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Observation about the Classical Electromagnetic Gauge Transformation and Its Quantum Correspondence

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Abstract

Using the Landau and symmetric gauges for the vector potential of a constant magnetic field and the quantum problem of a charged particle moving on a flat surface, we show the classical electromagnetic gauge transformation does not correspond to a one-dimensional unitary group transformation U(1) of the wave function for the quantum case. In addition, with the re-examination of the relation between the magnetic field $\boldsymbol{B}$ and its vector potential $\boldsymbol{A}$, we found that, in order to have a consistent formulation of the dynamics of the charged particle with both expressions, we must have that $\boldsymbol{B}=\nabla\times\boldsymbol{A}$ if and only if $\boldsymbol{B}\neq\boldsymbol{0}$.

Keywords

Gauge Transformation, Harmonic Oscillator, Quantum Hall Effect, Electromagnetic Potentials

1. Introduction

Gauge Theory is a mathematical tool which has been used in Physics to study the interaction between matter and fields [1] through a Lagrangian which is invariant under local transformations (symmetry), defined by a Lie Group [2]. The elements of the associated Lie Algebra of the group are called gauge fields, and these gauge fields, in turn, are called gauge bosons when they are quantized. For example, Quantum Electrodynamics is considered an Abelian gauge theory with the symmetric group U (1) and having the photons as the gauge bosons [3] [4]. The Standard Model of elementary particles is considered a non-Abelian gauge theory with the symmetric group $U(1)\times SU(2)\times SU(3)$ where the photon (γ), three weak bosons (W_+,W_-,Z), and eight gluons (g_i, $i=1,\cdots,8$) are their gauge bosons. In addition, there are other examples of applications as Yang-Mills theory [5], gravity [6], condense matter [3] [7], and nuclear physics [8].

One could say that the source of all gauge theories is the relation between the gauge transformation in Classical Electrodynamics [9] and the associated non-relativistic quantum gauge transformation of the wave function [10], which could be stated (locally) in the following way [11] [12] [13] [14] [15].

Proposition: Let $\chi(\boldsymbol{x},t)$ be the phase associated with a local gauge transformation, symmetry group U(1), of the wave function $\Psi(\boldsymbol{x},t)$, that is,

$$\Psi'(\boldsymbol{x},t) = e^{\frac{iq}{\hbar c}\chi(\boldsymbol{x},t)}\Psi(\boldsymbol{x},t), \tag{1}$$

of the non-relativistic Quantum Mechanics. If the Schrödinger equation is of the form

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{1}{2m}\left(\hat{\boldsymbol{p}} - \frac{q}{c}\boldsymbol{A}\right)^2\Psi + qV\Psi, \tag{2}$$

where $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{x},t)$ is a vector field, $V = V(\boldsymbol{x},t)$ is a scalar field, q is the particle charge, $\hbar$ is the Plank's constant, m is the particle mass and c is the speed of light. Then, this equation remains invariant under this transformation if

$$\nabla^2\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = 0, \tag{3}$$

$$\boldsymbol{A}'(\boldsymbol{x},t) = \boldsymbol{A}(\boldsymbol{x},t) + \nabla\chi, \tag{4}$$

and

$$V'(\boldsymbol{x},t) = V(\boldsymbol{x},t) - \frac{1}{c}\frac{\partial\chi}{\partial t}. \tag{5}$$

Thus, the Schrödinger equation for Ψ' would be

$$i\hbar\frac{\partial\Psi'}{\partial t} = \frac{1}{2m}\left(\hat{\boldsymbol{p}} - \frac{q}{c}\boldsymbol{A}'\right)^2\Psi' + qV'\Psi'. \tag{6}$$

The expressions (4) and (5) are just the usual gauge transformation of the electromagnetic field where the electric and magnetic fields are given by

$$\boldsymbol{E} = -\nabla V - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} \qquad B = \nabla\times\boldsymbol{A}. \tag{7}$$

For the case when one is dealing with time-independent fields and zero scalar potential, one would say that a local gauge transformation

$$\Psi'(\boldsymbol{x},t) = e^{\frac{iq}{\hbar c}\chi(\boldsymbol{x})}\Psi(\boldsymbol{x},t) \tag{8}$$

will leave the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{1}{2m}\left(\hat{\boldsymbol{p}} - \frac{q}{c}\boldsymbol{A}\right)^2\Psi \tag{9}$$

invariant, if the vector potential $\boldsymbol{A}$ is transformed according to

$$\boldsymbol{A}'(\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{x}) + \nabla\chi \quad \text{with } \nabla^2\chi = 0. \tag{10}$$

For this particular case, in this paper we want to see whether or not the inverse proposition is satisfied, that is, if one has a vector field $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{x})$ satisfying the relations (10), and the Schrödinger equation is of the form (9), would the solution of the equation

$$i\hbar \frac{\partial \Psi'}{\partial t} = \frac{1}{2m}\left(\hat{\boldsymbol{p}} - \frac{q}{c}\boldsymbol{A}'\right)^2 \Psi' \tag{11}$$

be of the form (8)? Because of the time independence problem, there is a separation of variables, and one can propose $\Psi(\boldsymbol{x},t) = \mathrm{e}^{-iEt/\hbar}\Phi(\boldsymbol{x})$ to reduce the problem and to an eigenvalue problem

$$\frac{1}{2m}\left(\hat{\boldsymbol{p}} - \frac{q}{c}\boldsymbol{A}'\right)^2 \Phi' = E\Phi', \tag{12}$$

where the eigenvalues will be Landau's leves $E_n = \hbar\omega_c(n+1/2)$ which are gauge invariant, being $\omega_c = qB/mc$ the cyclotron frequency, and one wants to see whether or not the solution is of the form

$$\Phi'(\boldsymbol{x}) = \mathrm{e}^{\frac{iq}{\hbar c}\chi(\boldsymbol{x})}\Phi(\boldsymbol{x}), \tag{13}$$

where $\Phi(\boldsymbol{x})$ is the solution of the eigenvalue problem

$$\frac{1}{2m}\left(\hat{\boldsymbol{p}} - \frac{q}{c}\boldsymbol{A}\right)^2 \Phi = E\Phi. \tag{14}$$

2. Analysis

For a charged particle moving on a flat surface ($x, y, 0$) in a constant transversal magnetic field $\boldsymbol{B} = (0,0,B)$, one can select Landau's gauges $\boldsymbol{A} = B(-y,0,0)$ and $\boldsymbol{A}' = B(0,x,0)$ or the symmetric gauge $\boldsymbol{A}'' = B(-y,x,0)/2$ since they bring about the same magnetic field $\boldsymbol{B} = \nabla\times\boldsymbol{A} = \nabla\times\boldsymbol{A}' = \nabla\times\boldsymbol{A}''$, and one has that

$$\boldsymbol{A}'(\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{x}) + \nabla\chi \quad \text{and} \quad \boldsymbol{A}''(\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{x}) + \nabla\tilde{\chi}, \tag{15}$$

where χ and $\tilde{\chi}$ are given by

$$\chi(\boldsymbol{x}) = Bxy \quad \text{and} \quad \tilde{\chi}(\boldsymbol{x}) = Bxy/2. \tag{16}$$

One needs to point out here that Landau' solution for this problem is separable on the variables x and y, which comes from the used property that the operator $\hat{p}_x$ commutes with the Hamiltonian. However, this does not mean that the solution must be separable on the variables x and y since the associated equation is not of separable variables, as clearly can be seen on reference [16] equation (10), and in this same reference also a non-separable solution was given for this problem. It is not difficult to see that the non-separable solution of the eigenvalue problem

$$\frac{1}{2m}\left(\hat{\boldsymbol{p}} - \frac{q}{c}\boldsymbol{A}\right)^2 \Phi = E\Phi \tag{17}$$

was given in [16] as

$$\Phi_n(\boldsymbol{x}) \sim e^{-i\beta xy}\psi_n\left(\sqrt{\beta}x\right), \quad E_n = \hbar\omega_c\left(n+\frac{1}{2}\right), \tag{18}$$

as it was mentioned before, where one has omitted the normalization constant since it is no relevant for the analysis, the constants β is defined as $\beta = m\omega_c/\hbar$, and $\omega_c = qB/mc$ is the cyclotron frequency. The energies E_n are called Landau's levels, which are gauge invariant, and the function ψ_n is the solution of 1-D eigenfunctions of the harmonic oscillator. In this case, one does not have a continuous degeneration for each Landau's level (which is due to separation of the variables x and y) but one obtains an infinity discrete degeneration [17], and it is enough to see our statement for a single eigenvalue, without to lose any generalization. On the other hand, the solution of the eigenvalue problem

$$\frac{1}{2m}\left(\hat{\boldsymbol{p}} - \frac{q}{c}\boldsymbol{A}'\right)^2 \Phi' = E\Phi' \tag{19}$$

is just obtained through a rotation where the variables are changed as follow $x \to y$ and $y \to -x$

$$\Phi'_n(\boldsymbol{x}) \sim e^{i\beta xy}\psi_n\left(\sqrt{\beta}y\right), \quad E_n = \hbar\omega_c\left(n+\frac{1}{2}\right), \tag{20}$$

having the constants the same meaning as above. As one can see from (18) and (20), there is not way to mach both solutions through an U(1) local transformation, that is,

$$\Phi'_n(\boldsymbol{x}) \neq e^{\frac{iq}{\hbar c}\chi(\boldsymbol{x})}\Phi_n(\boldsymbol{x}). \tag{21}$$

Now, the solution of the eigenvalue problem

$$\frac{1}{2m}\left(\hat{\boldsymbol{p}} - \frac{q}{c}\boldsymbol{A}''\right)^2 \Phi'' = E\Phi'' \tag{22}$$

is given by [18] (ones again, this represents a non-separable solution of the problem) as

$$\Phi''_n(\boldsymbol{x}) \sim e^{-\alpha\left(x^2+y^2\right)-\lambda(x+iy)}\left[2\alpha(x-iy)+\lambda\right]^n \quad \text{and} \quad E_n = \hbar\omega_c\left(n+\frac{1}{2}\right), \tag{23}$$

where $\alpha = m\omega_c/4\hbar$ and λ being a complex constant. Once a gain appears the obvious situation

$$\Phi''_n(\boldsymbol{x}) \neq e^{\frac{iq}{\hbar c}\chi(\boldsymbol{x})}\Phi_n(\boldsymbol{x}). \tag{24}$$

3. Dynamics with *B* and *A*

Given the electric field $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{x},t)$ and the magnetic field $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{x},t)$ it is well known that the dynamics of a charged particle under these field is given (CGS units) by the equation [19]

$$\frac{d\boldsymbol{p}}{dt} = q\boldsymbol{E} + \frac{q}{c}\boldsymbol{v}\times\boldsymbol{B}, \tag{25}$$

where c is the speed of light, q is the charge (we adopted a different sign as previous section), $\boldsymbol{v}$ is its velocity, and $\boldsymbol{p}$ can be $m\boldsymbol{v}$ (non relativistic motion) or $\gamma m\boldsymbol{v}$ (relativistic motion), with $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$, m the mass of the charge, and $v = |\boldsymbol{v}|$ its speed. The known relations between these field and the vector potential $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{x}, t)$ and the scalar potential $V = V(\boldsymbol{x}, t)$ are

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}. \tag{26}$$

Now, using these relations in (25), the dynamics of the charged particle is given by

$$\frac{\mathrm{d}\left(\boldsymbol{p} + q\boldsymbol{A}/c\right)}{\mathrm{d}t} = -\nabla\left(qV - \frac{q}{c}\boldsymbol{v}\cdot\boldsymbol{A}\right), \tag{27}$$

from which the Lagrangian and Hamiltonian are obtained. Note that in order to get this expression, it is necessary that $\partial \boldsymbol{A}/\partial t \neq \boldsymbol{0}$. From equation (25) one can see that if $\boldsymbol{B} = \boldsymbol{0}$ and $\boldsymbol{E} = \boldsymbol{0}$, one gets a free motion for the charged particle and the linear momentum $\boldsymbol{p}$ is a constant of motion. However, let us not from equation (27) that if $V = constant$ and $\nabla(\boldsymbol{v}\cdot\boldsymbol{A}) = \boldsymbol{0}$ the resulting constant of motion would be

$$\boldsymbol{P} = \boldsymbol{p} + q\boldsymbol{A}/c. \tag{28}$$

One might think that given the gauge (4) the function χ can be chosen such that $\boldsymbol{A} = \boldsymbol{0}$ and then $\boldsymbol{P} = \boldsymbol{p}$, but this is fact why $\boldsymbol{A}$ must be zero if the magnetic field is also zero. Let us note that there is not even a chance that the vector potential $\boldsymbol{A}$ could be of the form $\boldsymbol{A} = \nabla\chi$, being the scalar function χ solution of the wave equation (3). If the electrostatic field is zero, it is no possible classically to have $\boldsymbol{p}$ and $\boldsymbol{P} = \boldsymbol{p} + q\boldsymbol{A}/c$ as constants of motion simultaneously. Therefore, in order for (25) and (27) to keep the same classical dynamics, one must have that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ is an acceptable expression if and only if $\boldsymbol{B} \neq \boldsymbol{0}$.

4. Conclusion

We have shown that the inverse statement given by the above proposition (1)-(5) is not satisfied for the solutions of a charged particle moving in a flat surface with transversal constant magnetic field, and this suggests that the use of the gauge invariance on quantum field theories must be carefully used. In addition, we have shown that the expression $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ also must be used only for $\boldsymbol{B} \neq \boldsymbol{0}$ in order to have invariant the classical dynamics of a charged particle, suggesting that experiments where $\boldsymbol{B} = \boldsymbol{0}$ and still considering the vector potential should be carefully analyzed within classical or quantum theory [20] [21].

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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