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Spherically Symmetric Problem of General Relativity for a Fluid Sphere

Valery V. Vasiliev, Leonid V. Fedorov

Institute of Applied Mechanics of Russian Academy of Sciences, Moscow, Russia
Email: kkamenarova@mmcbg.org

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Abstract

The paper is devoted to a spherically symmetric problem of General Relativity (GR) for a fluid sphere. The problem is solved within the framework of a special geometry of the Riemannian space induced by gravitation. According to this geometry, the four-dimensional Riemannian space is assumed to be Euclidean with respect to the space coordinates and Riemannian with respect to the time coordinate. Such interpretation of the Riemannian space allows us to obtain complete set of GR equations for the external empty space and the internal spaces for incompressible and compressible perfect fluids. The obtained analytical solution for an incompressible fluid is compared with the Schwarzchild solution. For a sphere consisting of compressible fluid or gas, a numerical solution is presented and discussed.

Keywords

General Relativity, Spherically Symmetric Problem, Fluid Sphere

1. Introduction

The paper is concerned with the spherically symmetric GR problem for a sphere consisting of perfect fluid. This problem is of primary importance for the theory, because the model of an incompressible fluid allows us to obtain the exact analytical solution which is hardly reachable for more complicated models. The first solution of the problem was found by K. Schwarzchuld [1] who discovered than for a certain radius of the sphere (8/9 of the gravitation radius) the pressure at the sphere center can become infinitely high. This result was later associated with the existence of Black Holes [2]. However, the model of an incompressible fluid is not compatible with GR [3], because the velocity of sound in such fluid is infinitely high, whereas in GR it is limited by the velocity of light. The numerical solution for a sphere consisting of a compressible gas was obtained by J.R. Oppenheimer and G.M. Volkoff [4]. A class of inverse solutions was constructed by R.C. Tolman [5]. Since three field equations (for the energy-momentum tensor components T^{11}, T^{22} and T^{44}) include four unknown functions (two metric coefficients g_{11} and g_{44}, pressure and density), the metric coefficients were linked by an assumed equation and the obtained solutions were studied with respect to physical consistency. Further results are presented elsewhere [6] [7] [8] [9].

Though a sphere of perfect fluid is frequently used as an idealized model of stellar objects, particular neutron stars [4], the problem considered in this paper corresponds to the conventional phenomenological GR theory based on the traditional model of space as a homogeneous isotropic continuum whose actual microstructure is ignored.

The aforementioned results are based on the traditional form of the line element in which space and time metric coefficients correspond to the Riemannian space. This paper demonstrates the alternative approach based on the proposed special model of the Riemannian space which is Euclidean with respect to space coordinates and is Riemannian with respect to time only. Within the framework of this model, analytical solutions of the external and internal problems for a sphere of perfect incompressible fluid are obtained and the numerical solution of the internal problem for a sphere consisting of compressible fluid or gas is constructed.

2. Special Riemannian Space Model

Within the framework of the traditional GR theory, the continuum is characterized with the energy-momentum tensor $T_i^j \ (i, j = 1,2,3,4)$ introduced in a 4-dimensional Riemannian space with the line element

$$ds^2 = g_{ij} dx^i dx^j \tag{1}$$

The energy-momentum tensor must satisfy four conservation equations

$$\nabla_k T_i^k = 0 \quad (i, k = 1,2,3,4) \tag{2}$$

If $x^k \ (k = 1,2,3)$ are the space coordinates and x^4 is the time coordinate, the first three Equations (2) are the motion equations, whereas the fourth Equation (2) provides the conservation of mass.

To obtain the expressions for T_i^j, consider a spherically symmetric problem in coordinates r, θ, φ for the Newton gravitation theory. The field equations including the motion equation and the conservation equationin the Euler coordinates are [10]

$$\frac{\partial \sigma_r}{\partial r} + \frac{2}{r}(\sigma_r - \sigma_\theta) - \mu \frac{\partial \psi_i}{\partial r} = \mu \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} \right), \quad \frac{\partial (\mu v_r)}{\partial r} + \frac{2}{r} \mu v_r + \frac{\partial \mu}{\partial t} = 0 \tag{3}$$

Here, σ_r and σ_θ are the radial and the circumferential stresses, v_r is the radial velocity, μ is the density, t is time, and ψ is the Newton gravitation potential. Using identical transformations and the second equation in Equations (3), we can reduce the right-hand part of the first equation to the following form [10]:

$$\mu\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}\right)=\frac{\partial(\mu v_r)}{\partial t}+\frac{\partial\left(\mu v_r^2\right)}{\partial r}-v_r\left[\frac{\partial \mu}{\partial t}+\frac{\partial(\mu v_r)}{\partial r}\right]$$
$$=\frac{\partial(\mu v_r)}{\partial t}+\frac{\partial\left(\mu v_r^2\right)}{\partial r}+\frac{2}{r}\mu v_r^2$$

Using this result, we can present Equations (3) as

$$\begin{aligned}&\frac{\partial}{\partial r}\left(\sigma_r-\mu v_r^2\right)+\frac{2}{r}\left[\left(\sigma_r-\mu v_r^2\right)-\sigma_\theta\right]-\mu\frac{\partial\psi}{\partial r}-\frac{\partial(\mu v_r)}{\partial t}=0,\\&\frac{\partial}{\partial r}(\mu c v_r)+\frac{2}{r}\mu c v_r+\frac{\partial\left(\mu c^2\right)}{c\partial t}=0\end{aligned}\tag{4}$$

Here, c is the velocity of light. To apply Equations (2) for the energy-momentum tensor to the spherically symmetric problem of the Newton theory, we should take the following form of the line element in Equation (1):

$$\mathrm{d}s^2=\mathrm{d}r^2+r^2\mathrm{d}\Omega^2-(1-f)c^2\mathrm{d}t^2,\quad \mathrm{d}\Omega^2=\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2\tag{5}$$

in which the amplitude value of function f is much smaller than unity. Applying Equations (2) for the line element in Equation (5) and undertaking linearization of these equations with respect to f, we arrive at

$$\frac{\partial T_1^1}{\partial r}+\frac{2}{r}\left(T_1^1-T_2^2\right)+\frac{1}{2}\frac{\partial f}{\partial r}T_4^4+\frac{\partial T_1^4}{c\partial t}=0,\quad \frac{\partial T_4^1}{\partial r}+\frac{2}{r}T_4^1+\frac{\partial T_4^4}{c\partial t}=0\tag{6}$$

Matching Equations (4) to Equations (6), we can conclude that

$$T_1^1=\sigma_r-\mu v_r^2,\ T_2^2=\sigma_\theta,\ \ T_4^4=\mu c^2,\ T_4^1=\mu v_r c,\ T_1^4=-\mu v_r c,\ \ \frac{\partial f}{\partial r}=-\frac{2}{c^2}\frac{\partial\psi}{\partial r}\tag{7}$$

The last of these equations in which f is the component of the metric tensor of the Riemannian space and ψ is the Newton gravitation potential shows that the analogy between the gravitation and the Riemannian geometry actually follows from the Newton gravitation theory. In GR, Equations (7) are generalized as

$$T_i^j=\sigma_i^j-\mu v_i v^j,\ T_4^4=\mu c^2,\ T_4^i=\mu c v^i,\ T_i^4=-\mu c v_i\tag{8}$$

It is assumed that μ is the same in the first and the rest equations which follows from the principle of equivalence of the gravitation and the inertia masses. It is important to take into account that in the spherical coordinates mixed tensor components coincide with physical components. So, for a spherically symmetric problem, we have

$$T_1^1=\sigma_r-\mu v_1 v^1,\ T_2^2=\sigma_\theta,\ \ T_4^4=\mu c^2,\ T_4^1=\mu c v^1,\ T_1^4=-\mu c v_1\tag{9}$$

To determine the geometry of the Riemannian space corresponding to the Newton gravitation theory, the trajectory of a particle in a Newton gravitation field is compared to the equation that specifies a geodesic line in a Riemannian space with the line element

$$\mathrm{d}s^2 = g_{11}\mathrm{d}r^2 + g_{22}\mathrm{d}\Omega^2 - g_{44}c^2\mathrm{d}t^2 \tag{10}$$

The result of comparison yields the following expressions for the components of the metric tensor [11]:

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{44} = 1 - \frac{r_g}{r} \tag{11}$$

in which

$$r_g = \frac{2mG}{c^2} \tag{12}$$

is the so-called gravitation radius that depends on the sphere mass m and the classical gravitation constant G. Note that Equations (10) and (11) correspond to the Riemannian space which is Euclidean with respect to the space coordinates r,θ,φ and Riemannian only with respect to the time coordinate tc.

The obtained solution is not confirmed with experiment. The angle of deviation of the light beam from the straight line in the vicinity of Sun calculated for the space with the metric coefficients in Equations (11) turns out to be only one half of the measured value. A more general GR theory is based on equation

$$E_i^j = \chi T_i^j \tag{13}$$

according to which the energy-momentum tensor in Equations (8) is proportional to the Einstein tensor

$$E_i^j = R_i^j - \frac{1}{2} g_i^j R = \chi T_i^j \tag{14}$$

in which R_i^j are the components of the Ricci curvature tensor depending on the metric tensor of the four-dimensional Riemannian space. The coefficient

$$\chi = 8\pi G / c^4 \tag{15}$$

is the relativity gravitation constant.

Thus, in accordance to Equations (13) and (14), gravitation is associated with the curvature of the Riemannian space. However, as follows from the foregoing derivation, only one component of the energy-momentum tensor, namely T_4^4, allows for gravitation. The rest components in Equations (8) include mechanical stresses and velocities which induce the curvature of the Riemannian space as well. So, we can conclude that according to GR the space can be Euclidean only in the absence of gravitation, stresses and motion. Since this is not the case for the real continuum, the corresponding space is Riemannian and the Euclidean space does not exist. It should be noted that three-dimensional Riemannian space can exist in the Euclidean space with six dimensions [12], so the actual space is Euclidean but six-dimensional. This strange result follows directly from the GR theory. Fortunately, there is a more simple and realistic model of space. Recall that in the Newton theory Equations (10) and (11) correspond to the space which is Euclidean with respect to space coordinates and Riemannian with respect to time. Thus, introduce, in general, the special Riemannian space which is Euclidean with respect to space coordinates x^1, x^2, x^3 and Riemannian with respect to the time coordinate $x^4 = ct$ only. In the absence of gravitation, the space is three-dimensional and Euclidean, whereas the continuum is described by the classical theories of deformable solids, fluids and gasses. Gravitation results in a four-dimensional space with space coordinates supplemented with time whereas the continuum is described by GR theory Equations (2), (13) and (14) in which six components of the metric tensor, namely $g_{ij}\left(i, j = 1,2,3\right)$, correspond to the Euclidean space and are known. The rest four metric coefficients, namely $g_{i4}\left(i = 1,2,3,4\right)$, are found from equations, following from Equations (13) and (14). In general, there are ten such equations which describe the gravitation in vacuum. However, since the tensor E_i^j is proportional to tensor T_i^j and hence satisfies four Equations (2), only six of these ten equations are mutually independent. The traditional set of GR equations is not complete [11] [13] [14] [15] and should be supplemented with the so-called coordinate conditions the general form of which is not known. For the proposed special Riemannian space, the set of independent GR equations consists of six equations and includes four unknown metric coefficients. In the theory of partial differential equations, the case in which the number of equations exceeds the number of unknown functions is less critical than the case in which the number of unknown functions exceeds the number of equations, because some equations can be used to determine the integration functions which enter the solutions of the other equations. Moreover, it can occur that some of equations are satisfied identically. This is the case for a spherically symmetric problem considered further for which the set of GR equations is complete. For this problem, the line element corresponding to the proposed special Riemannian space is [16]

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 + 2g_{14}c\mathrm{d}r\mathrm{d}t - g_{44}c^2\mathrm{d}t^2 \tag{16}$$

As can be seen, in contrast to the traditional Equation (19), there are two unknown metric coefficients and the space is not “orthogonal” to time.

3. The Schwarzchild Solution

The first solution of the GR problem for a sphere consisting of a perfect incompressible fluid was obtained by K. Schwarzcchild. The set of the Einstein equations, Equations (13), (14), for a spherically symmetric static problem with the line element in Equation (10) has the following form [17]:

$$E_1^1 = \frac{1}{g_{22}} - \frac{1}{g_{11}}\left[\frac{1}{4}\left(\frac{g'_{22}}{g_{22}}\right)^2 + \frac{g'_{22}g'_{44}}{2g_{22}g_{44}}\right] = \chi T_1^1 \tag{17}$$

$$E_2^2 = -\frac{1}{2g_{11}}\left[\frac{g''_{44}}{g_{44}} - \frac{1}{2}\left(\frac{g'_{44}}{g_{44}}\right)^2 + \frac{g''_{22}}{g_{22}} - \frac{1}{2}\left(\frac{g'_{22}}{g_{22}}\right)^2 + \frac{g'_{22}}{2g_{22}}\left(\frac{g'_{44}}{g_{44}} - \frac{g'_{11}}{g_{11}}\right) - \frac{g'_{11}g'_{44}}{2g_{11}g_{44}}\right] = \chi T_2^2 \tag{18}$$

$$E_4^4 = \frac{1}{g_{22}} - \frac{1}{g_{11}}\left[\frac{g''_{22}}{g_{22}} - \frac{1}{4}\left(\frac{g'_{22}}{g_{22}}\right)^2 - \frac{g'_{11}g'_{22}}{2g_{11}g_{22}}\right] = \chi T_4^4 \tag{19}$$

where $(\cdot)' = \mathrm{d}(\cdot)/\mathrm{d}r$. For the sphere consisting of a perfect incompressible fluid, $\sigma_r = \sigma_\theta = -p$, where p is the pressure and the density is constant, *i.e.* $\mu = \mu_0$. For a static problem, $v_1 = v^1 = 0$ and Equations (8) yield

$$T_1^1 = -p,\ T_2^2 = -p,\ T_4^4 = \mu_0 c^2,\ T_1^4 = T_4^1 = 0$$

Consider the empty space surrounding the sphere with radius *R*. For this space, $p = 0$, $\mu_0 = 0$ and Equations (17)-(19) are homogeneous. As can be seen, Equations (19) and (17) allow us to express g_{11} and g_{44} in terms of g_{22}. The general solution of these equations is [18]

$$g_{11} = \frac{\left(g_{22}'\right)^2}{4\left(g_{22} + C_1\sqrt{g_{22}}\right)},\ g_{44} = C_2\left(1 + \frac{C_1}{\sqrt{g_{22}}}\right) \tag{20}$$

in which C_1 and C_2 are the integration constants. Substituting Equations (20) in Equation (18), we can conclude that this equation is satisfied identically with any function $g_{22}(r)$. This result looks natural because the set of the GR equations is not complete. The Schwarzchild solution was obtained under the additional coordinate condition $g_{22} = r^2$. Originally, K. Schwarzchild used a different condition which can reduced to the written above [19]. The integration constants can be found from the asymptotic conditions according to which Equations (20) should reduce to Equations (11) corresponding to the Newton theory for $r \to \infty$. Taking $g_{22} = r^2$, we finally get

$$g_{11}^e = \frac{1}{1 - r_g/r},\ g_{44}^e = 1 - \frac{r_g}{r} \tag{21}$$

Here r_g is specified by Equation (12) and index "*e*" corresponds to the external space. The solution in Equations (21) formally has two singular points—$r = r_g$ and $r = 0$. The second singularity does not appear because Equations (21) are valid for $r \ge R$ in which R is the radius of a fluid sphere. The first singularity takes place at the minimum possible value of r which is $r = R$. Thus, for the problem under study, the so-called Schwarzchild singularity appears on the sphere surface. Naturally, r_g cannot be referred to as the radius of the horizon of events because the penetration through this surface is physically not possible.

Consider the internal problem for $0 \le r \le R$. For $g_{22} = r^2$ and $T_4^4 = \mu_0 c^2$, integration of Equation (19) yields

$$g_{11}^i = \frac{1}{1 - \left(\chi\mu_0 c^2/3\right)r^2 + C_3/r} \tag{22}$$

in which index "*i*" corresponds to the internal space and C_3 is the integration constant. We should take $C_3 = 0$, otherwise, g_{11}^i becomes singular at the center of the sphere of any radius. Thus,

$$g_{11} = \frac{1}{1 - \left(\chi\mu_0 c^2/3\right)r^2} \tag{23}$$

As can be seen this expression does not include an integration constant. This is associated with the coordinate condition $g_{22} = r^2$. Indeed, substitution of this condition in Equation (19) which has initially the second order reduces it to the equation of the first order whose solution (22) includes only one constant. As a result, we arrive at the problem associated with the boundary condition on the sphere surface according to which

$$g_{11}^i(R) = g_{11}^e(R) \tag{24}$$

Substituting Equations (21) and (23) and using Equations (12) for r_g and (15) for χ, we arrive at the following expression for the sphere mass:

$$m = \frac{4}{3}\pi\mu_0 R^3 \tag{25}$$

which corresponds to the Euclidean space. However, the space inside the sphere is Riemannian and the mass can be found using Equation (23) as

$$m = 4\pi\mu_0 \int_0^R \sqrt{g_{11}^i} r^2 \mathrm{d}r \approx \frac{4}{3}\pi\mu_0 R^3 \left[1 + \frac{3r_g}{10R} + \frac{9}{56}\left(\frac{r_g}{R}\right)^2 + \cdots\right] \tag{26}$$

This result coincides with Equation (25) if $r_g = 0$ which means the absence of gravitation. Naturally, the actual mass is specified by Equation (26) and the boundary condition (24) is not satisfied in the Schwarzchild solution.

To proceed, transform Equation (23) Equations (12), (15) and (25) for r_g, χ and m which yield

$$\frac{1}{3}\chi\mu_0 c^2 = \frac{r_g}{R^3} \tag{27}$$

Then, Equation (23) becomes

$$g_{11}^i = \frac{1}{1 - r_g r^2 / R^3} \tag{28}$$

Consider Equation (2) which for the problem under study has the following form:

$$\frac{\mathrm{d}T_1^1}{\mathrm{d}r} + \frac{g'_{22}}{g_{22}}\left(T_1^1 - T_2^2\right) + \frac{g'_{44}}{2g_{44}}\left(T_1^1 - T_4^4\right) = 0$$

Substituting Equations (20) for the energy-momentum tensor, we get

$$p' + \frac{g'_{44}}{2g_{44}}\left(p + \mu_0 c^2\right) = 0 \tag{29}$$

Taking $T_1^1 = -p$ in Equation (17) and substituting g_{11}^i from Equation (28), we can find

$$\frac{1}{g_{44}^i}\frac{\mathrm{d}g_{44}^i}{\mathrm{d}\overline{r}} = \overline{r}_g \overline{r} \frac{3\overline{p} + 1}{1 - \overline{r}_g \overline{r}^2} \tag{30}$$

where

$$\overline{r} = \frac{r}{R}, \quad \overline{r}_g = \frac{r_g}{R}, \quad \overline{p} = \frac{p}{\mu_0 c^2} \tag{31}$$

Equations (29) and (30) yield the final equation for the normalized pressure

$$\frac{\mathrm{d}\overline{p}}{\mathrm{d}\overline{r}}+\frac{1}{2}\overline{r}_g\overline{r}\frac{3\overline{p}+1}{1-\overline{r}_g\overline{r}^2}\left(\overline{p}+1\right)=0$$

The solution of this equation which satisfies the boundary condition $\overline{p}\left(\overline{r}=1\right)=0$ is [17]

$$\overline{p}=\frac{\sqrt{1-\overline{r}_g\overline{r}^2}-\sqrt{1-\overline{r}_g}}{3\sqrt{1-\overline{r}_g}-\sqrt{1-\overline{r}_g\overline{r}^2}} \tag{32}$$

Using this result and integrating Equation (30) under boundary condition $g_{44}^i\left(\overline{r}=1\right)=g_{44}^e\left(\overline{r}=1\right)$, we can obtain the following expression for the time metric coefficient [17]:

$$g_{44}^i=\frac{1}{4}\left(3\sqrt{1-\overline{r}_g}-\sqrt{1-\overline{r}_g\overline{r}^2}\right)^2 \tag{33}$$

Substituting the obtained solution, Equations (28), (32) and (33), in the remaining field equation, Equation (18), we can prove that this equation is satisfied identically. Thus, Equations (21), (28), (32) and (33) specify the Schwarzchild solution of the spherically symmetric problem for a sphere consisting of a perfect incompressible fluid. To analyze this solution, calculate the pressure at the sphere center. Taking $r=0$ in Equation (32), we get

$$\overline{p}_0=\frac{1-\sqrt{1-\overline{r}_g}}{3\sqrt{1-\overline{r}_g}-1} \tag{34}$$

The denominator of this expression is zero at $\overline{r}_g^s=8/9$. This means that the pressure $\overline{p}_0$ becomes infinitely high for the sphere with radius $R_s=9/8\,r_g=1.125r_g$. It is natural to suppose that the Schwarzchild solution is not valid for the spheres whose radius is less than R_s. Indeed, for $R=1.11r_g$ ($\overline{r}_g=0.9$) Equation (33) gives negative pressure ($\overline{p}_0=-11.35$) which has no physical sense. Thus, the Schwarchild singularity following from Equations (21) and taking place at $R=r_g$ is not reached in the Schwarzchild solution for a fluid sphere.

4. GR Theory Equations for the Special Riemannian Space

Consider spherically symmetric problem in the Riemannian space with the line element in Equation (16). The Einstein equations, Equations (13) and (14), have the following form:

$$E_1^1=\frac{1}{r^2g^2}\left[g\left(g_{14}^2-rg_{44}'\right)+rg_{14}g_{44}\frac{\partial}{c\partial t}\ln\frac{g_{44}}{g_{14}^2}\right]=\chi T_1^1 \tag{35}$$

$$\begin{aligned}E_2^2=&\frac{1}{4rg^2}\left(4g_{14}g_{44}g_{14}'-4g_{14}^2g_{44}'-2g_{44}g_{44}'-2rg_{44}g_{44}''+rg_{44}'^{\,2}-2rg_{14}^2g_{44}''\right.\\&\left.+2rg_{14}g_{14}'g_{44}'-4g_{44}\dot{g}_{14}+2g_{14}\dot{g}_{44}+4rg_{14}g_{14}'\dot{g}_{14}+2rg_{14}'\dot{g}_{44}-4rg\dot{g}_{14}'\right)\\&=\chi T_2^2\end{aligned} \tag{36}$$

$$E_4^4=\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{rg_{14}^2}{g}\right)=\chi T_4^4 \tag{37}$$

$$E_1^4 = \frac{g_{14}}{r}\frac{\partial}{\partial r}\left(\frac{1}{g}\right) = \chi T_1^4 \tag{38}$$

$$E_4^1 = \frac{g_{14}^2 g_{44}}{rg^2}\frac{\partial}{c\partial t}\ln\frac{g_{44}}{g_{14}^2} = \chi T_4^1, \quad g = g_{44} + g_{14}^2 \tag{39}$$

Here, $(\cdot)' = \partial(\cdot)/\partial r$ and $(\dot{\cdot}) = \partial/c\partial t$. The energy-momentum tensor satisfies conservation Equations (2) whose explicit form is

$$\left(T_1^1\right)' + \frac{2}{r}\left(T_1^1 - T_2^2\right) + \frac{g'_{44}}{2g}\left(T_1^1 - T_4^4\right) + \frac{g_{14}g'_{44}}{2g}T_1^4 + \frac{g'_{14}}{g}T_4^1 + \dot{T}_1^4 + \frac{\dot{g}}{2g}T_1^4 = 0 \tag{40}$$

$$\begin{gathered} rg'_{44}\left[g_{14}\left(T_1^1 - T_4^4\right) - g_{44}T_1^4\right] + 2g\left[r\dot{T}_4^4 + r\left(T_4^1\right)' + 2T_4^1\right] \\ + 2rg_{14}g'_{14}T_4^1 - 2rg_{44}\dot{g}_{14}T_1^4 + rg_{14}\dot{g}_{44}T_1^4 = 0 \end{gathered} \tag{41}$$

The energy-momentum tensor is specified by Equations (9) which need some additional comments. The GR problem cannot be strictly static even if there is no motion, because Equations (9) include the velocity of light. If the radial and the time coordinate axes are orthogonal as in Equation (10) used in the previous Section, the velocity of light directed along the time axis does give the projection on the radial axis and for a static problem $v_1 = v^1 = 0$. However, for an oblique coordinate frame the situation is different [20]. The velocity of light gives zero contravariant projection on the radial axis, whereas the covariant projection is not zero. Thus, we must take $v^1 = 0$ and $v_1 \neq 0$ in Equations (9) which take the following form for a sphere consisting of a perfect fluid:

$$T_1^1 = -p, \ T_2^2 = -p, \ T_4^4 = \mu c^2, \ T_4^1 = 0, \ T_1^4 = -\mu c v_1 \tag{42}$$

Simplify Equations (35)-(39). Since $T_4^1 = 0$, Equation (39) yields

$$g_{14}^2 = f(r)g_{44}$$

where $f(r)$ is an unknown function. It is natural to assume that for a static problem the functions in Equations (35)-(38) and (40)-(42) do not depend on time. Then, taking into account Equations (41), we can present Equations (35)-(38) and (40), (41) as

$$E_1^1 = \frac{1}{r^2 g}\left(g_{14}^2 - rg'_{44}\right) = -\chi p \tag{43}$$

$$E_2^2 = \frac{1}{4rg^2}\left[4g_{14}\left(g_{44}g'_{14} - g_{14}g'_{44}\right) - 2g_{44}\left(rg'_{44}\right)' + rg'^2_{44} - 2rg_{14}\left(g''_{44} - g'_{14}g'_{44}\right)\right] = -\chi p \tag{44}$$

$$E_4^4 = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{rg_{14}^2}{g}\right) = \chi\mu c^2 \tag{45}$$

$$E_1^4 = \frac{g_{14}}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{g}\right) = -\chi\mu v_1 c, \quad g = g_{44} + g_{14}^2 = g_{44}\left[1 + f(r)\right] \tag{46}$$

$$p' - \frac{g'_{44}}{2g}\left(p + \mu c^2\right) - \frac{g_{14}g'_{44}}{2g}\mu c v_1 = 0 \tag{47}$$

$$g_{14}\left(p+\mu c^{2}\right)+g_{44}\mu c v_{1}=0 \tag{48}$$

Since we have two Equations (47) and (48), only two of four Equations (43)-(46) are mutually independent. Thus, we have four equations for four functions g_{14}, g_{44}, p, v_1 and the system of GR equations is complete.

5. External Solution

Consider the empty space surrounding the sphere with radius *R*. In this space, $p=0$ and $\mu=0$, Equations (47) and (48) are satisfied identically, whereas (43)-(46) are homogeneous. Taking $\mu=0$ in Equations (45) and (46), we get

$$g_{14}^{2}=C_{1}\frac{g}{r},\quad g=C_{2},\quad g_{44}=g-g_{14}^{2}=C_{2}-\frac{C_{1}C_{2}}{r}$$

For $r\to\infty$, the obtained solution must reduce to the solution following from the Newton theory, *i.e.*,to $g_{14}=0$, $g_{44}=1-\left(r_{g}/r\right)$. Thus,

$$g_{14}^{e}=\pm\sqrt{\frac{r_{g}}{r}},\quad g_{44}^{e}=1-\frac{r_{g}}{r} \tag{49}$$

The obtained solution is not singular and specifies two equivalent spaces that correspond to the positive and the negative values of g_{14}^{e}. Recall that solution (49) is found from Equations (45) and (46). Direct substitution in the rest homogeneous Equations (43) and (44) satisfies these equations identically. Equations (49) correspond to the so-called Gullstand-Painlever coordinates [21] [22] found as a result of coordinate transformation of the Schwarzchild solution. Here, the metric coefficients in Equations (49) are not associated with the Schwarzchild solution and follow from the proposed model of the Riemannian space.

6. Internal Solution for a Sphere Consisting of a Perfect Incompressible Fluid

Consider an internal space of a fluid sphere for which $\mu=\mu_{0}$. Integration in Equation (45) yields

$$g_{14}^{2}=g\left(\frac{1}{3}\chi\mu_{0}c^{2}r^{2}+\frac{C_{3}}{r}\right) \tag{50}$$

in which $C_{3}=0$. We can transform this result with aid of Equation (27). Note that in contrast to the Schwarzchild solution considered in Section 2, for the problem under study this equation is exact, because the space in coordinates r,θ,φ is Euclidean and the sphere mass is specified by Equation (25). Thus,

$$g_{14}^{2}=\frac{g}{R^{3}}r_{g}r^{2}$$

Applying Equation (42) and Equation (46) for *g*, we get

$$f=\frac{r_{g}r^{2}}{R^{3}-r_{g}r^{2}} \tag{51}$$

Consider Equation (48) and express

$$v_1 = -\frac{g_{14}}{g_{44}\mu_0 c}\left(p + \mu_0 c^2\right) \tag{52}$$

Substituting this result in equation (47), we arrive at the following equation:

$$p' + \frac{g'_{44}}{2g_{44}}\left(p + \mu_0 c^2\right) = 0 \tag{53}$$

which is the same as Equation (29) in the Schwarzchild solution. Transforming Equation (43) with the aid of Equations (27), (42) and (46), we find

$$\frac{g'_{44}}{g_{44}} = \frac{r_g r}{R^3 - r_g r^2}\left(\frac{3p}{\mu_0 c^2} + 1\right) \tag{54}$$

This equation is also the same as Equation (30) in the Schwarzchild solution. Since Equations (53) and (55) coincide with Equations (29) and (30), and the boundary conditions are the same in the Schwarzchild solution, the pressure is specified by Equation (32) and the time metric coefficient g_{44} by Equation (33). As in the Schwarzchild solution, the pressure at the sphere center becomes infinitely high for the sphere with radius $R_s = 9/8 r_g$. For spheres whose radii are less than R_s, the solution does not exist. It should be noted that, in contrast to the Schwarzchild solution, the obtained external solution is not singular for $R = r_g$.

The metric coefficient g_{14} can be found from Equations (42) and (51), *i.e.*

$$g_{14} = \pm\frac{1}{2}\left(3\sqrt{1-\overline{r}_g} - \sqrt{1-\overline{r}_g\overline{r}^2}\right)\frac{\overline{r}\sqrt{\overline{r}_g}}{\sqrt{1-\overline{r}_g\overline{r}^2}} \tag{55}$$

Taking $\overline{r} = 1$, we get $g_{14} = \pm\sqrt{\overline{r}_g}$ and the boundary condition on the sphere surface is satisfied. To complete the analysis, determine the velocity v_1. Substituting the obtained results in Equation (52), we arrive at

$$v_1 = \mp\frac{4c\overline{r}\sqrt{\overline{r}_g\left(1-\overline{r}_g\right)}}{\sqrt{1-\overline{r}_g\overline{r}^2}\left(3\sqrt{1-\overline{r}_g} - \sqrt{1-\overline{r}_g\overline{r}^2}\right)^2} \tag{56}$$

The maximum absolute value takes place on the sphere surface, *i.e.*,

$$v_1^m = \frac{c\sqrt{\overline{r}_g}}{1-\overline{r}_g}$$

Formally, $v_1^m > c$ if $\overline{r}_g > 0.382$. The same result was obtained for a solid elastic sphere [20]. However, as follows from Equation (56), any point of a fluid has two equal velocities with different signs and we can suppose that v_1 is not associated with any real motion. The obtained solution identically satisfies the field equations, Equations (43)-(48).

7. Internal Solution for a Sphere consisting of a Perfect Compressible Fluid or Gas

The model of a perfect incompressible fluid considered in Sections 2 and 5 is not compatible with GR, because the velocity of sound in this model is infinitely high, whereas in the GR theory it cannot be higher than *c*. To avoid this shortcoming, consider the case of a compressible fluid or gas and assume that the density is a linear function of the pressure, *i.e.*,

$$\mu = \mu_0 (1 + kp) \tag{57}$$

in which k is a given coefficient. Recall that only four of six field Equations (43)-(48) are mutually independent. To study the problem, *i.e.*, to determine g_{14}, g_{44}, v_1 and *p*, we use Equations (43), (45), (47) and (48). In addition to Equations (31), introduce the following normalized parameters: $\bar{\mu} = \mu / \mu_0$ and $\bar{k} = k\mu_0 c^2$. Then, Equation (57) can be presented as

$$\bar{\mu} = 1 + \bar{k}\bar{p} \tag{58}$$

Consider Equations (12), (25) and (20) for r_g, m and χ. The first of them, *i.e.* $r_g = 2mG/c^2$, is valid for a compressible sphere material, because the GR Equations (2) provide the conservation of mass which has the form

$$m = 4\pi \int_0^R \mu r^2 \mathrm{d}r = 4\pi\mu_0 \int_0^R (1 + kp) r^2 \mathrm{d}r = 4\pi\mu_0 R^3 \int_0^1 (1 + \bar{k}\bar{p}) \bar{r}^2 \mathrm{d}\bar{r} \tag{59}$$

Introduce a formal parameter $r_g^0 = 2m_o G/c^2$ which is the gravitation radius of the homogeneous sphere with radius *R*, density μ_0 and mass $m_0 = (4/3)\pi\mu_0 R^3$. Then, Equation (27) yields

$$\chi = \frac{3\bar{r}_g^0}{\mu_0 c^2 R^2} \tag{60}$$

in which $\bar{r}_g^0 = r_g^0 / R$.

The first equation of the governing set follows from Equation (43) which can be presented with the aid of Equation (60) as

$$\bar{r}\frac{\mathrm{d}g_{44}}{\mathrm{d}\bar{r}} - g_{14}^2 = 3\bar{r}_g^0 g\bar{p}\bar{r}^2 \tag{61}$$

Taking into account Equations (58), (61) and using Equation (45), we can obtain the second equation of the governing set

$$\frac{\mathrm{d}}{\mathrm{d}\bar{r}}\left(\frac{\bar{r}g_{14}^2}{g}\right) = 3\bar{r}_g^0 \bar{r}^2 (1 + \bar{k}\bar{p}), \quad g = g_{44} + g_{14}^2 \tag{62}$$

Equation (48) can be used to determine v_1, *i.e.*,

$$v_1 = -\frac{cg_{14}}{g_{44}} \frac{2 + \bar{k}\bar{p}}{1 + \bar{k}\bar{p}}$$

Substituting this result in Equation (47), we arrive at the third equation of the governing set

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}\bar{r}} - \frac{1}{2g_{44}} \frac{\mathrm{d}g_{44}}{\mathrm{d}\bar{r}} (1 + \bar{p} + \bar{k}\bar{p}) = 0 \tag{63}$$

Thus, we arrived at three Equations (62), (63) and (64) for g_{14}, g_{44} and $\bar{p}$.

In these equations, $\bar{r} = r/R$ and $0 \le \bar{r} \le 1$. Note that we do not know the sphere radius R which depends on pressure. To obtain the solution, we need three boundary conditions according to which

$$g_{14}(\bar{r}=0)=0,\quad \bar{p}(\bar{r}=1)=0,\quad g_{44}(\bar{r}=1)=1-\frac{r_g}{R}=1-\bar{r}_g \tag{64}$$

Since we do not know $\bar{r}_g = r_g/R$, we apply the following iteration procedure. For the preliminary selected parameter $\bar{r}_g^0$, we take $\bar{r}_g = \bar{r}_g^0$ and undertake numerical integration of the equations. Substituting the obtained solution for $\bar{p}(\bar{r})$ in Equation (59), we find

$$\bar{r}_g = \bar{r}_g^0 \frac{m}{m_0} = 3\bar{r}_g^0 \int_0^1 \left(1+\bar{k}\bar{p}\right)\bar{r}^2 \mathrm{d}\bar{r} = \bar{r}_g^0 \left(1 + 3\bar{k}\int_0^1 \bar{p}\bar{r}^2 \mathrm{d}\bar{r}\right) = \lambda \bar{r}_g^0 \tag{65}$$

The integration is repeated with this value of $\bar{r}_g$ and so on. The procedure is terminated when the fourth boundary condition following from the solution for the external space in Equations, *i.e.*,

$$g_{14}(\bar{r}=1)=\sqrt{\bar{r}_g} \tag{66}$$

is satisfied with the given accuracy.

To demonstrate the procedure, take $\bar{k}=1$ and $\bar{r}_g^0 = 0.5$. Then, for the first iteration step, we have initially $g_{14}^0(1)=0.707$. Integration yields $\lambda = 1.190385$ and $\bar{r}_g = 0.59519$. We find $g_{14}(1) = 0.857 \ne g_{14}^0$ and the boundary condition in Equation (66) is not satisfied. For the second step, we take $\bar{r}_g = 0.59519$ in Equations (65), so that $g_{14}^1 = 0.771487$. The result of integration is $g_{14} = 0.771488$ and the boundary condition in Equation (66) is practically satisfied. Thus, for $\bar{r}_g^0 = 0.5$ we arrive at $\bar{r}_g = 0.59519$ and we need only two iteration steps to obtain the solution with the reasonable accuracy.

For $\bar{k}=0$, we have the analytical solution presented in Section 4. According to this solution, the pressure at the sphere center becomes infinitely high for the sphere with the normalized gravitation radius $\bar{r}_g = 8/9 = 0.888(8)$. Numerical integration for $\bar{k} = 10^{-14}$ allows us to conclude that the numerical procedure does not converge if $\bar{r}_g \ge 0.885$. Thus, we can associate the absence of the process convergence with the pressure singularity at the sphere center. Assume a relatively small value, e.g. $\bar{k} = 0.0001$. The maximum value of $\bar{r}_g^0$ for which the process converges is 0.884 which corresponds to $\bar{r}_g = 0.8842$. The results of the parametric analysis are presented in **Table 1**.

As can be seen, the higher is the fluid or gas compressibility, the lower is the maximum gravitation radius. In general, the maximum gravitation radius depends on the model of the sphere material. For example, for a solid elastic sphere, it is $\bar{r}_g = 1$ [20].

The dependence of the normalized pressure on the radial coordinate is shown in **Figure 1**. Solid lines correspond to the compressible fluid or gas, whereas dotted lines demonstrate the analytical solution for an incompressible fluid (the numbers on curves correspond to **Table 1**).

Table 1. The maximum values of $\bar{r}_g^0$ and $\bar{r}_g$ for various values of parameter $\bar{k}$.

$\bar{k}$	0.0001	0.001	0.01	0.1	0.5	1.0	1.5	2.0
$\bar{r}_g^0$	0.8840	0.8820	0.8650	0.7710	0.6110	0.5120	0.4470	0.3990
$\bar{r}_g$	0.8842	0.8836	0.8803	0.8276	0.7197	0.6492	0.5877	0.5359

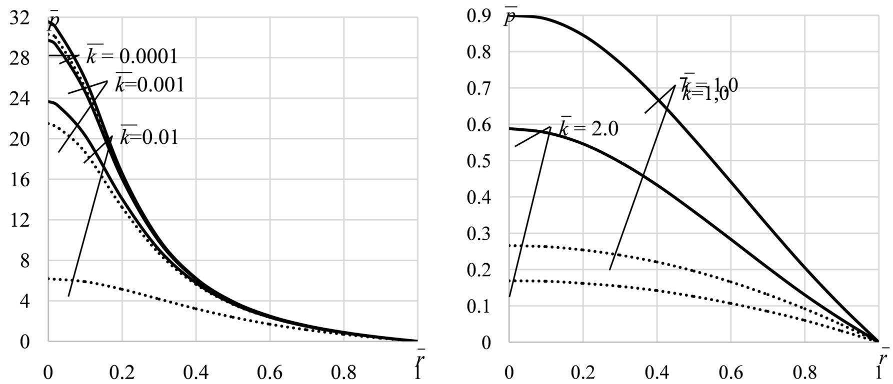

Figure 1. Dependences of the normalized pressure on the radial coordinate for various values of parameter $\bar{k}$, —: compressible fluid or gas, ···: incompressible fluid.

8. Conclusion

The proposed model of the Riemannian space according to which the space is Euclidean with respect to space coordinates of the continuum and is Riemannian with respect to the time coordinate only is applied to a spherically symmetric GR problem for a perfect fluid or gas sphere. The solution to the external problem results in the so-called Gullstand-Painlever metrics. The solution of the internal problem for an incompressible fluid gives the pressure which coincides with the Schwarzchild solution for the same problem. The solution for the fluid or gas whose density is a linear function of pressure is obtained numerically and compared to the Schwarzchild solution for an incompressible fluid.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Schwarzchild, K. (1016) *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, **1916**, 424-432.

[2] Thorne, K.S. (1994) Black Holes and Time Wraps. W.W. Norton, New York.

[3] Einstein, A. (1939) *Annals of Mathematics*, **40**, 922-936. https://doi.org/10.2307/1968902

[4] Oppenheimer, J.R. and Volkoff, G.M. (1939) *Physical Review*, **55**, 374-389. https://doi.org/10.1103/PhysRev.55.374

[5] Tolman, R.C. (1939) *Physical Review*, **55**, 364-373. https://doi.org/10.1103/PhysRev.55.364

[6] Buchdahl, H.A. (1959) *Physical Review*, **116**, 1027-1034. https://doi.org/10.1103/PhysRev.116.1027

[7] Adler, R.J. (1974) *Journal of Mathematical Physics*, **15**, 727-729. https://doi.org/10.1063/1.1666717

[8] Fodor, G. (2000) Generating Spherically Symmetric Static Perfect Fluid Solutions. http://www.arxiv.org/abs/gr-qc/0011040

[9] Vasiliev, V.V. and Fedorov, L.V. (2014) *Applied Physics Research*, **6**, 40-49. https://doi.org/10.5539/apr.v6n3p40

[10] Kilchevskii, N.A. (1972) Foundations of Tensor Calculus with Applications to Mechanics. Naukova Dumka, Kiev. (In Russian)

[11] Landau, L.D. and Lifshitz, E.M. (1971) The Classical Theory of Fields. Pergamon Press, London.

[12] Rashevskii, P.K. (1967) Riemannian Geometry and Tensor Analysis. Nauka, Moscow. (In Russian)

[13] Misner, C.W., Thorne, K.S. and Wheeler, J.A. (1973) Gravitation. W.H. Freeman and Co., San Francisco.

[14] Weinberg, S. (1972) Gravitation and Cosmology. John Wiley and Sons, Inc., New York.

[15] Hawking, S.W. and Ellis, G.F.R. (1977) The Large Scale Structure of Space-Time. Cambridge Univ. Press, Cambridge.

[16] Vasiliev, V.V. and Fedorov, L.V. (2023) *Journal of Modern Physics*, **14**, 147-159. https://doi.org/10.4236/jmp.2023.142010

[17] Sing, J.L. (1960) Relativity: The General Theory. North Holland, Amsterdam.

[18] Vasiliev, V.V. (2017) *Journal of Modern Physics*, **8**, 1087-1100. https://doi.org/10.4236/jmp.2017.87070

[19] Vasiliev, V.V. and Fedorov, L.V. (2018) *Journal of Modern Physics*, **9**, 2482-2494. https://doi.org/10.4236/jmp.2018.914160

[20] Vasiliev, V.V. and Fedorov, L.V. (2023) *Journal of Modern Physics*, **14**, 818-832. https://doi.org/10.4236/jmp.2023.146047

[21] Painleve, P. (1921) *Comptes Rendus de l'Académie des Sciences* (*Paris*), **173**, 677-680.

[22] Gullstrand, A. (1922) *Arkiv for Matematik, Astronomioch Fysik*, **16**, 1-15.