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The Yukawa and Exponential Potentials for a Galaxy's Spherical Central Bulge*

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Abstract

By adding an extra term to the Newtonian potential, matter outside the orbit of a star adds to the gravitational acceleration acting on that star. In this work, we solve the Poisson equation for non-Newtonian potentials of a spherically symmetric distribution of mass. We derive equations for calculating the centripetal acceleration and velocity of galactic disk stars that are due to the Newtonian and exponential potentials of the galaxy's central bulge.

Keywords

Galaxies: Kinematics and Dynamics, Gravitation, General Physics (Physics Education)

1. Introduction

In Book III of Principia, Newton [1] published proofs of two theorems that afford an easy calculation of the gravitational potential of any spherically symmetric distribution of matter and to calculate excellent approximations of the potential for approximately spherical distributions of matter such as the central bulge of a spiral galaxy. Thus, since 1687 we have known that the Newtonian acceleration of an orbiting object depends only on the mass inside the orbit. Any uniform spherical shell of matter outside the orbit has no effect on the orbital dynamics. These useful Newtonian theorems are not valid for other gravitational potentials proposed over the past decade to explain anomalous galactic dynamics. For example, with a linear potential [2], the centripetal acceleration of a disk star is sensitive to the halo mass outside its orbit and, even more inconveniently, to the mass of the entire Universe.

*This article is dedicated to the fond memory of Don Eckhardt, who passed away in the morning of 21st October 2023. He was my friend and collaborator, and a true visionary.

The foci of our exposition are two other non-Newtonian gravitational models that are more tractable than the linear potential. The first is a Yukawa potential [3] which, for a point mass M at $r = 0$, has the form

$$V_{\mathrm{Y}}(r) = -\alpha G \mathrm{e}^{-\mu r} / r,$$

where G is the Newtonian gravitational constant, α is a dimensionless constant, and μ is the wave number ($2\pi/\mu$ is the Compton wavelength). The second is an exponential potential [4] [5] which is actually the superposition of two Yukawa potentials with coupling constants $\pm\alpha G$ that are identical except for their signs: an attractive potential, $V_{\mathrm{Y}}^{\mathrm{A}} = -\alpha G \mathrm{e}^{-\mu_{\mathrm{A}} r}$, and a repulsive potential, $V_{\mathrm{Y}}^{\mathrm{R}} = \alpha G \mathrm{e}^{-\mu_{\mathrm{R}} r}$. Their wave numbers are different, but almost identical: $\delta\mu = \mu_{\mathrm{R}} - \mu_{\mathrm{A}} > 0$ and $\mu = (\mu_{\mathrm{A}} + \mu_{\mathrm{R}})/2 \gg \delta\mu$. Then

$$V_{\mathrm{E}}(r) = (\partial V_{\mathrm{Y}} / \partial\mu)\delta\mu. \tag{1}$$

In this paper, we present the mathematical development that has allowed us to resolve for the exponential potential the same problem that Newton solved for the Newtonian potential. That is, given the mass distribution of a spherically symmetric central bulge and halo, then what are their contributions to the specific forces on disk matter that are due to the exponential potential? With this distribution, how much does the exponential potential contribute to the orbital velocity of disk matter? In fact, we provide a route to the potential, and then to the acceleration, that is often more convenient than the iteration by Eckhardt [5]. This model assumes that the central bulge of a galaxy is spherically symmetric, that the mass of the galaxy disk is negligible or is treated separately [6], and that there is an exponential potential as well as a Newtonian potential. To make it clear, the same approach is used to model the shell for the Newtonian potential, then the Yukawa potential, and finally the exponential potential. Although the solution for the Newtonian potential is trivial, it is presented as a guide for the approach for solving the other potentials.

2. Newtonian Potential

The governing equation for the [1] potential V_{N}, which is mediated by the massless graviton, is Poisson's equation:

$$\nabla^2 V_{\mathrm{N}} = 4\pi G\rho,$$

The potential of a point source with mass $\mathrm{d}M$ at a distance R from the source is $-G\mathrm{d}M/R$. (That is, the appropriate Green's function that vanishes at $R = \infty$ is $-G/R$.) Thus the potential at $r = 0$ of a shell at $r = a$ with mass

$$\mathrm{d}M = \mathrm{d}M(a) = 4\pi\rho(a)a^2\mathrm{d}a$$

is

$$\mathrm{d}V_{\mathrm{N}}(0) = -Ga^{-1}\mathrm{d}M.$$

Because of the spherical symmetry, Laplace's Equation ($\rho = 0$) is

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\frac{\mathrm{d}V_\mathrm{N}}{\mathrm{d}r}\right]=0.$$

The two solutions to this equation are of the form $constant$ and $constant\times r^{-1}$. The solution that remains finite at $r=0$ is $V_\mathrm{N}=constant$, so for $r\le a$, we have

$$\mathrm{d}V_\mathrm{N}(r)=\mathrm{d}V_\mathrm{N}(0)=-Ga^{-1}\mathrm{d}M,\quad r\le a. \tag{2}$$

The solution that remains finite at $r=\infty$ is $V_\mathrm{N}=constant\times r^{-1}$, so for $r\ge a$, we have

$$\mathrm{d}V_\mathrm{N}(r)=\frac{a}{r}\mathrm{d}V_\mathrm{N}(a)=\frac{a}{r}\mathrm{d}V_\mathrm{N}(0)=-Gr^{-1}\mathrm{d}M,\quad r\ge a. \tag{3}$$

3. Differential Equation for the Exponential Potential

We define the operators

$$\mathcal{D}=\frac{\mathrm{d}}{\mathrm{d}r}$$

and

$$\mathcal{L}=\mathcal{D}^2+\frac{2}{r}\mathcal{D}-\mu^2.$$

With spherical symmetry, the differential equations for the Yukawa potentials $V_\mathrm{Y}^\mathrm{A}(r)$ and $V_\mathrm{Y}^\mathrm{R}(r)$ are then

$$\left[\mathcal{L}+\left(\mu^2-\mu_\mathrm{A}^2\right)\right]V_\mathrm{Y}^\mathrm{A}=\left[\mathcal{L}+\mu\delta\mu\right]V_\mathrm{Y}^\mathrm{A}=4\pi\alpha G\rho$$

and

$$\left[\mathcal{L}+\left(\mu^2-\mu_\mathrm{R}^2\right)\right]V_\mathrm{Y}^\mathrm{R}=\left[\mathcal{L}-\mu\delta\mu\right]V_\mathrm{Y}^\mathrm{R}=-4\pi\alpha G\rho.$$

In addition to $V_\mathrm{E}=V_\mathrm{A}+V_\mathrm{R}$, we define $V_\mathrm{D}=V_\mathrm{A}-V_\mathrm{R}$. We then have

$$\mathcal{L}V_\mathrm{E}+\mu\delta\mu V_\mathrm{D}=0,$$

and

$$\mathcal{L}V_\mathrm{D}+\mu\delta\mu V_\mathrm{E}=8\pi\alpha G\rho.$$

It is obvious that $|V_\mathrm{D}|\gg|V_\mathrm{E}|$, so we neglect the $\mu\delta\mu V_\mathrm{E}$ term and then define

$$V_\mathrm{C}=\mu\delta\mu V_\mathrm{D},$$

and finally derive

$$\mathcal{L}V_\mathrm{E}=-V_\mathrm{C}, \tag{4}$$

and

$$\mathcal{L}V_\mathrm{C}=8\pi\gamma\mu^2G\rho. \tag{5}$$

The exponential potential satisfies the fourth order differential equation

$$\mathcal{L}^2V_\mathrm{E}=-8\pi\gamma\mu^2G\rho.$$

4. The Yukawa and Exponential Potentials of a Thin Spherical Shell

For a thin spherical shell of mass M at $r=a$, the Yukawa potential (coupling constant αG) at the origin is

$$V_{\mathrm{Y}}(0)=-\alpha GM\frac{\exp(-\mu a)}{a}.$$

In free space, the Yukawa potential satisfies the homogeneous equation $\mathcal{L}V_{\mathrm{Y}}=0$, whose solutions are linear combinations of $\exp(\pm\mu r)/r$. The only combination that is finite and equal to unity at the origin is $\sinh(\mu r)/(\mu r)$, so the Yukawa potential inside the shell is

$$V_{\mathrm{Y}}(r)=-\alpha GM\frac{\exp(-\mu a)}{a}\frac{\sinh(\mu r)}{\mu r},\quad r\le a. \tag{6}$$

The solution interval is closed at the shell because the potential (but not its derivative) is continuous there. Thus

$$V_{\mathrm{Y}}(a)=-\alpha GM\frac{\exp(-\mu a)\sinh(\mu a)}{\mu a^2}.$$

The homogeneous solution that is equal to unity at the shell boundary and remains finite outside the shell is $(a/r)\exp[-\mu(r-a)]$, so the Yukawa potential outside the shell is

$$V_{\mathrm{Y}}(r)=-\alpha GM\frac{\sinh(\mu a)}{\mu a}\frac{\exp(-\mu r)}{r},\quad r\ge a. \tag{7}$$

Note that Equation (6) becomes Equation (7) on interchanging r and a.

The derivative of $V_{\mathrm{Y}}(r)$ is discontinuous at $r=a$. Let δa be the shell thickness, and let $a_-=a-\delta a/2$ represent r just inside the shell, while $a_+=a+\delta a/2$ represents r just outside the shell. Then

$$\mathcal{D}V_{\mathrm{Y}}(r)\big|_{a_+}-\mathcal{D}V_{\mathrm{Y}}(r)\big|_{a_-}=4\pi\alpha G\rho\delta a.$$

For $\mu=0$, this is the Gauss divergence theorem.

Setting $\gamma=\alpha\delta\mu/\mu$ in Equation (1), the exponential potential is

$$V_{\mathrm{E}}(r)=-\gamma\mu GM\frac{\exp(-\mu a)}{\mu a}\left[(1+\mu a)\frac{\sinh(\mu r)}{\mu r}-\cosh(\mu r)\right],\quad r\le a, \tag{8}$$

and, on interchanging r and a and then rearranging terms,

$$V_{\mathrm{E}}(r)=-\gamma\mu GM\left[\frac{\sinh(\mu a)}{\mu a}\exp(-\mu r)+\left(\frac{\sinh(\mu a)}{\mu a}-\cosh(\mu a)\right)\frac{\exp(-\mu r)}{\mu r}\right],\quad r\ge a. \tag{9}$$

The operators $\mathcal{D}$ and $\partial/\partial\mu$ commute, so

$$\begin{aligned}\mathcal{D}V_{\mathrm{E}}(r)\big|_{a_+}-\mathcal{D}V_{\mathrm{E}}(r)\big|_{a_-}&=\frac{\partial}{\partial\mu}\left[\mathcal{D}V_{\mathrm{Y}}(r)\big|_{a_+}-\mathcal{D}V_{\mathrm{Y}}(r)\big|_{a_-}\right]\delta\mu\\&=\frac{\partial(4\pi\alpha G\rho\delta a)}{\partial\mu}\delta\mu=0.\end{aligned}$$

Thus, unlike $\mathcal{D}V_{\mathrm{Y}}(r)$, the derivative of $V_{\mathrm{E}}(r)$ is continuous at $r=a$. This is the reason why $V_{\mathrm{E}}(r)$ cannot be modeled with a second order differential equation. Note that in the far field, where $1/(\mu r)$ is negligible, Equation (9) becomes

$$V_{\mathrm{E}}(r)=-\gamma\mu GM\frac{\sinh(\mu a)}{\mu a}\exp(-\mu r);$$

and, for $\mu a \ll 1$,

$$V_{\mathrm{E}}(r)=-\gamma\mu GM\exp(-\mu r).$$

All the variables in Equations (8) and (9) are linear combinations of $\exp(\pm\mu r)$ and $\exp(\pm\mu r)/r$. Thus, because

$$\mathcal{L}\exp(\pm\mu r)=\left[\mathcal{D}^2-\mu^2\right]\exp(\pm\mu r)+\frac{2}{r}\mathcal{D}\exp(\pm\mu r)$$
$$=\frac{2}{r}\mathcal{D}\exp(\pm\mu r)=\pm2\mu\frac{\exp(\pm\mu r)}{r}$$

and

$$\mathcal{L}\left[\frac{\exp(\pm\mu r)}{r}\right]=0,$$

the equation

$$\mathcal{L}^2V_{\mathrm{E}}(r)=0$$

is satisfied wherever $\rho=0$.

5. Specific Forces and Galaxy Rotation Velocities

The centripetal specific force (acceleration) due to the Newtonian and exponential potentials of the spherically symmetric bulge is

$$g=\frac{\partial\left[V_{\mathrm{N}}+V_{\mathrm{E}}\right]}{\partial r},$$

where

$$V_{\mathrm{N}}+V_{\mathrm{E}}=\int_0^r\mathrm{d}\left[V_{\mathrm{N}}+V_{\mathrm{E}}\right]+\int_r^\infty\mathrm{d}\left[V_{\mathrm{N}}+V_{\mathrm{E}}\right].$$

From Equations (3) and (9),

$$\int_0^r\mathrm{d}\left[V_N+V_E\right]=-r^{-1}G\int_0^r\mathrm{d}M-\gamma\mu G\exp(-\mu r)\int_0^r\frac{\sinh(\mu a)}{\mu a}\mathrm{d}M$$
$$-\gamma\mu G\frac{\exp(-\mu r)}{\mu r}\int_0^r\frac{\sinh(\mu a)}{\mu a}\mathrm{d}M$$
$$+\gamma\mu G\frac{\exp(-\mu r)}{\mu r}\int_0^r\cosh(\mu a)\mathrm{d}M,$$

and from Equations (2) and (8),

$$\int_r^\infty\mathrm{d}\left[V_{\mathrm{N}}+V_{\mathrm{E}}\right]=-G\int_r^\infty a^{-1}\mathrm{d}M-\gamma\mu G\frac{\sinh(\mu r)}{\mu r}\int_r^\infty\frac{\exp(-\mu a)}{\mu a}\mathrm{d}M$$
$$-\gamma\mu G\frac{\sinh(\mu r)}{\mu r}\int_r^\infty\exp(-\mu a)\mathrm{d}M$$
$$+\gamma\mu G\cosh(\mu r)\int_r^\infty\frac{\exp(-\mu a)}{\mu a}\mathrm{d}M.$$

Thus

$$
\begin{aligned}
g = {} & G r^{-2}\int_0^r \mathrm{d}M + G\gamma\left(1+\mu r+\mu^2 r^2\right)\frac{\exp(-\mu r)}{r^2}\int_0^r \frac{\sinh(\mu a)}{\mu a}\mathrm{d}M \\
& - G\gamma(1+\mu r)\frac{\exp(-\mu r)}{r^2}\int_0^r \cosh(\mu a)\mathrm{d}M \\
& - G\frac{\gamma\left[\mu r\cosh(\mu r)-\sinh(\mu r)-\mu^2 r^2\sinh(\mu r)\right]}{r^2}\int_r^\infty \frac{\exp(-\mu a)}{\mu a}\mathrm{d}M \\
& - G\frac{\gamma\left[\mu r\cosh(\mu r)-\sinh(\mu r)\right]}{r^2}\int_r^\infty \exp(-\mu a)\mathrm{d}M.
\end{aligned}
$$

The sum of the two last terms (dominated by the last one), which is negative, represents a very small outward force due to the exponential potential of matter further from the origin than r; they vanish at the origin. In modeling spiral galaxies with a combination of Newtonian and exponential potentials [7], we have estimated that $\gamma = 12.5$ and $\lambda = \mu^{-1} = 20\ \text{kpc}$, roughly one order of magnitude larger than the radius of the central bulge, so a good approximation is

$$g = \left[r^{-2} + \gamma\mu^2\exp(-\mu r)\right]GM(r),$$

where

$$M(r) = \int_0^r \mathrm{d}M.$$

If a star is in a circular orbit a distance r from the origin with velocity v, then $v^2 = gr$. Setting $\xi = \mu r$, we then have

$$v = \left\{\mu GM(r)\left[\xi^{-1} + \gamma\xi\mathrm{e}^{-\xi}\right]\right\}^{1/2}.$$

6. Conclusions

Whereas the Poisson equation for a spherically symmetric Newtonian potential is of second order in r, the corresponding equation for the exponential potential is of fourth order. We have derived rigorous and precise mathematical solutions to calculate the exponential potential of a spherically symmetric distribution of mass such as a galactic bulge or halo. Moreover, we have derived a simple and efficient algorithm to closely approximate the associated radial gradients in galaxies. Using the Abel transform, all integrations of spherically symmetric models are relatively simple.

Combined with our technique for modeling the gravitational field of a galactic disk [6], we have found it easy to use in evaluating the effects of Newtonian and exponential potentials for numerous spiral galaxies [7]. Furthermore, it is evident that our approach will be convenient to apply at other distance scales, smaller and larger, such as globular clusters, elliptical galaxies and galaxy clusters.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Newton, I. (1687) Philosophiae Naturalis Principia Mathematica. Royal Society. https://doi.org/10.5479/sil.52126.39088015628399

[2] Mannheim, P.D. (1993) *ApJ*, **419**, 150-154. https://doi.org/10.1086/173468

[3] Yukawa, H. (1935) *Progress of Theoretical Physics*, **17**, 48.

[4] Fischbach, E., Talmadge, C. and Krause, D. (1991) *Physical Review D*, **43**, 460. https://doi.org/10.1103/PhysRevD.43.460

[5] Eckhardt, D.H. (1993) *Physical Review D*, **48**, 3762. https://doi.org/10.1103/PhysRevD.48.3762

[6] Eckhardt, D.H. and Pestaña, J.L.G. (2002) *ApJ*, **572**, L135. https://doi.org/10.1086/341745

[7] Pestaña, J.L.G. and Eckhardt, D.H. (2011) *ApJ*, **741**, L31. https://doi.org/10.1088/2041-8205/741/2/L31