravity in high-density regions. This indicates that the observed galactic rotation curves—traditionally attributed to dark matter—could also be interpreted through geometric nonlinearities in the gravitational sector, presenting an alternative to the hypothesis of exotic matter components.

The strength of this framework lies in its achievement of unification at the geometric level classically, indicating that the unity of interactions is rooted in geometric structure. In classical gauge theory gravity (GTG), general relativity can already be analogized with gauge field theory (e.g., via Cartan geometry [25] [33]). The nonlinear GS extension provides a natural mechanism for addressing one of cosmology's most persistent puzzles while maintaining geometric consistency. Theories based on Weyl geometry or teleparallel gravity in the literature also support this path of prioritizing classical geometric unification [34]. Therefore, the unity of the world is primarily geometric, with the nonlinear GS framework offering concrete solutions to outstanding observational challenges.

4) Quantum Gravity as an Effective Approach: Considerations on Mainstream Frameworks

Based on the above discussion, we contend that quantum gravity theory is primarily a methodology for addressing specific problems of general relativity in the microscopic domain. The success of the nonlinear GS approach in resolving the dark matter problem demonstrates the value of exploring geometric extensions to classical gravity that yield testable predictions at astronomical scales.

Importantly, the GS-based quantization framework has been demonstrated to be asymptotically safe [24], ensuring UV-completeness and mathematical consistency at high energy scales. This addresses a fundamental challenge in quantum gravity research and strengthens the theoretical foundation of the GS approach.

In many practices, gravity quantization is performed within the linearized weak-field approximation ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, (where $|h| \ll 1$)), treating gravitons as excitation quanta [32]. This avoids the nonlinear self-interactions of GR, but in strong-field regions (such as within black holes or near singularities), perturbative methods fail, necessitating non-perturbative approaches [32]. This raises important questions about the most fruitful paths forward in quantum gravity research.

However, this does not negate the value of related research. On the contrary, it points to new directions: 1) exploring non-perturbative quantization schemes (e.g., area and volume quantization in loop quantum gravity [31] or perturbative expansions in string theory [32]); 2) considering extensions of classical geometry that address observational puzzles like dark matter through frameworks like the nonlinear GS equations; and 3) developing effective descriptions that bridge quantum and classical gravitational behavior.

Therefore, diverse approaches should be valued in quantum gravity research. The nonlinear GS framework demonstrates how geometric extensions to classical gravity can resolve major observational puzzles while maintaining theoretical consistency. Mainstream theories (e.g., string theory, loop quantum gravity) and the geometric perspective proposed here are not opposed but represent different paths exploring the unknown, potentially converging in a more profound future theory. The "gravitational spinor (GS)" concept and its nonlinear extensions offer new possibilities for this convergence, with final validation awaiting further experimental exploration.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A. Solution Process of Quantum Cartan Structural Equation

We follow the idea of "semiclassical approximation" or "background field quantization" to decompose the quantum operator into classical background values plus quantum fluctuations [35]-[37].

Step 1: Decomposition of the quantum field

- **Quantum Metric and Quantum Vierbein:**

The quantum metric operator is $\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \hat{h}_{\mu\nu}$. We decompose the corresponding quantum Vierbein operator $\hat{e}^a$ as follows:

$$\hat{e}^a{}_\mu = e^a{}_\mu + \delta\hat{e}^a{}_\mu \tag{A1}$$

where $e^a{}_\mu$ is the classical background Vierbein, defined by the classical soliton metric $H = \text{sech}^2(ku)\left[A\left(x^2 - y^2\right) + 2Bxy\right]$ (whose components have been given above). $\delta\hat{e}^a{}_\mu$ is the quantum fluctuation operator, whose expectation value $\left\langle \delta\hat{e}^a{}_\mu \right\rangle = 0$.

- **Quantum Lorentz connection:**

Similarly, we decompose the desired quantum Lorentz connection operator into:

$$\hat{\omega}^a_{\mu b} = \omega^a_{\mu b} + \delta\hat{\omega}^a_{\mu b} \tag{A2}$$

where $\omega^a_{\mu b}$ is the classical background connection, *i.e.*, the one solved by the classical Cartan first structure equation $de^a + \omega^a_b \wedge e^b = 0$ (its components include $\omega^0_{u2} = -\frac{1}{\sqrt{2}}\text{sech}^2(ku)(Ax + By)$). $\delta\hat{\omega}^a_{\mu b}$ is the quantum fluctuation operator of the connection.

Step 2: Substitute into the quantized equation

Substitute the decomposed operator into the quantized Cartan first structure equation:

$$de^a + \omega^a_b \wedge e^b = 0$$

after expansion, we get:

$$d\left(e^a + \delta\hat{e}^a\right) + \left(\omega^a_b + \delta\hat{\omega}^a_b\right) \wedge \left(e^b + \delta\hat{e}^b\right) = 0 \tag{A3}$$

Step 3: Separate the equations by perturbation order

This is an operator equation. We separate it by the order of quantum fluctuations.

- Zero-order term (pure classical background term):

$$de^a + \omega^a_b \wedge e^b = 0$$

This is exactly the classic Cartan equation, which we already satisfied when we constructed Vierbein. Therefore, this term is always zero.

- First-order term (linear fluctuation term):

After the expansion, all terms containing a fluctuation operator ($\delta\hat{e}$ or $\delta\hat{\omega}$) and any number of classical quantities are first-order terms. Extract these terms:

$$d\left(\delta\hat{e}^a\right)+\omega^a_{\ b}\wedge\delta\hat{e}^b+\delta\hat{\omega}^a_{\ b}\wedge e^b=0 \tag{A4}$$

This equation is the core equation for solving quantum fluctuations $\delta\hat{\omega}^a_{\ b}$.

- Higher-order terms:

Terms involving the product of two or more fluctuation operators (e.g., $\delta\hat{\omega}\wedge\delta\hat{e}$) are nonlinear terms and higher-order contributions in perturbation theory, which we will not consider here.

Step 4: Solve the first-order equation to obtain the connection fluctuation

Our goal is to solve the first-order equation for $\delta\hat{\omega}^a_{\ b}$. To do this, we need to rewrite the equation so that we explicitly solve for $\delta\hat{\omega}$. This is a system of linear differential equations.

A standard and effective method is to choose a specific gauge, such as the Lorentz gauge ($\omega^a_{[bc]}=0$), which can greatly simplify the calculation. After the gauge is chosen, the equation:

$$\delta\hat{\omega}^a_{\ b}\wedge e^b=-d\left(\delta\hat{e}^a\right)-\omega^a_{\ b}\wedge\delta\hat{e}^b \tag{A5}$$

The right side of is completely known (the classical connection ω is known, and the Vierbein fluctuations $\delta\hat{e}^a$ are fundamental quantum fields). This is a linear equation with $\delta\hat{\omega}^a_{\ b}$ as the unknown.

Since this form is differential, we can project it onto the coordinate basis. Applying both sides to the vector basis ∂_μ and exploiting the form-component relationship, we obtain a system of equations for the components $\delta\hat{\omega}^a_{\mu b}$. Through a lengthy but straightforward calculation involving metrics, Vierbein equations, and their inverses, we can inversely solve for $\delta\hat{\omega}^a_{\mu b}$ in terms of $\hat{e}^a_{\ \mu}$ and its derivatives as follows:

$$\delta\hat{\omega}_{\mu ab}\sim\frac{1}{2}\left(e^\nu_a\left(\partial_\mu\delta\hat{e}_{b\nu}-\partial_\nu\delta\hat{e}_{b\mu}\right)-e^\nu_b\left(\partial_\mu\delta\hat{e}_{a\nu}-\partial_\nu\delta\hat{e}_{a\mu}\right)-e^\rho_a e^\sigma_b\left(\partial_\rho\delta\hat{e}_{\sigma c}-\partial_\sigma\delta\hat{e}_{\rho c}\right)e^c_\mu\right) \tag{A6}$$
$$+\left(\text{terms involving the classical background connection }\omega\right)$$

Step 5: Combine to get a complete quantum connection

Combining the classical solution and the quantum fluctuation solution, we get the complete quantum Lorentz connection operator:

$$\hat{\omega}^a_{\mu b}=\omega^a_{\mu b}+\delta\hat{\omega}^a_{\mu b} \tag{A2}$$

Now, let's look at the specific component form given. For $\hat{\omega}^0_{u2}$:

$$\hat{\omega}^0_{u2}=\underbrace{\left(-\frac{1}{\sqrt{2}}\operatorname{sech}2(ku)(Ax+By)\right)}_{\text{Classic background value }\omega^0_{u2}}+\underbrace{\delta\hat{\omega}^0_{u2}}_{\text{Quantum Fluctuations}} \tag{A7}$$

- Classical part: comes directly from classical calculations, the coefficients A and B are parameters, and x and y here are classical spacetime coordinates.
- Quantum Fluctuation Part $(\cdots)$: $\delta\hat{\omega}^0_{u2}$ is an operator that can ultimately be expressed as a linear combination of the fundamental quantum field $\delta\hat{e}^a_{\ \mu}$ and its derivatives. Its specific form depends on the specific pattern of $\delta\hat{e}^a_{\ \mu}$ and the quantum state we choose. This is why it is denoted by the ellipsis $\cdots$

—it is not a universal c-number function, but an operator that depends on quantum details.

So the expression:

$$\hat{\omega}_{u2}^{0} = -\frac{1}{\sqrt{2}}\operatorname{sech}^2(ku)(A\hat{x}+B\hat{y})+\cdots$$

Its physical meaning is:

- $-\frac{1}{\sqrt{2}}\operatorname{sech}^2(ku)(Ax+By)$: This is the classical background value of the quantum connection. It determines the soliton profile of the quantum connection and the main expectation value $\langle \hat{\omega}_{u2}^{0} \rangle$.
- $(A\hat{x}+B\hat{y})$: Where, $\hat{x}$ and $\hat{y}$ should be understood as coordinate operators. In quantum field theory, the coordinates are parameters, and the field is the operator. A more accurate understanding is that A and B are parameters, while $\operatorname{sech}^2(ku)$ is a classical function. The entire classical part contains no operators.
- $\cdots$: represents the quantum fluctuation operator $\delta\hat{\omega}_{u2}^{0}$. It contains all operator terms induced by the Vierbein quantum fluctuation $\delta\hat{e}^{a}{}_{\mu}$, and its expectation value is zero $\langle \delta\hat{\omega}_{u2}^{0} \rangle = 0$, but its existence means that the quantum connection is an operator, not a classical number.

This result proves that the expectation value of the quantum connection perfectly reproduces the classical soliton connection, while it also has quantum fluctuation characteristics, which is exactly what we expect.

Appendix B. First-Principles Derivation of the Nonlinear Term $|\nabla\psi|^2$

1) Path integral representation and effective action:

The basic statement of quantum gravity begins with the path integral:

$$Z = \int D\left[g_{\mu\nu}\right]\exp\left(iS\left[g_{\mu\nu}\right]\right) \tag{B1}$$

where $iS\left[g_{\mu\nu}\right]$ is the Einstein-Hilbert action. In the background field method, the metric is decomposed into:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{B2}$$

where $\bar{g}_{\mu\nu}$ is the classical background field and $h_{\mu\nu}$ is the quantum fluctuation. After performing path integration on the quantum fluctuation, we get the effective action:

$$\exp\left(i\Gamma[\bar{g}]\right) = \int D\left[h_{\mu\nu}\right]\exp\left(iS[\bar{g}+h]\right) \tag{B3}$$

2) Quantum corrections and higher-order curvature terms:

Single-loop quantum correction produces effective action:

$$\Gamma[\bar{g}] = S[\bar{g}] + (i/2)Tr\ln\left[\delta^2 S/\delta g\delta g\right] + \text{Counter term} \tag{B4}$$

here, $Tr\ln\left[\delta^2 S/\delta g\delta g\right]$ denotes the operation of taking the logarithm of the operator $\delta^2 S/\delta g \delta g$ followed by taking its trace. This term originates from the Gaussian path integral over the quantum fluctuation fields and represents the one-loop graviton fluctuation contribution. This contribution typically contains ultraviolet divergences [37] [38]. The "counterterm" in the expression is introduced precisely to cancel these divergences via renormalization. To eliminate these UV divergences, the required form of the counterterm is dominated by curvature-squared terms in the high-energy limit. Consequently, the resulting low-energy effective action naturally incorporates higher-order curvature corrections of the following form:

$$S_{eff} = \int \mathrm{d}^4 x \sqrt{-g} \Big[\gamma_1 \nabla_\alpha C_{\mu\nu\rho\sigma} \nabla^\alpha C^{\mu\nu\rho\sigma} + \gamma_2 C_{\mu\nu\rho\sigma} \nabla^\alpha C^{\mu\nu\rho\sigma} + \beta_1 \nabla_\alpha R \nabla^\alpha R + \beta_2 \nabla_\alpha R_{\mu\nu} \nabla^\alpha R^{\mu\nu} + \beta_3 \nabla_\alpha R_{\mu\nu\rho\sigma} \nabla^\alpha R^{\mu\nu\rho\sigma} + \alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \cdots \Big] \tag{B5}$$

Analysis shows that in vacuum, $\nabla_\alpha C_{\mu\nu\rho\sigma} \nabla^\alpha C^{\mu\nu\rho\sigma}$ is the only curvature derivative term that contributes to the momentum-dependent correction at the single-loop order, and is dominant, while other curvature derivative terms either disappear in vacuum or contribute high-order small quantities.

3) The transition from quantum operators to classical fields

In the effective action framework, the expectation value of a quantum operator is given by the path integral:

$$\langle \hat{\psi}_{ABCD} \rangle = \int D\left[g_{\mu\nu} \right] \psi_{ABCD} \exp\left(iS\left[g_{\mu\nu} \right] \right) \tag{B6}$$

Under the saddle point approximation, the dominant contribution comes from the classical solution:

$$\langle \hat{\psi}_{ABCD} \rangle \approx \psi_{ABCD} \left[\bar{g}_{\mu\nu} \right] \tag{B7}$$

This explains why the classical field ψ_{ABCD} can be used to describe the original quantum operator in the effective theory.

4) Spiror Representation of Weyl Tensor

Using the composite particle Gravitational Spinor (GS) theory proposed in this paper, we employ the spinor representation of the Weyl tensor as a starting point:

$$C_{\mu\nu\rho\sigma} = \kappa_{geom} \psi_{ABCD} \left(\sigma^{AB}_{\mu\rho} \sigma^{CD}_{\nu\sigma} - \sigma^{AB}_{\mu\sigma} \sigma^{CD}_{\nu\rho} \right) \tag{B8}$$

where ψ_{ABCD} is a totally symmetric left-handed spinor field. The coefficient κ_{geom} is a dimensionful constant introduced to ensure the dimensional consistency between the spinor field ψ and the Weyl tensor. Its value can be, in principle, fixed by matching the scales in the quantum gravity framework, and it is distinct from the GGE-related coupling discussed in the main text for the gravitational-electromagnetic interaction.

$$\nabla_\alpha C_{\mu\nu\rho\sigma} = \kappa_{geom} \left(\nabla_\alpha \psi_{ABCD} \right) \left(\sigma^{AB}_{\mu\rho} \sigma^{CD}_{\nu\sigma} - \sigma^{AB}_{\mu\sigma} \sigma^{CD}_{\nu\rho} \right) \tag{B9}$$

5) Cantraction calculation and emergence of nonlinear terms

Using the contraction identity of the spin matrix:

$$\sigma^{AB}_{\mu\nu} \sigma^{\mu\nu}_{CD} = \epsilon^A_C \epsilon^B_D + \epsilon^A_D \epsilon^B_C \tag{B10}$$

We compute $\nabla_\alpha C_{\mu\nu\rho\sigma}\nabla^\alpha C^{\mu\nu\rho\sigma}$ by substituting Equation (B8), its dual form, and Equation (B9), and systematically contracting the terms using (B10). The calculation shows that all cross terms cancel out, with the non-vanishing contribution coming entirely from the diagonal terms. The complete derivation yields:

$$\nabla_\alpha C_{\mu\nu\rho\sigma}\nabla^\alpha C^{\mu\nu\rho\sigma} = 8\kappa_{geom}^2\left(\nabla_\alpha \psi_{ABCD}\right)\left(\nabla^\alpha \psi^{ABCD}\right) = 8\kappa_{geom}^2\left|\nabla\psi\right|^2 \quad \text{(B11)}$$

Here, the numerical factor 8 arises from the complete contraction of the spinor matrices, and κ_{geom}^2 comes from the geometric constant in the definition of the Weyl tensor.

6) Detailed derivation process

Step 1: Starting from the Weyl-spinor representation

According to the Weyl-spinor representation (B8), the real Weyl tensor can be written in terms of the totally symmetric left-handed spinor ψ_{ABCD} and its complex conjugate right-handed part:

$$C_{\mu\nu\rho\sigma} = \kappa_{geom}\psi_{ABCD}T_{\mu\nu\rho\sigma}^{ABCD} + \kappa_{geom}\bar{\psi}_{A'B'C'D'}\bar{T}_{\mu\nu\rho\sigma}^{A'B'C'D'} \quad \text{(B12)}$$

where we define the tensor-spinor structure:

$$T_{\mu\nu\rho\sigma}^{ABCD} = \sigma_{\mu\rho}^{AB}\sigma_{\nu\sigma}^{CD} - \sigma_{\mu\sigma}^{AB}\sigma_{\nu\rho}^{CD} \quad \text{(B13)}$$

and $\bar{T}_{\mu\nu\rho\sigma}^{A'B'C'D'}$ is its complex conjugate.

Step 2: Calculate the derivative

Assuming that the covariant derivatives act mainly on the dynamical fields ψ_{ABCD} and $\bar{\psi}_{A'B'C'D'}$, we obtain:

$$\nabla_\alpha C_{\mu\nu\rho\sigma} = \kappa_{geom}\left(\nabla_\alpha\psi_{ABCD}\right)T_{\mu\nu\rho\sigma}^{ABCD} + \kappa_{geom}\left(\nabla_\alpha\bar{\psi}_{A'B'C'D'}\right)\bar{T}_{\mu\nu\rho\sigma}^{A'B'C'D'} \quad \text{(B14)}$$

Step 3: Construct the complete contraction terms

The scalar term we want to calculate is:

$$\nabla_\alpha C_{\mu\nu\rho\sigma}\nabla^\alpha C^{\mu\nu\rho\sigma}$$

Substitute the derivative expression (B14) from step 2. Since the left-hand part and the right-hand part are orthogonal under the contraction (*i.e.*, the cross term is zero), the complete contraction consists of two independent parts:

$$\nabla_\alpha C_{\mu\nu\rho\sigma}\nabla^\alpha C^{\mu\nu\rho\sigma} = \kappa_{geom}^2\left(\nabla_\alpha\psi_{ABCD}\right)\left(\nabla^\alpha\psi_{EFGH}\right)T_{\mu\nu\rho\sigma}^{ABCD}T_{EFGH}^{\mu\nu\rho\sigma} + \left(\text{Complex conjugate part}\right)$$

The complex conjugate part will give exactly the same form, so we only need to calculate one of the parts in detail and multiply the final result by 2.

Step 4: Expand and simplify the core contraction items

Now, we compute the core spinor-tensor contraction:

$$S_{EFGH}^{ABCD} = T_{\mu\nu\rho\sigma}^{ABCD}T_{EFGH}^{\mu\nu\rho\sigma} \quad \text{(B15)}$$

Substitute the definition of T and expand it:

$$S_{EFGH}^{ABCD} = \left(\sigma_{\mu\rho}^{AB}\sigma_{\nu\sigma}^{CD} - \sigma_{\mu\sigma}^{AB}\sigma_{\nu\rho}^{CD}\right)\times\left(\sigma_{\mu\rho}^{EF}\sigma_{\nu\sigma}^{GH} - \sigma_{\mu\sigma}^{EF}\sigma_{\nu\rho}^{GH}\right)$$

After expansion, we get four items:

$$S^{ABCD}_{EFGH} = \sigma^{AB}_{\mu\rho}\sigma^{CD}_{\nu\sigma}\sigma^{EF}_{\mu\rho}\sigma^{GH}_{\nu\sigma} - \sigma^{AB}_{\mu\rho}\sigma^{CD}_{\nu\sigma}\sigma^{EF}_{\mu\sigma}\sigma^{GH}_{\nu\rho} - \sigma^{AB}_{\mu\sigma}\sigma^{CD}_{\nu\rho}\sigma^{EF}_{\mu\rho}\sigma^{GH}_{\nu\sigma} + \sigma^{AB}_{\mu\sigma}\sigma^{CD}_{\nu\rho}\sigma^{EF}_{\mu\sigma}\sigma^{GH}_{\nu\rho}$$

Step 5: Apply the spinor matrix contraction relation

Using the spin matrix contraction identity (B10):

$$\sigma^{AB}_{\mu\nu}\sigma^{\mu\nu}_{CD} = \epsilon^A_C\epsilon^B_D + \epsilon^A_D\epsilon^B_C$$

Perform a systematic reduction calculation on the four terms from the previous step. The calculation results show that all cross terms (the second and third terms) cancel each other out, and the non-zero contribution comes entirely from the two "diagonal terms" (the first and fourth terms). Ultimately, the reduction simplifies to:

$$S^{ABCD}_{EFGH} = 4\left(\epsilon^A_E\epsilon^B_F\epsilon^C_G\epsilon^D_H + \epsilon^A_E\epsilon^B_F\epsilon^C_H\epsilon^D_G + \epsilon^A_F\epsilon^B_E\epsilon^C_G\epsilon^D_H + \epsilon^A_F\epsilon^B_E\epsilon^C_H\epsilon^D_G\right) \quad \text{(B16)}$$

Step 6: Combine with the $\nabla\psi$ term and obtain the final factor

Substitute the contraction result (B16) back into the expression from step 3. The first part (from the left-handed spinors) becomes:

$$\kappa^2_{geom}\nabla_\alpha C_{\mu\nu\rho\sigma}\nabla^\alpha C^{\mu\nu\rho\sigma}S^{ABCD}_{EFGH}$$
$$= 4\kappa^2_{geom}\left(\nabla_\alpha\psi_{ABCD}\right)\left(\nabla^\alpha\psi_{EFGH}\right)\left(\epsilon^A_E\epsilon^B_F\epsilon^C_G\epsilon^D_H + \text{symmetric permutations}\right)$$

Using the property of the spinor metric ϵ^{AB} to raise indices, *i.e.*, $\psi^{ABCD} = \epsilon^{AE}\epsilon^{BF}\epsilon^{CG}\epsilon^{DH}\psi_{EFGH}$, the expression in the parentheses precisely contracts with the two $\nabla\psi$ fields to yield $\left(\nabla_\alpha\psi_{ABCD}\right)\left(\nabla^\alpha\psi_{EFGH}\right)$. Therefore:

$$\text{First Part} = 4\kappa^2_{geom}\left(\nabla_\alpha\psi_{ABCD}\right)\left(\nabla^\alpha\psi^{ABCD}\right) = 4\kappa^2_{geom}\left|\nabla\psi\right|^2$$

An identical calculation for the complex conjugate part (from the right-handed spinors) yields another $4\kappa^2_{geom}\left|\nabla\bar{\psi}\right|^2$. For a real metric and the specific form of ψ used in our theory, which ensures the reality of the geometric constructs, these two contributions are equal. Summing them gives the final result (B11):

$$\nabla_\alpha C_{\mu\nu\rho\sigma}\nabla^\alpha C^{\mu\nu\rho\sigma} = 8\kappa^2_{geom}\left|\nabla\psi\right|^2$$

where $\left|\nabla\psi\right|^2 \equiv \left(\nabla_\alpha\psi_{ABCD}\right)\left(\nabla^\alpha\psi^{ABCD}\right)$.

7) Quantum-determined coupling strength

Since the single-loop quantum correction determination coefficient:

$$\gamma = (N-4)/\left(960\pi^2\right) + O\left(1/N^2\right) \quad \text{(B17)}$$

Therefore, nonlinear terms emerge in the classical equations of motion:

$$\Box\psi_{ABCD} - 4\lambda\left|\nabla\psi\right|^2 B_{ABCD} = 0 \quad \text{(B18)}$$

where $\lambda = \gamma/4$ is determined by quantum effects.

It is important to emphasize that this derivation establishes the fundamental origin of the nonlinear term $\left|\nabla\psi\right|^2$ from pure quantum gravitational considerations. In the full GGE framework discussed in the main text, this geometrically derived nonlinearity provides the foundation for the gravitational self-interaction, while the specific relation $C_{\mu\nu\rho\sigma} = \kappa_{coupl}\left(F_{\mu\rho}F_{\nu\sigma} - F_{\mu\sigma}F_{\nu\rho}\right)$ emerges as a particular physical solution of the coupled system. The constant κ_{geom} derived here serves as the geometric normalization, while $\kappa_{coupl} \sim 8\pi\alpha$ represents the physical coupling strength in specific solitonic configurations.