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Nonlinear Conformal Gravitation

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Abstract

In 1909 the brothers E. and F. Cosserat discovered a new nonlinear group theoretical approach to elasticity (EL), with the only experimental need to measure the EL constants. In a modern framework, they used the nonlinear Spencer sequence instead of the nonlinear Janet sequence for the Lie groupoid defining the group of rigid motions of space. Following H. Weyl, our purpose is to compute for the first time the linear and nonlinear Spencer sequences for the Lie groupoid defining the conformal group of space-time in order to provide the mathematical foundations of both electromagnetism (EM) and gravitation (GR), with the only experimental need to measure the EM and GR constants. With a manifold of dimension $n \geq 3$, the difficulty is to deal with the n nonlinear transformations that have been called "elations" by E. Cartan in 1922. Using the fact that dimension $n = 4$ has very specific properties for the computation of the Spencer cohomology, we also prove that there is no conceptual difference between the (nonlinear) Cosserat EL field or induction equations and the (linear) Maxwell EM field or induction equations. As for gravitation, the dimension $n = 4$ also allows to have a conformal factor defined everywhere but at the central attractive mass because the inversion law of the isotropy subgroupoid made by second order jets transforms attraction into repulsion. The mathematical foundations of both electromagnetism and gravitation are thus only depending on the structure of the conformal pseudogroup of space-time.

Keywords

Nonlinear Differential Sequences, Linear Differential Sequences, Lie Groupoids, Lie Algebroids, Conformal Geometry, Spencer Cohomology, Maxwell Equations, Cosserat Equations

1. Introduction

The famous Special Relativity paper of A. Einstein published in 1905 contains two specific parts, the first one dealing with kinematics while the second is applying to electrodynamics the results obtained in the first part ([1]). With more details, the central result of the first part is sketched as follows in a footnote:

"When c is the speed of light, the Lorentz transformation $(x,y,z,t)\rightarrow(\xi,\eta,\zeta,\tau)$ can be deduced in a more direct way by supposing that the relation $\xi^2+\eta^2+\zeta^2-c^2\tau^2=0$ must bring $x^2+y^2+z^2-c^2t^2=0$".

However, the basic underlying assumption has been to suppose that the transformation was only depending on the relative speed v of the frames and to restrict the study to a linear group reducing to the Galilée group when the dimensionless number v/c was going to zero. As a byproduct, it must be noticed that people did believe that Einstein had not been influenced in 1905 by the Michelson and Morley experiment of 1887 till the rather recent discovery of hand written notes taken during lectures given by Einstein in Chicago (1921) and Kyoto (1922). As for other books on Special Relativity, each writer is similarly avoiding the use of the conformal group of space-time implied by the Michelson and Morley experiment, only caring about the Poincaré or Lorentz subgroups, sometimes claiming that the conformal factor could eventually depend on the local property of space-time, adding however that, if there was no surrounding electromagnetism or gravitation, the situation should be reduced to the preceding one but nothing was said otherwise.

Similarly, using standard notations of differential geometry for the exterior derivative on forms when $n=4$, the *second Maxwell operator* $\wedge^4T^*\otimes\wedge^2T\xrightarrow{ad(d)}\wedge^4T^*\otimes T:(\mathcal{F}^{ij})\rightarrow(\partial_i\mathcal{F}^{ij}=\mathcal{J}^j)$ is the adjoint of the parametrizing operator $T^*\xrightarrow{d}\wedge^2T^*:A\rightarrow dA=F$ in electromagnetism in such a way that F is killed by the *first Maxwell operator* $\wedge^2T^*\xrightarrow{d}\wedge^3T^*$, independently of the Minkowski constitutive relations $F\rightarrow\mathcal{F}$ between field and induction, that may depend on the Minkowksi metric $\omega\in S_2T^*$. *The two sets of Maxwell equations are thus separately invariant by any diffeomorphism*.

Though surprising it may look like, the conformal group of space-time is only the maximum group of invariance of the Minkowski constitutive law in vacuum. Indeed, this law is *not at all* $F_{ij}=\mu_0\omega_{ir}\omega_{js}\mathcal{F}^{rs}$ where μ_0 is the magnetic constant because such a relation is *not* tensorial as F is a 2-form, that is a 2-covariant tensor, but $\mathcal{F}$ is a 2-contravariant tensor *density*. Hence, introducing the *metric density* $\hat{\omega}_{ij}=\left(\left|\det(\omega)\right|\right)^{-1/n}\omega_{ij}$, we must set $F_{ij}=\mu_0\hat{\omega}_{ir}\hat{\omega}_{js}\mathcal{F}^{rs}$. Accordingly, this constitutive law is only invariant by diffeomorphisms preserving $\hat{\omega}$ and this is exactly the definition of the Lie pseudogroup of conformal transformations (see Section 3.1 for details).

With more details, if group theory must be used, the underlying group of transformations of space-time *must* be related to the propagation of light *by itself* rather than by considering tricky signals between observers, thus *must* have to do with the biggest group of invariance of Maxwell equations ([2] [3]). However, at the time we got the solution of this problem with the publication of ([4]) in 1988 (see [5] for recent results), a deep confusion was going on, which is still not acknowledged though it can be explained in a few lines (compare to [6]).

Using standard notations of differential geometry, if the 2-form $F \in \wedge^2 T^*$ describing the EM field is satisfying the *first set* of Maxwell equations, it amounts to say that it is closed, that is killed by the exterior derivative $d : \wedge^2 T^* \to \wedge^3 T^*$ as we said. The EM field can be thus (locally) parametrized by the EM potential 1-form $A \in T^*$ with $dA = F$ where $d : T^* \to \wedge^2 T^*$ is again the exterior derivative, because $d^2 = d \circ d = 0$. Now, if E is a vector bundle over a manifold X of dimension n, then we may define its *adjoint* vector bundle $ad(E) = \wedge^n T^* \otimes E^*$ where E^* is obtained from E by inverting the transition rules, like T^* is obtained from $T = T(X)$ and such a construction can be extended to linear partial differential operators between (sections of) vector bundles. When $n = 4$, it follows that the *second set* of Maxwell equations for the EM induction is just described by $ad(d) : \wedge^4 T^* \otimes \wedge^2 T \to \wedge^4 T^* \otimes T$, *independently of any Minkowski constitutive relation between field and induction*. Using *Hodge duality* with respect to the volume form $dx = dx^1 \wedge \cdots \wedge dx^4$, this operator is isomorphic to $d : \wedge^2 T^* \to \wedge^3 T^*$. It follows that *both the first set and second set of Maxwell equations are invariant by any diffeomorphism* and that the conformal group of space-time is the biggest group of transformations preserving the Minkowski constitutive relations *in vacuum* where the speed of light is truly c as a universal constant.

It is thus natural to believe that the mathematical structure of electromagnetism and gravitation have only to do with such a group having:

$$\boxed{4\ \text{translations} + 6\ \text{rotations} + 1\ \text{dilatation} + 4\ \text{elations} = 15\ \text{parameters}}$$

the main difficulty being to deal with these later non-linear 4 transformations. Of course, such a challenge could not be solved without the help of the non-linear theory of partial differential equations and Lie pseudogroups combined with homological algebra, that is before 1995 *at least* ([7] [8]).

From a purely physical point of view, these new nonlinear methods have been introduced for the first time in 1909 by the brothers E. and F. Cosserat for studying the mathematical foundations of elasticity theory ([9] [10] [11] [12] [13]). We have presented their link with the *nonlinear Spencer differential sequences* existing in the formal theory of Lie pseudogroups at the end of our book "Differential Galois Theory" published in 1983 ([14]). Similarly, the conformal methods have been introduced by H. Weyl in 1918 for revisiting the mathematical foundations of electromagnetism ([3]). We have presented their link with the above approach through a *unique* differential sequence only depending on the structure of the *conformal group* in our book "Lie Pseudogroups and Mechanics" published in 1988 ([4]). However, the Cosserat brothers were only dealing with *translations* and *rotations* while Weyl was only dealing with *dilatation* and *elations*. Also, as an additional condition not fulfilled by the classical Einstein-Fokker-Nordström theory ([15]), if the conformal factor has to do with gravitation, it *must* be defined everywhere but at the central attractive mass as we already said in the abstract.

Let G be a Lie group with coordinates $(a^\rho) = (a^1, \cdots, a^p)$ be a Lie group acting on X with a local action map $y = f(x,a)$. According to the *second fundamental theorem* of Lie, if $\theta_1,\cdots,\theta_p$ are the infinitesimal generators of the effective action of a lie group G on X, then $[\theta_\rho,\theta_\sigma] = c^\tau_{\rho\sigma}\theta_\tau$ where the $c = (c^\tau_{\rho\sigma} = -c^\tau_{\sigma\rho})$ are the *structure constants* of a Lie algebra of vector fields which can be identified with $\mathcal{G} = T_e(G)$ the tangent space to $\mathcal{G}$ at the identity $e \in G$ by using the action. Equivalently, introducing the non-degenerate inverse matrix $\alpha = \omega^{-1}$ of right invariant vector fields on G, we obtain from crossed-derivatives the *compatibility conditions* (CC) for the previous system of partial differential (PD) equations called *Maurer-Cartan equations* or simply MC equations, namely:

$$\boxed{\partial\omega^\tau_s/\partial a^r - \partial\omega^\tau_r/\partial a^s + c^\tau_{\rho\sigma}\omega^\rho_r\omega^\sigma_s = 0}$$

(*care to the sign used*) or equivalently $[\alpha_\rho,\alpha_\sigma] = c^\tau_{\rho\sigma}\alpha_\tau$ (see [7] for more details).

Using again crossed-derivatives, we obtain the corresponding *integrability conditions* (IC) on the structure constants and the Cauchy-Kowaleski theorem finally provides *the third fundamental theorem of Lie* saying that, for any Lie algebra $\mathcal{G}$ defined by structure constants $c = (c^\tau_{\rho\sigma})$ satisfying:

$$\boxed{c^\tau_{\rho\sigma} + c^\tau_{\sigma\rho} = 0, \quad c^\lambda_{\mu\rho}c^\mu_{\sigma\tau} + c^\lambda_{\mu\sigma}c^\mu_{\tau\rho} + c^\lambda_{\mu\tau}c^\mu_{\rho\sigma} = 0}$$

one can construct an analytic group G such that $\mathcal{G} = T_e(G)$ by recovering the MC forms from the MC equations.

EXAMPLE 1.1: Considering the affine group of transformations of the real line $y = a^1x + a^2$, the orbits are defined by $x = a^1x_0 + a^2$, a definition leading to $dx = da^1x_0 + da^2$ and thus $dx = \left(\left(1/a^1\right)da^1\right)x + \left(da^2 - \left(a^2/a^1\right)da^1\right)$. We obtain therefore $\theta_1 = x\partial_x$, $\theta_2 = \partial_x \Rightarrow [\theta_1,\theta_2] = -\theta_2$ and $\omega^1 = \left(1/a^1\right)da^1$, $\omega^2 = da^2 - \left(a^2/a^1\right)da^1 \Rightarrow d\omega^1 = 0$, $d\omega^2 - \omega^1\wedge\omega^2 = 0 \Leftrightarrow [\alpha_1,\alpha_2] = -\alpha_2$ with $\alpha_1 = a^1\partial_1 + a^2\partial_2$, $\alpha_2 = \partial_2$.

Now, if $x = a(t)x_0 + b(t)$ with $a(t)$ a time depending orthogonal matrix (*rotation*) and $b(t)$ a time depending vector (*translation*) describes the movement of a rigid body in $\mathbb{R}^3$, then the projection of the *absolute speed* $v = \dot{a}(t)x_0 + \dot{b}(t)$ in an orthogonal frame fixed in the body is the so-called *relative speed* $a^{-1}v = a^{-1}\dot{a}x_0 + a^{-1}\dot{b}$ and the kinetic energy/Lagrangian is a quadratic function of the 1-forms $A = \left(a^{-1}\dot{a}, a^{-1}\dot{b}\right)$. Meanwhile, taking into account the preceding example, the *Eulerian speed* $v = v(x,t) = \dot{a}a^{-1}x + \dot{b} - \dot{a}a^{-1}b$ only depends on the 1-forms $B = \left(\dot{a}a^{-1}, \dot{b} - \dot{a}a^{-1}b\right)$. We notice that $a^{-1}\dot{a}$ and $\dot{a}a^{-1}$ are both 3 × 3 skew symmetric time depending matrices that may be quite different.

REMARK 1.2: An easy computation in local coordinates for the case of the movement of a rigid body shows that the action of the 3×3 skew-symmetric matrix $\dot{a}a^{-1}$ on the position x at time t just amounts to the vector product by the *vortex vector* $\omega = \frac{1}{2}curl(v)$.

The above particular case, well known by anybody studying the analytical mechanics of rigid bodies, can be generalized as follows. If X is a manifold and G

is a lie group (*not acting necessarily on X*), let us consider maps $a: X \to G: (x) \to (a(x))$ or equivalently sections of the trivial (principal) bundle $X \times G$ over *X*. If $x + dx$ is a point of *X* close to *x*, then $T(a)$ will provide a point $a + da = a + \frac{\partial a}{\partial x} dx$ close to *a* on *G*. We may bring *a* back to *e* on *G* by acting on *a* with a^{-1}, *either on the left or on the right*, getting therefore a 1-form $a^{-1}da = A$ or $(da)a^{-1} = B$ with value in $\mathcal{G}$. As $aa^{-1} = e$ we also get $(da)a^{-1} = -ada^{-1} = -b^{-1}db$ if we set $b = a^{-1}$ as a way to link *A* with *B*. When there is an action $y = ax$, we have $x = a^{-1}y = by$ and thus $dy = dax = (da)a^{-1}y$, a result leading through the first fundamental theorem of Lie to the equivalent formulas:

$$a^{-1}da = A = \left(A_i^\tau(x)dx^i = -\omega_\sigma^\tau(b(x))\partial_i b^\sigma(x)dx^i\right)$$

$$(da)a^{-1} = B = \left(B_i^\tau(x)dx^i = \omega_\sigma^\tau(a(x))\partial_i a^\sigma(x)dx^i\right)$$

Introducing the induced bracket $[A, A](\xi, \eta) = [A(\xi), A(\eta)] \in \mathcal{G}, \forall \xi, \eta \in T$, we may define the *curvature* 2-form $dA - [A, A] = F \in \wedge^2 T^* \otimes \mathcal{G}$ by the local formula (*care again to the sign*):

$$\partial_i A_j^\tau(x) - \partial_j A_i^\tau(x) - c_{\rho\sigma}^\tau A_i^\rho(x) A_j^\sigma(x) = F_{ij}^\tau(x)$$

This definition can also be adapted to *B* by using $dB + [B, B]$ and we obtain from the second fundamental theorem of Lie:

THEOREM 1.3: There is a *nonlinear gauge sequence*:

$$\boxed{\begin{array}{ccccc} X \times G & \to & T^* \otimes \mathcal{G} & \xrightarrow{MC} & \wedge^2 T^* \otimes \mathcal{G} \\ a & \to & a^{-1}da = A & \to & dA - [A, A] = F \end{array}} \quad (1)$$

Choosing *a* "close" to *e*, that is $a(x) = e + t\lambda(x) + \cdots$ and linearizing as usual, we obtain the linear operator $d: \wedge^0 T^* \otimes \mathcal{G} \to \wedge^1 T^* \otimes \mathcal{G}: (\lambda^\tau(x)) \to (\partial_i \lambda^\tau(x))$ leading to (see [7] for more details):

COROLLARY 1.4: There is a *linear gauge sequence*:

$$\boxed{\wedge^0 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^1 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^2 T^* \otimes \mathcal{G} \xrightarrow{d} \cdots \xrightarrow{d} \wedge^n T^* \otimes \mathcal{G} \to 0} \quad (2)$$

which is the tensor product by $\mathcal{G}$ of the Poincaré sequence for the exterior derivative.

It just remains to introduce the previous results into a variational framework. For this, we may consider a lagrangian on $T^* \otimes \mathcal{G}$, that is an *action* $W = \int w(A)dx$ where $dx = dx^1 \wedge \cdots \wedge dx^n$ and to vary it. With $A = a^{-1}da = -(db)b^{-1}$ we may introduce $\lambda = a^{-1}\delta a = -(\delta b)b^{-1} \in \mathcal{G} = \wedge^0 T^* \otimes \mathcal{G}$ with local coordinates $\lambda^\tau(x) = -\omega_\sigma^\tau(b(x))\delta b^\sigma(x)$ and we obtain in local coordinates:

$$\boxed{\delta A = d\lambda - [A, \lambda] \Leftrightarrow \delta A_i^\tau = \partial_i \lambda^\tau - c_{\rho\sigma}^\tau A_i^\rho \lambda^\sigma} \quad (3)$$

Then, setting $\partial w / \partial A = \mathcal{A} = (\mathcal{A}_\tau^i) \in \wedge^{n-1} T^* \otimes \mathcal{G}$, we get:

$$\delta W = \int \mathcal{A} \delta A dx = \int \mathcal{A}(d\lambda - [A, \lambda])dx$$

and therefore, after integration by part, the Euler-Lagrange (EL) equations of

Poincaré ([7] [16]):

$$\boxed{\partial_i \mathcal{A}_\tau^i + c_{\rho\tau}^\sigma A_i^\rho \mathcal{A}_\sigma^i = 0} \quad (4)$$

Such a linear operator for $\mathcal{A}$ has non constant coefficients linearly depending on A and is the adjoint of the previous operator. However, setting $(\delta a)a^{-1} = \mu \in G$, we get $\lambda = a^{-1}((\delta a)a^{-1})a = Ad(a)\mu$ while, setting $a' = ab$, we get the *gauge transformation*:

$$\boxed{A \to A' = (ab)^{-1} d(ab) = b^{-1}a^{-1}((da)b + adb) = Ad(b)A + b^{-1}db, \forall b \in G} \quad (5)$$

Setting $b = e + t\lambda + \cdots$ with $t \ll 1$, then δA becomes an infinitesimal gauge transformation. Finally, $a' = ba$ $\Rightarrow A' = a^{-1}b^{-1}((db)a + a(db)) = a^{-1}(b^{-1}db)a + A \Rightarrow \delta A = Ad(a)d\mu$ when $b = e + t\mu + \cdots$ with $t \ll 1$. Therefore, introducing $\mathcal{B}$ such that $\mathcal{B}\mu = \mathcal{A}\lambda$, we get the divergence-like equations:

$$\boxed{\partial_i \mathcal{B}_\sigma^i = 0} \quad (6)$$

In a completely different local setting, if G acts on X and Y is a copy of X with an action graph $X \times G \to X \times Y : (x,a) \to (x, y = f(x,a))$, we may use the theorems of S. Lie in order to find a basis $\{\theta_\tau \mid 1 \le \tau \le p = dim(G)\}$ of infinitesimal generators of the action. If $\mu = (\mu_1, \cdots, \mu_n)$ is a multi-index of length $|\mu| = \mu_1 + \cdots + \mu_n$ and $\mu + 1_i = (\mu_1, \cdots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \cdots, \mu_n)$, we may introduce the system of infinitesimal Lie equations or Lie algebroid $R_q \subset J_q(T)$ with sections defined by $\xi_\mu^k(x) = \lambda^\tau(x)\partial_\mu \theta_\tau^k(x)$ for an arbitrary section $\lambda \in \wedge^0 T^* \otimes \mathcal{G}$ and finally obtain the *Spencer operator* through the chain rule for derivatives:

$$\boxed{(d\xi_{q+1})_{\mu,i}^k(x) = \partial_i \xi_\mu^k(x) - \xi_{\mu+1_i}^k(x) = \partial_i \lambda^\tau(x)\partial_\mu \theta_\tau^k(x)} \quad (7)$$

THEOREM 1.5: When q is large enough to have an isomorphism $R_{q+1} \simeq R_q \simeq \wedge^0 T^* \otimes \mathcal{G}$ and the following (trivial) *linear Spencer sequence* in which the operators D_r are induced by d:

$$\boxed{R_q \xrightarrow{D_1} T^* \otimes R_q \xrightarrow{D_2} \wedge^2 T^* \otimes R_q \xrightarrow{D_3} \cdots \xrightarrow{D_n} \wedge^n T^* \otimes R_q \to 0} \quad (8)$$

is isomorphic to the linear gauge sequence but with a completely different meaning because G is now acting on X.

EXAMPLE 1.6: (Weyl group of transformations) For an arbitrary dimension n, the conformal group has n translations, $n(n-1)/2$ rotations, 1 dilatation and n nonlinear elations, that is a total of $(n+1)(n+2)/2$ parameters and the Weyl subgroup has only $(n^2+n+2)/2$ parameters. When $n=2$ and the standard Euclidean metric, we may choose the infinitesimal generators $\theta_1 = \partial_1$, $\theta_2 = \partial_2$, $\theta_3 = x^1\partial_2 - x^2\partial_1$, $\theta_4 = x^1\partial_1 + x^2\partial_2$ of the Weyl subgroup with $2+1+1=4$ parameters by taking out the elations. Setting $\xi_\mu^k = \lambda^\tau \partial_\mu \theta_\tau^k$ with $\lambda = (\lambda^\tau) \in \mathcal{G}$, we have the 4×4 full rank matrix allowing to describe the isomorphism $R_2 \simeq R_1 \simeq \wedge^0 T^* \otimes \mathcal{G}$:

$$\xi^1 = \lambda^1 - x^2\lambda^3 + x^1\lambda^4, \xi^2 = \lambda^2 + x^1\lambda^3 + x^2\lambda^4, \xi_1^2 = -\xi_2^1 = \lambda^3, \xi_1^1 = \xi_2^2 = \lambda^4$$

$$\begin{pmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{pmatrix}$$

$$\begin{pmatrix} \xi^1 \\ \xi^2 \\ -\xi_2^1 = \xi_1^2 \\ \xi_2^2 = \xi_1^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -x^2 & x^1 \\ 0 & 1 & x^1 & x^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \\ \lambda^4 \end{pmatrix}$$

Now, in order to determine $ad(D_1)$, we have to integrate by parts the duality summation:

$$\sigma_1^1\left(\partial_1\xi^1 - \xi_1^1\right) + \sigma_2^1\left(\partial_1\xi^2 - \xi_1^2\right) + \sigma_1^2\left(\partial_2\xi^1 - \xi_2^1\right) + \sigma_2^2\left(\partial_2\xi^2 - \xi_2^2\right)$$
$$+ \mu^{12,r}\partial_r\xi_1^2 + \nu^r\partial_r\xi_1^1$$

in which we have taking into account the Medolaghi equations $\xi_2^2 - \xi_1^1 = 0$, $\xi_2^1 + \xi_1^2 = 0$, $\xi_{ij}^k = 0$ defining the Weyl algebroid. We get the 4 Cosserat/Clausius equations describing the adjoint of the first Spencer operator in which we may have $\sigma^{1,2} \neq \sigma^{2,1}$:

$$\partial_r\sigma^{i,r} = f^i, \partial_r\mu^{12,r} + \sigma^{2,1} - \sigma^{1,2} = m^{12}, \partial_r\nu^r + \sigma_r^r = u$$

that we can transform into the 4 *pure divergence* equations by comparing $ad(D_1)$ with $ad(d)$:

$$\partial_r\left(\sigma^{i,r}\right) = f^i, \partial_r\left(\mu^{12,r} + x^1\sigma^{2,r} - x^2\sigma^{1,r}\right) = m^{12} + x^1 f^2 - x^2 f^1,$$
$$\partial_r\left(\nu^r + x^i\sigma_i^r\right) = u + x^i f_i$$

a result not completely evident at first sight. When $n = 2$, the conformal group has therefore 6 parameters and we should follow the same procedure after adding the two elations:

$$\theta_5 = \frac{1}{2}\left(\left(x^1\right)^2 - \left(x^2\right)^2\right)\partial_1 + x^1x^2\partial_2, \theta_6 = x^1x^2\partial_1 + \frac{1}{2}\left(\left(x^2\right)^2 - \left(x^1\right)^2\right)\partial_2$$

in such a way that $\partial_r\theta_5^r = 2x^1, \partial_r\theta_6^r = 2x^2$ (see [17] for the relation with the Clausius virial theorem and [18] for the relation with the conformal group). The only difference is that we have now to deal with the 6 right members $\left(f^1, f^2, m^{12}, u, v^1, v^2\right)$. As we have been only using the Spencer bundles C_0 and C_1, these results have strictly nothing to do with C_2 involving 2-forms and the so-called Cartan curvature, a result also proving that the mathematical foundations of Gauge theory must be revisited as we have no longer any link with the unitary group $U(1)$.

From a purely mathematical point of view, the concept of a finite length *differential sequence*, now called *Janet sequence*, has been described for the first time as a footnote by M. Janet in 1920 ([19]). Then, the work of D. C. Spencer in 1970 has been the first attempt to use the formal theory of systems of partial differential equations that he developed himself in order to study the formal theory of Lie pseudogroups ([20] [21] [22]). In 1978 we have provided a link between these two sequences in our *Fundamental Diagram I* ([4] [7]). The *nonlinear Spencer sequences for Lie pseudogroups*, though never used in physics, largely supersede the "*Cartan structure equations*" introduced by E. Cartan in 1905

([23]) and are quite different from the "*Vessiot structure equations*" introduced by E. Vessiot in 1903 ([24] [25] [26] [27] [28]) for the same purpose but still not known today after more than a century because they have never been acknowledged by Cartan himself or even by all his successors.

The purpose of the previous paper ([29]) has been to apply these new methods for studying the conformal origin of electromagnetism *in a purely mathematical way* by constructing explicitly the corresponding nonlinear Spencer sequence for the conformal group when the dimension of the ground manifold X is equal to 4, a *very specific* value as we have seen. All the more specific physical consequences concerning gravitation are presented in the present paper and we shall discover that this dimension is also quite specific but for a completely different reason.

2. Variational Calculus

It remains to graft a variational procedure adapted to the previous results obtained in ([28]). Contrary to what happens in analytical mechanics or elasticity for example, *the main idea is to vary sections but not points*. Hence, we may introduce the variation $\delta f^k(x)=\eta^k(f(x))$ and set $\eta^k(f(x))=\xi^i(x)\partial_i f^k(x)$ along the canonical "*vertical machinery*" ([7]) but *notations like* $\delta x^i=\xi^i$ *or* $\delta y^k=\eta^k$ *have no meaning at all*.

If Y is a copy of X with local coordinates (y^k) and $\mathcal{E}=X\times Y$, we shall denote by Π_q the open sub-fibered manifold of the q-jet bundle $J_q(X\times Y)$ defined independently of the coordinate system by $det(y_i^k)\neq 0$ with *source projection* $\alpha_q:\Pi_q\to X:(x,y_q)\to(x)$ and *target projection* $\beta_q:\Pi_q\to Y:(x,y_q)\to(y)$. We denote by $id:X\to Y:x\to y=x$ the *identity* map and we have the identification $T=id^{-1}(V(\mathcal{E}))$. In order to construct another nonlinear sequence, we need a few basic definitions on *Lie groupoids* and *Lie algebroids* that will become substitutes for Lie groups and Lie algebras. Introducing the operator $j_q:X\times Y\to J_q(X\times Y)=f(x)\to(f^k(x),\partial_i f^k(x),\partial_{ij}f^k(x),\cdots)$, the first idea is to use the chain rule for derivatives $j_q(g\circ f)=j_q(g)\circ j_q(f)$ whenever $f,g\in aut(X)$ can be composed and to replace both $j_q(f)$ and $j_q(g)$ respectively by f_q and g_q in order to obtain the new section $g_q\circ f_q$. This kind of "composition" law can be written in a pointwise symbolic way by introducing another copy Z of X with local coordinates (z) as follows:

$$\gamma_q:\Pi_q(Y,Z)\times_Y\Pi_q(X,Y)\to\Pi_q(X,Z):\left(y,z,\frac{\partial z}{\partial y},\cdots\right),\left(x,y,\frac{\partial y}{\partial x},\cdots\right)\to\left(x,z,\frac{\partial z}{\partial y}\frac{\partial y}{\partial x},\cdots\right)$$

We may also define $j_q(f)^{-1}=j_q(f^{-1})$ and obtain similarly an "inversion" law $\iota_q:\Pi_q\to\Pi_q$.

A fibered submanifold $\mathcal{R}_q\subset\Pi_q$ is called a *system of finite Lie equations* or a *Lie groupoid* of order q if we have an induced *source projection* $\alpha_q:\mathcal{R}_q\to X$, *target projection* $\beta_q:\mathcal{R}_q\to Y$, *composition* $\gamma_q:\mathcal{R}_q\times_Y\mathcal{R}_q\to\mathcal{R}_q$, *inversion* $\iota_q:\mathcal{R}_q\to\mathcal{R}_q$ and *identity* $j_q(id)=id_q:X\to\mathcal{R}_q$. In the sequel we shall only consider *transitive* Lie groupoids such that the map $(\alpha_q,\beta_q):\mathcal{R}_q\to X\times Y$ is an epimorphism and we shall denote by $\mathcal{R}_q^0 = id^{-1}(\mathcal{R}_q)$ the *isotropy Lie group bundle* of $\mathcal{R}_q$. Also, one can prove that the new system $\rho_r(\mathcal{R}_q) = \mathcal{R}_{q+r}$ obtained by differentiating *r* times all the defining equations of $\mathcal{R}_q$ is a Lie groupoid of order $q+r$.

The vector sub-bundle $R_q = id_q^{-1}(V(\mathcal{R}_q)) \subset J_q(T)$ is called a *system of infinitesimal Lie equations* or a *Lie algebroid* of order q.

As a major result first discovered in specific cases by the brothers Cosserat in 1909 and by Weyl in 1916, we shall prove and apply the following key result:

THE PROCEDURE ONLY DEPENDS ON THE LINEAR SPENCER OPERATOR AND ITS ADJOINT, recalling that $\delta: g_{q+1} \to T^* \otimes g_q$ is *minus* the restriction of *d* to the symbols.

Let the non-linear operator $\bar{D}: \mathcal{R}_{q+1} \to T^* \otimes R_q : f_{q+1} \to f_{q+1}^{-1} \circ j_1(f_q) - id_{q+1} = \chi_q$ be defined as in ([7], p. 224 and [22]). For compositions like $f'_{q+1} = g_{q+1} \circ f_{q+1} = f_{q+1} \circ h_{q+1}$, we get:

$$\begin{aligned} \bar{D}f'_{q+1} &= f_{q+1}^{-1} \circ g_{q+1}^{-1} \circ j_1(g_q) \circ j_1(f_q) - id_{q+1} = f_{q+1}^{-1} \circ \bar{D}g_{q+1} \circ j_1(f_q) + \bar{D}f_{q+1} \\ &= h_{q+1}^{-1} \circ f_{q+1}^{-1} \circ j_1(f_q) \circ j_1(h_q) - id_{q+1} = h_{q+1}^{-1} \circ \bar{D}f_{q+1} \circ j_1(h_q) + \bar{D}h_{q+1} \end{aligned}$$

Using the local exactness of the first nonlinear Spencer sequence ([7], p. 215 or [25], p. 176), we get:

LEMMA 2.1: For any section $f_{q+1} \in \mathcal{R}_{q+1}$, the *finite gauge transformation*:

$$\chi_q \in T^* \otimes R_q \to \chi'_q = f_{q+1}^{-1} \circ \chi_q \circ j_1(f_q) + \bar{D}f_{q+1} \in T^* \otimes R_q$$

exchanges the solutions of the *field equations* $\bar{D}'\chi_q = 0$.

Introducing the *bilinear algebraic bracket* $\{j_{q+1}(\xi), j_{q+1}(\eta)\} = j_q([\xi,\eta])$, we may then introduce both the *formal Lie derivative* and the *differential algebroid bracket* on $J_q(T)$ by the formulas:

$$L(\xi_{q+1})\eta_q = \{\xi_{q+1},\eta_{q+1}\} + i(\xi)d\eta_{q+1} = [\xi_q,\eta_q] + i(\eta)d\xi_{q+1}$$

$$\big(L(j_1(\xi_{q+1}))\chi_q\big)(\zeta) = L(\xi_{q+1})(\chi_q(\zeta)) - \chi_q([\xi,\zeta])$$

in such a way that $[R_q, R_q] \subset R_q$.

LEMMA 2.2: Passing to the limit *over the source* with $\chi_q = \bar{D}f_{q+1}$ and $h_{q+1} = id_{q+1} + t\xi_{q+1} + \cdots$ for $t \to 0$, we get an *infinitesimal gauge transformation* leading to the *infinitesimal variation*:

$$\boxed{\begin{aligned} &\delta\chi_q = d\xi_{q+1} + L(j_1(\xi_{q+1}))\chi_q \\ &\delta\chi_q(\zeta) = i(\zeta)d\xi_{q+1} + \{\xi_{q+1}, \chi_{q+1}(\zeta)\} + i(\xi)d\chi_{q+1}(\zeta) - \chi_q([\xi,\zeta]) \\ &\qquad\quad = i(\zeta)d\xi_{q+1} + L(\xi_{q+1})(\chi_q(\zeta)) - \chi_q([\xi,\zeta]) \end{aligned}} \quad (9)$$

which *only depends on* χ_q but *does not depend on* the parametrization of χ_q.

LEMMA 2.3: Passing to the limit *over the target* with $\chi_q = \bar{D}f_{q+1}$ and $g_{q+1} = id_{q+1} + t\eta_{q+1} + \cdots$ for $t \to 0$, we get the other *infinitesimal variation*:

$$\boxed{\delta\chi_q = f_{q+1}^{-1} \circ d\eta_{q+1} \circ j_1(f_q)} \quad (10)$$

which *highly depends on the parametrization* of χ_q.

EXAMPLE 2.4: We obtain for $q=1$:

$$\boxed{\begin{array}{l}\delta\chi^k_{,i}=\left(\partial_i\xi^k-\xi^k_i\right)+\left(\xi^r\partial_r\chi^k_{,i}+\chi^k_{,r}\partial_i\xi^r-\chi^r_{,i}\xi^k_r\right)\\ \delta\chi^k_{j,i}=\left(\partial_i\xi^k_j-\xi^k_{ij}\right)+\left(\xi^r\partial_r\chi^k_{j,i}+\chi^k_{j,r}\partial_i\xi^r+\chi^k_{r,i}\xi^r_j-\chi^r_{j,i}\xi^k_r-\chi^r_{,i}\xi^k_{jr}\right)\end{array}} \quad (11)$$

Introducing the inverse matrix $B=A^{-1}$, we obtain therefore equivalently:

$$\delta A^k_i=\xi^r\partial_r A^k_i+A^k_r\partial_i\xi^r-A^r_i\xi^k_r \Leftrightarrow \delta B^i_k=\xi^r\partial_r B^i_k-B^r_k\partial_r\xi^i+B^i_r\xi^r_k$$

both with:

$$\delta\chi^k_{j,i}=\left(\partial_i\xi^k_j-A^r_i\xi^k_{jr}\right)+\left(\xi^r\partial_r\chi^k_{j,i}+\chi^k_{j,r}\partial_i\xi^r+\chi^k_{r,i}\xi^r_j-\chi^r_{j,i}\xi^k_r\right)$$

For the Killing system $R_1\subset J_1(T)$ with $g_2=0$, these variations are *exactly* the ones that can be found in ([10], (50) + (49), p. 124 with a printing mistake corrected on p. 128) when replacing a 3 × 3 skew-symmetric matrix by the corresponding vector. *The three last unavoidable Lemmas are thus essential in order to bring back the nonlinear framework of finite elasticity to the linear framework of infinitesimal elasticity that only depends on the linear Spencer operator.*

For the conformal Killing system $\hat{R}_1\subset J_1(T)$ (see next section) we obtain:

$$\begin{aligned}\alpha_i=\chi^r_{r,i}\Rightarrow\delta\alpha_i&=\left(\partial_i\xi^r_r-\xi^r_{ri}\right)+\left(\xi^r\partial_r\alpha_i+\alpha_r\partial_i\xi^r-\chi^s_{,i}\xi^r_{rs}\right)\\ &=\left(\partial_i\xi^r_r-A^s_i\xi^r_{rs}\right)+\left(\alpha_r\partial_i\xi^r+\xi^r\partial_r\alpha_i\right)\end{aligned}$$

$$\varphi_{ij}=\partial_i\alpha_j-\partial_j\alpha_i\Rightarrow\delta\varphi_{ij}=\left(\partial_j\left(A^s_i\xi^r_{rs}\right)-\partial_i\left(A^s_j\xi^r_{rs}\right)\right)+\left(\varphi_{rj}\partial_i\xi^r+\varphi_{ir}\partial_j\xi^r+\xi^r\partial_r\varphi_{ij}\right)$$

These are *exactly* the variations obtained by Weyl ([3], (76), p. 289) who was assuming implicitly $A=0$ when setting $\bar{\xi}^r_r=0\Leftrightarrow\xi^r_r=-\alpha_i\xi^i$ by introducing a connection. Accordingly, ξ^r_{ri} is the variation of the EM potential itself, that is the δA_i of engineers used in order to exhibit the Maxwell equations from a variational principle ([3], § 26) but the introduction of the Spencer operator is new in this framework. If $f_1=id_1$, we have $\chi_0=0$ and $\delta\alpha_i=\left(\partial_i\xi^r_r-\xi^r_{ri}\right)+\left(\alpha_r\partial_i\xi^r+\xi^r\partial_r\alpha_i\right)$.

The explicit general formulas of the three previous lemmas cannot be found somewhere else (The reader may compare them to the ones obtained in [22] by means of the so-called "diagonal" method that cannot be applied to the study of explicit examples). The following unusual difficult proposition generalizes well known variational techniques used in continuum mechanics and will be crucially used for applications:

PROPOSITION 2.5: The same variation is obtained whenever $\eta_q=f_{q+1}\left(\xi_q+\chi_q(\xi)\right)$ with $\chi_q=\bar{D}f_{q+1}$, a transformation which only depends on $j_1(f_q)$ and is invertible if and only if $det(A)\neq0$.

Proof: First of all, setting $\bar{\xi}_q=\xi_q+\chi_q(\xi)$, we get $\bar{\xi}=A(\xi)$ for $q=0$, a transformation which is invertible if and only if $det(A)\neq0$ and thus $\Delta\neq0$. In the nonlinear framework, we have to keep in mind that there is no need to vary the object ω which is given but only the need to vary the section f_{q+1} as we already saw, using $\eta_q \in R_q(Y)$ *over the target* or $\xi_q \in R_q$ *over the source*. With $\eta_q = f_{q+1}(\xi_q)$, we obtain for example:

$$\delta f^k = \eta^k = f_r^k \xi^r$$
$$\delta f_i^k = \eta_u^k f_i^u = f_r^k \xi_i^r + f_{ri}^k \xi^r$$
$$\delta f_{ij}^k = \eta_{uv}^k f_i^u f_j^v + \eta_u^k f_{ij}^u = f_r^k \xi_{ij}^r + f_{ri}^k \xi_j^r + f_{rj}^k \xi_i^r + f_{rij}^k \xi^r$$

and so on. Introducing the formal derivatives d_i for $i = 1,\cdots,n$, we have:

$$\delta f_\mu^k = \zeta_\mu^k(f_q,\eta_q) = d_\mu \eta^k = \eta_u^k f_\mu^u + \cdots = f_r^k \xi_\mu^r + \cdots + f_{\mu+1_r}^k \xi^r$$

We shall denote by $\#(\eta_q) = \zeta_\mu^k(y_q,\eta_q)\frac{\partial}{\partial y_\mu^k} \in V(\mathcal{R}_q)$ with $\zeta^k = \eta^k$ the corresponding vertical vector field, namely:

$$\#(\eta_q) = 0\frac{\partial}{\partial x^i} + \eta^k(y)\frac{\partial}{\partial y^k} + \left(\eta_u^k(y)y_i^u\right)\frac{\partial}{\partial y_i^k} + \left(\eta_{uv}^k(y)y_i^u y_j^v + \eta_u^k(y)y_{ij}^u\right)\frac{\partial}{\partial y_{ij}^k} + \cdots$$

However, the standard prolongation of an infinitesimal change of source coordinates described by the horizontal vector field ξ, obtained by replacing all the derivatives of ξ by a section $\xi_q \in R_q$ over $\xi \in T$, is the vector field:

$$\flat(\xi_q) = \xi^i(x)\frac{\partial}{\partial x^i} + 0\frac{\partial}{\partial y^k} - \left(y_r^k \xi_i^r(x)\right)\frac{\partial}{\partial y_i^k}$$
$$-\left(y_r^k \xi_{ij}^r(x) + y_{rj}^k \xi_i^r(x) + y_{ri}^k \xi_j^r(x)\right)\frac{\partial}{\partial y_{ij}^k} + \cdots$$

It can be proved that $\left[\flat(\xi_q),\flat(\xi_q')\right] = \flat\left(\left[\xi_q,\xi_q'\right]\right), \forall \xi_q,\xi_q' \in R_q$ *over the source*, with a similar property for $\#(.)$ *over the target* ([25] [27]). However, $\flat(\xi_q)$ *is not a vertical vector field and cannot therefore be compared to* $\#(\eta_q)$. The solution of this problem explains a strange comment made by Weyl in ([3], p. 289 + (78), p. 290) and which became a founding stone of classical gauge theory. Indeed, ξ_r^r is *not* a scalar because ξ_i^k is *not* a 2-tensor. However, when $A = 0$, then $-\chi_q$ is a R_q-connection and $\bar{\xi}_r^r = \xi_r^r + \chi_{r,i}^r \xi^i$ is a true scalar that may be set equal to zero in order to obtain $\xi_r^r = -\chi_{r,i}^r \xi^i$, a fact explaining why the EM-potential is considered as a connection in quantum mechanics instead of using the second order jets ξ_{ri}^r of the conformal system, with a *shift by one step in the physical interpretation of the Spencer sequence* (see [4] for more historical details).

The main idea is to consider the vertical vector field $T(f_q)(\xi) - \flat(\xi_q) \in V(\mathcal{R}_q)$ whenever $y_q = f_q(x)$. Passing to the limit $t \to 0$ in the formula $g_q \circ f_q = f_q \circ h_q$, we first get $g \circ f = f \circ h \Rightarrow f(x) + t\eta(f(x)) + \cdots = f(x + t\xi(x) + \cdots)$. Using the chain rule for derivatives and substituting jets, we get successively:

$$\delta f^k(x) = \xi^r \partial_r f^k, \quad \delta f_i^k = \xi^r \partial_r f_i^k + f_r^k \xi_i^r,$$
$$\delta f_{ij}^k = \xi^r \partial_r f_{ij}^k + f_{rj}^k \xi_i^r + f_{ri}^k \xi_j^r + f_r^k \xi_{ij}^r$$

and so on, replacing $\xi^r f_{\mu+1_r}^k$ by $\xi^r \partial_r f_\mu^k$ in $\eta_q = f_{q+1}(\xi_q)$ in order to obtain:

$$\delta f_\mu^k = \eta_r^k f_\mu^r + \cdots = \xi^i \left(\partial_i f_\mu^k - f_{\mu+1_i}^k \right) + f_r^k \xi_\mu^r + \cdots + f_{\mu+1_r}^k \xi^r$$

where the right member only depends on $j_1(f_q)$ when $|\mu| = q$.

Finally, we may write the symbolic formula
$f_{q+1}(\chi_q) = j_1(f_q) - f_{q+1} = df_{q+1} \in T^* \otimes V(\mathcal{R}_q)$ in the inductive form:

$$\boxed{f_r^k \chi_{\mu,i}^r + \cdots + f_{\mu+1_r}^k \chi_{,i}^r = \partial_i f_\mu^k - f_{\mu+1_i}^k}$$

Substituting in the previous formula provides $\eta_q = f_{q+1}\left(\xi_q + \chi_q(\xi)\right)$ and we just need to replace q by $q+1$ in order to achieve the proof. Checking directly the proposition is not evident even when $q = 0$ but cannot be done by hand when $q \geq 1$. □

3. Applications

Before studying gravitation in a specific way, we shall provide a technical result which, though looking like evident at first sight, is at the origin of a misunderstanding done by the brothers Cosserat and Weyl on the variational procedure used in the study of physical problems.

Setting $dx = dx^1 \wedge \cdots \wedge dx^n$ for simplicity and using the fact that the standard Lie derivative is commuting with any diffeomorphism, we obtain at once:

$$y = f(x), x = g(y) \Rightarrow dy = det\left(\partial_i f^k(x)\right) dx = \Delta(x) dx$$

$$\eta = T(f)\xi \Rightarrow \mathcal{L}(\eta) dy = \mathcal{L}(\xi)\left(\Delta(x) dx\right) \Rightarrow \delta\Delta = \Delta div_y(\eta) = \Delta div_x(\xi) + \xi^r \partial_r \Delta$$

The interest of such a presentation is to provide the right correspondence between the source/target and the Euler/Lagrange (actual/initial) choices. Indeed, if we use the way followed by most authors up to now in continuum mechanics, we should have source = Lagrange, target = Euler, a result leading to the conservation of mass $dm = \rho dy = \rho_0 dx = dx$ when ρ_0 is the original initial mass per unit volume. We may set $\rho_0 = 1$ and obtain therefore $\rho(f(x)) = 1/\Delta(x)$, a choice leading to:

$$\delta\rho + \eta^k \frac{\partial \rho}{\partial y^k} = -\frac{1}{\Delta^2}\delta\Delta \Rightarrow \delta\rho = -\rho \frac{\partial \eta^k}{\partial y^k} - \eta^k \frac{\partial \rho}{\partial y^k} = -\rho \frac{\partial \xi^r}{\partial x^r} \Rightarrow \delta\rho = -\frac{\partial\left(\rho \eta^k\right)}{\partial y^k}$$

but the concept of "*variation*" is not mathematically well defined, though this result is coherent with the classical formulas that can be found in the literature where "*points are moved*".

On the contrary, if we adopt the *unusual* choice source = Euler, target = Lagrange, we should get $\rho(x) = \Delta(x)$, a choice leading to $\delta\rho = \delta\Delta$ and thus:

$$\delta\rho = \rho \frac{\partial \eta^k}{\partial y^k} = \rho \frac{\partial \xi^r}{\partial x^r} + \xi^r \partial_r \rho = \partial_r\left(\rho \xi^r\right)$$

which is the right choice agreeing, up to the sign, with classical formulas but with the important improvement that this result becomes a purely mathematical one, obtained from a well defined variational procedure involving only the so-called "*vertical*" machinery. We obtain the *fundamental identity over the* *source and over the target*:

$$\frac{\partial}{\partial x^i}\left(\Delta(x)\frac{\partial g^i}{\partial y^k}\left(f(x)\right)\right)\equiv 0,\quad \forall x\in X$$

$$\Leftrightarrow \frac{\partial}{\partial y^k}\left(\frac{1}{\Delta\left(g(y)\right)}\partial_i f^k\left(g(y)\right)\right)\equiv 0,\quad \forall y\in Y$$

which becomes the conservation of mass when $n=4$ and $k=4$.

In addition, as many chases will be used through many diagrams in the sequel, we invite the reader not familiar with these technical tools to consult the books ([30] [31] [32]) that we consider as the best references for learning about homological algebra. As for *differential homological algebra*, one of the most difficult tools existing in mathematics today, and its link with applications, we refer the reader to the various references provided in ([8] [32]-[37]).

3.1. Poincare, Weyl and Conformal Groups

When constructing inductively the Janet and Spencer sequences for an involutive systems $R_q\subset J_q(E)$, we have to use the following commutative and exact diagrams in which we have set $F_0=J_q(E)/R_q$:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0\to & & \delta\left(\wedge^{r-1}T^*\otimes g_{q+1}\right) & \to & \wedge^r T^*\otimes R_q & \to & C_r & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0\to & & \delta(\wedge^{r-1}T^*\otimes S_{q+1}T^*\otimes T & \to & \wedge^r T^*\otimes J_q(E) & \to & C_r(E) & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0\to & & \wedge^r T^*\otimes R_q+\delta\left(\wedge^{r-1}T^*\otimes S_{q+1}T^*\otimes E\right) & \to & \wedge^r T^*\otimes F_0 & \to & F_r & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

It follows that the short exact sequences $0\to C_r\to C_r(E)\xrightarrow{\Phi_r}F_r\to 0$ are allowing to define the Janet and Spencer bundles inductively. If we consider *two* involutive systems $0\subset R_q\subset \hat{R}_q\subset J_q(E)$, it follows that the *kernels* of the induced canonical epimorphisms $F_r\to\hat{F}_r\to 0$ are isomorphic to the *cokernels* of the canonical monomorphisms $0\to C_r\to\hat{C}_r\subset C_r(E)$ and we may say that *Janet and Spencer play at see-saw* because we have the formula

$dim(C_r)+dim(F_r)=dim(C_r(E))$.

When dealing with applications, we have set $E=T$ and considered systems of finite type Lie equations determined by Lie groups of transformations. Accordingly, we have obtained in particular $C_r=\wedge^r T^*\otimes R_2\subset\wedge^r T^*\otimes\hat{R}_2=\hat{C}_r\subset C_r(T)$ when comparing the classical and conformal Killing systems, but *these bundles have never been used in physics*. However, instead of the classical Killing system $R_1\subset J_1(T)$ defined by the infinitesimal first order PD Lie equations $\Omega\equiv\mathcal{L}(\xi)\omega=0$ *and* its first prolongations $R_2\subset J_2(T)$ defined by the infinitesimal additional second order PD Lie equations $\Gamma\equiv\mathcal{L}(\xi)\gamma=0$ or the conformal Killing system $\hat{R}_2 \subset J_2(T)$ defined by $\Omega \equiv \mathcal{L}(\xi)\omega = 2A(x)\omega$ and $\Gamma \equiv \mathcal{L}(\xi)\gamma = \left(\delta_i^k A_j(x) + \delta_j^k A_i(x) - \omega_{ij}\omega^{ks}A_s(x)\right) \in S_2T^* \otimes T$, we may also consider the formal Lie derivatives for geometric objects:

$$\Omega_{ij} \equiv \left(L(\xi_1)\omega\right)_{ij} \equiv \omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r + \xi^r\partial_r\omega_{ij} = 0$$

$$\Gamma_{ij}^k \equiv \left(L(\xi_2)\gamma\right)_{ij}^k \equiv \xi_{ij}^k + \gamma_{rj}^k\xi_i^r + \gamma_{ir}^k\xi_j^r - \gamma_{ij}^r\xi_k^r + \xi^r\partial_r\gamma_{ij}^k = 0$$

We may now introduce the *intermediate differential system* $\tilde{R}_2 \subset J_2(T)$ defined by $\mathcal{L}(\xi)\omega = 2A(x)\omega$ and $\Gamma \equiv \mathcal{L}(\xi)\gamma = 0$, for the *Weyl group* obtained by adding the only dilatation with infinitesimal generator $x^i\partial_i$ to the Poincaré group. We have the relations $R_1 \subset \tilde{R}_1 = \hat{R}_1$ and the strict inclusions $R_2 = \rho_1(R_1)$, $\tilde{R}_2 = \rho_1(\tilde{R}_1)$, $\hat{R}_2 = \rho_1(\hat{R}_1) \Rightarrow R_2 \subset \tilde{R}_2 \subset \hat{R}_2$ but we have to notice that we must have $\partial_i A - A_i = 0$ for the conformal system and thus $A_i = 0 \Rightarrow A = cst$ if we do want to deal again with an involutive second order system $\tilde{R}_2 \subset J_2(T)$. However, we must not forget that the comparison between the Spencer and the Janet sequences can only be done for involutive operators, that is we can indeed use the involutive systems $R_2 \subset \tilde{R}_2$ but we have to use $\hat{R}_3$ even if it is isomorphic to $\hat{R}_2$. Finally, as $\hat{g}_2 \simeq T^*$ and $\hat{g}_3 = 0$, $\forall n \geq 3$, the first Spencer operator $\hat{R}_2 \xrightarrow{D_1} T^* \otimes \hat{R}_2$ is induced by the usual Spencer operator $\hat{R}_3 \xrightarrow{D} T^* \otimes \hat{R}_2 : \left(0,0,\xi_{rj}^r,\xi_{rij}^r = 0\right) \to \left(0,\partial_i 0 - \xi_{ri}^r, \partial_i\xi_{rj}^r - 0\right)$ and thus projects by cokernel onto the induced operator $T^* \to T^* \otimes T^*$. Composing with δ, it projects therefore onto $T^* \xrightarrow{d} \wedge^2 T^* : A \to dA = F$ as in EM and so on by using he fact that D_1 *and* d *are both involutive* or the composite epimorphisms $\hat{C}_r \to \hat{C}_r/\tilde{C}_r \simeq \wedge^r T^* \otimes \left(\hat{R}_2/\tilde{R}_2\right) \simeq \wedge^r T^* \otimes \hat{g}_2 \simeq \wedge^r T^* \otimes T^* \xrightarrow{\delta} \wedge^{r+1} T^*$. The main result we have obtained is thus to be able to increase the order and dimension of the underlying jet bundles and groups as we have the inclusions:

$$\boxed{\text{POINCARE GROUP} \subset \text{WEYL GROUP} \subset \text{CONFORMAL GROUP}}$$

that is $10 < 11 < 15$ when $n = 4$ and our aim is now to prove that *the mathematical foundation of gravitation only depends on the second order jets*, exactly like we have already proved in ([29]) that *the mathematical foundation of electromagnetism only depends on these second order jets*.

With more details, the Killing system $R_2 \subset J_2(T)$ is defined by the infinitesimal Lie equations in *Medolaghi form* with the well known *Levi-Civita isomorphism* $(\omega,\gamma) \simeq j_1(\omega)$ for geometric objects:

$$\begin{cases} \Omega_{ij} \equiv \omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r + \xi^r\partial_r\omega_{ij} = 0 \\ \Gamma_{ij}^k \equiv \gamma_{rj}^k\xi_i^r + \gamma_{ir}^k\xi_j^r - \gamma_{ij}^r\xi_r^k + \xi^r\partial_r\gamma_{ij}^k = 0 \end{cases}$$

We notice that $R_2(\bar{\omega}) = R_2(\omega) \Leftrightarrow \bar{\omega} = a\omega, a = cst, \bar{\gamma} = \gamma$ and refer the reader to ([28]) for more details about the link between this result and the deformation theory of algebraic structures. We also notice that R_1 is formally integrable and thus R_2 is involutive if and only if ω has constant Riemannian curvature along the results of L. P. Eisenhart ([4] [25] [27] [35]). The only structure constant c appearing in the corresponding Vessiot structure equations is such that $\bar{c} = c/a$

and the normalizer of R_1 is R_1 if and only if $c \neq 0$. Otherwise R_1 is of codimension 1 in its normalizer $\tilde{R}_1$ as we shall see below by adding the only dilatation. In all what follows, ω is assumed to be flat with $c = 0$ and vanishing Weyl tensor.

The Weyl system $\tilde{R}_2 \subset J_2(T)$ is defined by the infinitesimal Lie equations:

$$\begin{cases} \omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r + \xi^r\partial_r\omega_{ij} = 2A(x)\omega_{ij} \\ \xi_{ij}^k + \gamma_{rj}^k\xi_i^r + \gamma_{ri}^k\xi_j^r - \gamma_{ij}^r\xi_r^k + \xi^r\partial_r\gamma_{ij}^k = 0 \end{cases}$$

and is involutive if and only if $\partial_i A = 0 \Rightarrow A = cst$. Introducing for convenience the *metric density* $\hat{\omega}_{ij} = \omega_{ij} \big/ \left(\left|det(\omega)\right|\right)^{\frac{1}{n}}$, we obtain the *Medolaghi form* with the geometric objects $(\hat{\omega}, \gamma)$:

$$\begin{cases} \hat{\Omega}_{ij} \equiv \hat{\omega}_{rj}\xi_i^r + \hat{\omega}_{ir}\xi_j^r - \frac{2}{n}\hat{\omega}_{ij}\xi_r^r + \xi^r\partial_r\hat{\omega}_{ij} = 0 \\ \Gamma_{ij}^k \equiv \xi_{ij}^k + \gamma_{rj}^k\xi_i^r + \gamma_{ri}^k\xi_j^r - \gamma_{ij}^r\xi_r^k + \xi^r\partial_r\gamma_{ij}^k = 0 \end{cases}$$

Finally, the conformal system $\hat{R}_2 \subset J_2(T)$ is defined by the following infinitesimal Lie equations:

$$\begin{cases} \omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r + \xi^r\partial_r\omega_{ij} = 2A(x)\omega_{ij} \\ \xi_{ij}^k + \gamma_{rj}^k\xi_i^r + \gamma_{ri}^k\xi_j^r - \gamma_{ij}^r\xi_r^k + \xi^r\partial_r\gamma_{ij}^k = \delta_i^k A_j(x) + \delta_j^k A_i(x) - \omega_{ij}\omega^{kr}A_r(x) \end{cases}$$

and is involutive if and only if $\partial_i A - A_i = 0$ or, equivalently, if ω has vanishing Weyl tensor ([7] [26]).

However, introducing again the *metric density* $\hat{\omega}$ while substituting, we obtain after prolongation and division by $\left(\left|det(\omega)\right|\right)^{\frac{1}{n}}$ the second order system $\hat{R}_2 \subset J_2(T)$ in *Medolaghi form* with geometric objects $(\hat{\omega}, \hat{\gamma}) \simeq j_1(\hat{\omega})$ such that:

$$\hat{\gamma}_{ij}^k = \gamma_{ij}^k - \frac{1}{n}\left(\delta_i^k\gamma_{rj}^r + \delta_j^k\gamma_{ri}^r - \omega_{ij}\omega^{ks}\gamma_{rs}^r\right) \Rightarrow \hat{\gamma}_{ri}^r = 0$$

$$\begin{cases} \hat{\Omega}_{ij} \equiv \hat{\omega}_{rj}\xi_i^r + \hat{\omega}_{ir}\xi_j^r - \frac{2}{n}\hat{\omega}_{ij}\xi_r^r + \xi^r\partial_r\hat{\omega}_{ij} = 0 \\ \hat{\Gamma}_{ij}^k \equiv \xi_{ij}^k - \frac{1}{n}\left(\delta_i^k\xi_{rj}^r + \delta_j^k\xi_{ri}^r - \hat{\omega}_{ij}\hat{\omega}^{kr}\xi_{rs}^s\right) + \hat{\gamma}_{rj}^k\xi_i^r + \hat{\gamma}_{ri}^k\xi_j^r - \hat{\gamma}_{ij}^r\xi_r^k + \xi^r\partial_r\hat{\gamma}_{ij}^k = 0 \end{cases}$$

Contracting the first equations by $\hat{\omega}^{ij}$ we notice that ξ_r^r is no longer vanishing while, contracting in k and j the second equations, we now notice that ξ_{ri}^r is no longer vanishing. It is also essential to notice that the symbols $\hat{g}_1$ and $\hat{g}_2$ only depend on ω and not on any conformal factor. We let the reader exhibit similarly the finite *Lie forms* of the previous equations that will be presented when needed. We have to distinguish the strict inclusions $\Gamma \subset \tilde{\Gamma} \subset \hat{\Gamma} \subset aut(X)$ with:

- The Lie pseudogroup $\Gamma \subset aut(X)$ of isometries which is preserving the metric $\omega \in S_2T^*$ with $det(\omega) \neq 0$ *and thus also* γ.
- The Lie pseudogroup $\tilde{\Gamma}$ which is preserving $\hat{\omega}$ *and* γ.

- The Lie pseudogroup $\hat{\Gamma}$ of conformal isometries which is preserving $\hat{\omega}$ *and thus also* $\hat{\gamma}$.

3.2. Gravitation

In ([7] [29] [35]), we have proved that the EM field $F \in \wedge^2 T^*$ could be described by $n(n-1)/2$ components of the bundle $T^* \otimes \hat{g}_2$ of 1-forms with value in the conformal symbol $\hat{g}_2$, which is a sub-bundle of the first Spencer bundle for the conformal group described by the bundle $T^* \otimes \hat{R}_2$ of 1-forms with value in the Lie algebroid $\hat{R}_2$, with no relation at all with the second Spencer bundle $\wedge^2 T^* \otimes \hat{R}_2$ that can be identified with the Cartan curvature. Similarly, in this subsection 2, which is by far the most difficult of the whole paper because third order jets are involved, we shall prove that the substitute for the Riemann curvature is only described by $n(n+1)/2$ other linearly independent components of $T^* \otimes \hat{g}_2 \subset T^* \otimes \hat{R}_2$ in such a way that
$n(n-1)/2 + n(n+1)/2 = n^2 = dim\left(T^* \otimes \hat{g}_2\right)$.

Let us start with a preliminary mathematical comment, independently of what has already been said, and explain the main differences existing between the initial part of the Janet sequence for a formally integrable system
$C_0 = R_q \subset J_q(E) = C_0(E)$ with a 2-acyclic symbol $g_q \subset S_q T^* \otimes E$ such that $g_{q+1} = 0$ and the initial part of the corresponding Spencer sequence for the first order involutive system $R_{q+1} \subset J_1(R_q)$. First of all, we recall the following commutative diagram with short exact vertical sequences, only depending on the left lower commutative square:

$$
\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \to & \Theta & \underset{2}{\overset{j_q}{\to}} & C_0 & \underset{1}{\overset{D_1}{\to}} & C_1 & \underset{1}{\overset{D_2}{\to}} & C_2 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \to & E & \overset{j_q}{\to} & C_0(E) & \underset{1}{\overset{D_1}{\to}} & C_1(E) & \underset{1}{\overset{D_2}{\to}} & C_2(E) \\
 & & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \\
0 & \to & \Theta & \to & E & \underset{q}{\overset{\mathcal{D}}{\to}} & F_0 & \underset{1}{\overset{\mathcal{D}_1}{\to}} & F_1 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & &
\end{array}
$$

In this diagram, $\Phi = \Phi_0$ is defined by the canonical projection
$\Phi : J_q(E) \to J_q(E)/R_q = F_0$ with kernel R_q and
$F_1 = T^* \otimes J_q(E) \Big/ \left(T^* \otimes R_q + \delta\left(S_{q+1} T^* \otimes E\right)\right)$ is induced by Φ_0 after *only one* (care) prolongation. As D_1 is a first order operator because it is induced by the Spencer operator, it is essential to notice that such a result is coming from the fact that $\mathcal{D}_1$ is of order 1 because R_q is formally integrable and g_q is 2-acyclic (see [7], p. 116, 120, 165). This very delicate result cannot be extended to the right with $\mathcal{D}_2 : F_1 \to F_2$ unless g_q is involutive, a situation fulfilled by j_q

which is an involutive injective operator. Also the first order operator $D_1 : R_q \to J_1(R_q)/R_{q+1} = C_1 \simeq T^* \otimes R_q/\delta(g_{q+1}) = T^* \otimes R_q$ is trivially involutive because $g_{q+1} = 0$ and $C_1 \subset C_1(E)$ while $C_2 = \wedge^2 T^* \otimes R_q \subset C_2(E)$. Hence, the upper sequence is formally exact, a result that can be extended to the right side (see [28] for a nice counterexample). From a snake chase in this diagram, it follows that the (local) cohomology at C_1 in the upper sequence is the same as the (local) cohomology at F_0 in the lower sequence though there is no link at all between C_1 and F_0. Moreover, in the present situation, we have an isomorphism $R_{q+1} \simeq R_q$ and obtain therefore $D_1\xi_q = d\xi_{q+1}$, $\forall \xi_q \in R_q$.

For helping the reader, we provide the two long exact sequences allowing to define C_1 and C_2 in the Spencer sequence while proving the formal exactness of the upper sequence on the jet level if we set $J_r(E) = 0, \forall r < 0$ and $J_0(E) = E$ for any vector bundle E:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \to & S_{r+1}T^* \otimes C_0 & \to & S_r T^* \otimes C_1 & \to & S_{r-1}T^* \otimes C_2 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & R_{q+r+1} & \to & J_{r+1}(C_0) & \to & J_r(C_1) & \to & J_{r-1}(C_2) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & R_{q+r} & \to & J_r(C_0) & \to & J_{r-1}(C_1) & \to & J_{r-2}(C_2) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

It just remains to apply inductively the Spencer δ-operator to the various upper symbol sequences obtained by successive prolongations, starting from the case $r = 0$ already considered.

Similarly, if we define F_2 in the Janet sequence by the following commutative and exact diagram:

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & 0 & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & \to & S_{q+2}T^* \otimes C_0 & \to & S_2 T^* \otimes F_0 & \to & T^* \otimes F_1 & \to & F_2 & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & \\
0 & \to & R_{q+2} & \to & J_{q+2}(E) & \to & J_2(F_0) & \to & J_1(F_1) & \to & F_2 & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & R_{q+1} & \to & J_{q+1}(E) & \to & J_1(F_0) & \to & F_1 & \to & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
 & & 0 & & 0 & & 0 & & 0 & & &
\end{array}
$$

we have $F_2 \simeq C_2(E)/C_2$. If we apply the Spencer δ-operator to the long symbol sequence:

$$0 \to S_{q+3}T^* \otimes E \to S_3 T^* \otimes F_0 \to S_2 T^* \otimes F_1 \to T^* \otimes F_2$$

we discover, through a standard snake diagonal chase, that such a sequence may not be exact at $S_2 T^* \otimes F_1$ with a cohomology equal to $H^3(g_q)$ *that may not vanish*.

With $n=4, q=2$ and the conformal system $\hat{R}_2 \subset J_2(T)$, we provide below the fiber dimensions:

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & 15 & \underset{1}{\overset{D_1}{\rightarrow}} & 60 & \underset{1}{\overset{D_2}{\rightarrow}} & 90 \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 4 & \overset{j_2}{\rightarrow} & 60 & \underset{1}{\overset{D_1}{\rightarrow}} & 160 & \underset{1}{\overset{D_2}{\rightarrow}} & 180 \\
 & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \\
 & & 4 & \underset{2}{\overset{\mathcal{D}}{\rightarrow}} & 45 & \underset{1}{\overset{\mathcal{D}_1}{\rightarrow}} & 100 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Now $H^3(\hat{g}_2) \neq 0$ when $n=4$ as $\hat{g}_2$ is 3-acyclic only when $n \geq 5$ but no classical approach could even allow to imagine such a specific cohomological importance of $n=4$ ([17], pp. 26-28).

The *large infinitesimal equivalence principle* initiated by the Cosserat brothers becomes natural in this framework, namely an observer cannot measure sections of R_q but can only measure their images by D_1 or, equivalently, can only measure sections of C_1 killed by D_2. Accordingly, for a free falling particle in a constant gravitational field, we have successively:

$$\boxed{\partial_4 \xi^k - \xi_4^k = 0, \partial_4 \xi_4^k - \xi_{44}^k = 0, \partial_i \xi_{44}^k - 0 = 0, 1 \leq i, k \leq 3}$$

This result explains why the elations are sometimes called "accelerations" by physicists ([34]).

Our purpose is now to extend these comments to the nonlinear sequences and we start with a few useful but technical local computations ([29]). First of all, we may define:

$$\chi_{l,i}^k = g_u^k \left(\partial_i f_l^u - A_i^r f_{rl}^u\right) \Rightarrow \tau_{r,s}^k = B_s^i \chi_{r,i}^k = g_u^k \left(B_s^i \partial_i f_r^u - f_{rs}^u\right) = T_{r,s}^k - \Gamma_{rs}^k$$

$$A_i^r A_j^s \left(\tau_{r,s}^k - \tau_{s,r}^k\right) = \partial_i A_j^k - \partial_j A_i^k \Rightarrow \tau_{r,s}^k - \tau_{s,r}^k = B_r^i B_s^j \left(\partial_i A_j^k - \partial_j A_i^k\right)$$

LEMMA 3.2.1: Summing on k and r when $\gamma = 0$, we get successively:

$$\begin{aligned}
\left(\tau_{i,r}^r - \tau_{r,i}^r\right) det(A) &= B_i^r B_k^j \left(\partial_r A_j^k - \partial_j A_r^k\right) det(A) \\
&= B_i^r \left(B_k^j \partial_r A_j^k + A_r^j \partial_s B_j^s\right) det(A) \\
&= B_i^r \left(B_k^j \partial_r A_j^k\right) det(A) + det(A) \partial_r B_i^r \\
&= B_i^r \partial_r det(A) + det(A) \partial_r B_i^r \\
&= \partial_r \left(B_i^r det(A)\right)
\end{aligned}$$

$$\begin{aligned}
-\omega^{ij} \partial_r \left(B_i^r det(A)\right) \xi_{sj}^s - \omega^{ij} \tau_{j,i}^r det(A) \xi_{sr}^s &= \left[\omega^{ij} \left(\tau_{r,i}^r - \tau_{i,r}^r\right) \xi_{sj}^s - \omega^{ij} \tau_{j,i}^r \xi_{sr}^s\right] det(A) \\
&= \left[\omega^{ij} \tau_{r,i}^r - \left(\omega^{ij} \tau_{i,r}^r + \omega^{ir} \tau_{r,i}^j\right)\right] det(A) \xi_{sj}^s
\end{aligned}$$

$$=\left[\omega^{ij}\tau^r_{r,i}-\left(\omega^{ij}\tau^r_{i,r}+\omega^{ir}\tau^j_{i,r}\right)\right]det(A)\xi^s_{sj}$$
$$=\left[\omega^{ij}\tau^r_{r,i}-\frac{2}{n}\omega^{jr}\tau^t_{t,r}\right]det(A)\xi^s_{sj}$$
$$=\frac{n-2}{n}\omega^{ij}\tau^r_{r,i}det(A)\xi^s_{sj}$$

Using the "*vertical machinery*", namely the isomorphism $V(J_q(\mathcal{E}))\simeq J_q(V(\mathcal{E}))$, like in the preceding sections, we shall vary the sections $\delta f_q=(\delta f^k_\mu(x))$ while setting $\delta(\partial_i f^k_\mu(x))=\partial_i\delta(f^k_\mu(x))$ as it is done in analytical mechanics with the notations $\delta\dot{q}=\dot{\delta q}$ when studying the variation of a Lagrangian $L(t,q,\dot{q})$.

LEMMA 3.2.2: Let us compute directly the variation of the 1-form α over the target and over the source, recalling that $\alpha=\alpha_i dx^i$ with $\alpha_i=\chi^r_{r,i}=g^r_k\partial_i f^k_r-A^r_i g^s_k f^k_{rs}=n(\partial_i a-A^r_i a_r)$ and $na_i=g^r_k f^k_{ri}$. We have successively:

$$\delta f^k=\eta^k=\xi^r\partial_r f^k,\quad \delta f^k_i=\eta^k_u f^u_i=\xi^r\partial_r f^k_i+f^k_r\xi^r_i$$
$$\delta f^k_{ij}=\eta^k_{uv}f^u_i f^v_j+\eta^k_u f^u_{ij}=\xi^r\partial_r f^k_{ij}+f^k_{rj}\xi^r_i+f^k_{ri}\xi^r_j+f^k_r\xi^r_{ij}$$
$$n\delta a_i=g^r_k\delta f^k_{ri}+f^k_{ri}\delta g^r_k=g^r_k\left(\eta^k_{uv}f^u_i f^v_r+\eta^k_u f^u_{ir}\right)-f^u_{ri}g^r_k\eta^k_u=f^r_i\eta^s_{sr}$$
$$n\delta a_i=g^s_k\left(\xi^r\partial_r f^k_{is}+f^k_{rs}\xi^r_i+f^k_{ri}\xi^r_s+f^k_r\xi^r_{si}\right)-f^u_{si}g^s_k\left(g^t_u\left(\xi^r\partial_r f^k_t+f^k_r\xi^r_t\right)\right)$$
$$=n\left(\xi^r\partial_r a_i+a_r\xi^r_i\right)+\xi^r_{ri}$$
$$a_i=f^k_i b_k\Rightarrow n\delta a_i=f^r_i\eta^s_{sr}=nf^k_i\delta b_k+nb_k\delta f^k_i=nf^k_i\delta b_k+nb_k f^u_i\eta^k_u$$
$$\Rightarrow f^k_i\eta^s_{sk}=nf^k_i\delta b_k+nb_r f^k_i\eta^r_k$$
$$\Rightarrow n\delta b_k=\eta^r_{rk}-nb_r\eta^r_k$$

Then, using the definition of *a*, namely $det(f^k_i)=e^{na}$, we have:

$$n\delta a=\left(1/det(f^k_i)\right)\delta det(f^k_i)=g^i_k\delta f^k_i=\eta^s_s=g^i_k\left(\xi^r\partial_r f^k_i+f^k_r\xi^r_i\right)$$
$$=n\xi^r\partial_r a+\xi^r_r$$

Using the variation $\delta A^k_i=\xi^r\partial_r A^k_i+A^k_r\partial_i\xi^r-A^r_i\xi^k_r$, we finally get:

$$\delta\alpha_i=n\delta\partial_i a-nA^r_i\delta a_r-na_r\delta A^r_i$$
$$=\left(\partial_i\xi^r_r-\xi^r_{ri}\right)+\left(\xi^r\partial_r\alpha_i+\alpha_r\partial_i\xi^r\right)-\chi^s_{,i}\xi^r_{rs}$$
$$=\left(\partial_i\xi^r_r-A^s_i\xi^r_{rs}\right)+\left(\xi^r\partial_r\alpha_i+\alpha_r\partial_i\xi^r\right)$$

The terms $\partial_i\xi^r_r+\left(\xi^r\partial_r\alpha_i+\alpha_r\partial_i\xi^r\right)$ of the variation, including the variation of $\alpha=\alpha_i dx^i$ as a 1-form, are *exactly* the ones introduced by Weyl in ([3] formula (76), p. 289). We also recognize the variation δA_i of the 4-potential used by engineers now expressed by means of second order jets.

We have over the target:

$$f^k_r A^r_i=\partial_i f^k\Rightarrow f^k_r\delta A^r_i+A^r_i\eta^k_u f^u_r=\frac{\partial\eta^k}{\partial y^u}\partial_i f^u\Rightarrow\delta A^r_i=g^r_l\left(\frac{\partial\eta^l}{\partial y^k}-\eta^l_k\right)\partial_i f^k$$

$$\delta\alpha_i = \left[\frac{\partial\eta_s^s}{\partial y^k} - ng_l^r\left(\frac{\partial\eta^l}{\partial y^k} - \eta_k^l\right)a_r\right]\partial_i f^k - A_i^r f_r^k \eta_{sk}^s$$
$$= \left[\left(\frac{\partial\eta_s^s}{\partial y^k} - \eta_{sk}^s\right) - nb_l\left(\frac{\partial\eta^l}{\partial y^k} - \eta_k^l\right)\right]\partial_i f^k$$

a result only depending on the components of the Spencer operator, in a coherent way with the general variational formulas that could have been used otherwise. We notice that these formulas, which have been obtained with difficulty for second order jets, could not even be obtained by hand for third order jets. They show the importance and usefulness of the general formulas providing the Spencer non-linear operators for an arbitrary order, in particular for the study of the conformal group which is defined by second order lie equations with a 2-acyclic symbol. It is also important to notice that:

$$\alpha_i = n\left(\frac{\partial b}{\partial y^k} - b_k\right)\partial_i f^k = \beta_k \partial_i f^k$$
$$\Rightarrow \delta\alpha_i = (\delta\beta_k)\partial_i f^k + \beta_k\frac{\partial\eta^k}{\partial y^u}\partial_i f^u = \left(\delta\beta_k + \beta_r\frac{\partial\eta^r}{\partial y^k}\right)\partial_i f^k$$

and thus $\delta\beta$ does not only depend linearly on the Spencer operator, contrary to $\delta\alpha$.

LEMMA 3.2.3: We have over the source:

$$\delta det(A) = det(A)B_k^i \delta A_i^k$$
$$= det(A)B_k^i\left(\xi^r\partial_r A_i^k + A_r^k\partial_i\xi^r - A_i^r\xi_r^k\right) \quad \square$$
$$= \xi^r\partial_r\left(det(A)\right) + det(A)\left(\partial_r\xi^r - \xi_r^r\right)$$

Now, we recall the identities:

$$\boxed{\partial_i\chi_{l,j}^k - \partial_j\chi_{l,i}^k - \chi_{l,i}^r\chi_{r,j}^k + \chi_{l,j}^r\chi_{r,i}^k - A_i^r\chi_{lr,j}^k + A_j^r\chi_{lr,i}^k = 0}$$

that we may rewrite in the equivalent form:

$$\boxed{\begin{aligned}\tau_{lr,s}^k - \tau_{ls,r}^k &= B_r^i B_s^j\left(\partial_i\chi_{l,j}^k - \partial_j\chi_{l,i}^k - \chi_{l,i}^r\chi_{r,j}^k + \chi_{l,j}^r\chi_{r,i}^k\right)\\ &= B_r^i B_s^j\left(\partial_i\chi_{l,j}^k - \partial_j\chi_{l,i}^k\right) - \left(\tau_{l,r}^t\tau_{t,s}^k - \tau_{l,s}^t\tau_{t,r}^k\right)\end{aligned}}$$

Looking only at the terms not containing the jets of order 2 in the right member, *we have separately*:

$$B_r^i B_s^j\left(\partial_i\left(g_u^k\partial_j f_l^u\right) - \partial_j\left(g_u^k\partial_i f_l^u\right)\right) = B_r^i B_s^j\left(\left(\partial_i g_u^k\right)\left(\partial_j f_l^u\right) - \left(\partial_j g_u^k\right)\left(\partial_i f_l^u\right)\right)$$

$$\left(g_u^t B_r^i\partial_i f_l^u\right)\left(g_v^k B_s^i\partial_i f_t^v\right) - (r\leftrightarrow s) = B_r^i B_s^j\left(\left(g_u^t\partial_i f_l^u\right)\left(g_v^k\partial_j f_t^v\right)\right) - (r\leftrightarrow s)$$
$$= -\left(B_r^i B_s^j\left(g_u^t\partial_i f_l^u\right)\left(f_t^v\partial_j g_v^k\right) - (r\leftrightarrow s)\right)$$
$$= -\left(B_r^i B_s^j\left(\partial_j g_u^k\right)\left(\partial_i f_l^u\right) - (r\leftrightarrow s)\right)$$

and the total sum vanishes.

Looking at the terms linear in the second order jets $g_u^k f_{ij}^u$, we have *separately* (care to the sign):

$$B_r^i B_s^j \left(\partial_j A_i^t - \partial_i A_j^t\right) g_u^k f_{tl}^u = \left(\tau_{r,s}^t - \tau_{s,r}^t\right) g_u^k f_{tl}^u = g_v^t \left(B_s^i \partial_i f_r^v - B_r^i \partial_i f_s^v\right) g_u^k f_{tl}^u$$

$$\left(g_u^t B_r^i \left(\partial_i f_l^u\right) g_v^k f_{st}^v + g_v^k B_s^j \left(\partial_j f_t^v\right) g_u^t f_{rl}^u\right) - (r \leftrightarrow s) \tag{12}$$

The simplest and final checking concerns *the derivatives of the second order jets*. We get:

$$\begin{aligned} B_r^i B_s^j \left(\partial_i \chi_{l,j}^k - \partial_j \chi_{l,i}^k\right) &= B_r^i B_s^j \left(A_i^t \partial_j \left(g_u^k f_{tl}^u\right) - A_j^t \partial_i \left(g_u^k f_{tl}^u\right)\right) + \cdots \\ &= B_s^j \partial_j \left(g_u^k f_{rl}^u\right) - B_r^i \partial_i \left(g_u^k f_{sl}^u\right) + \cdots \end{aligned} \tag{13}$$

with $y = f(x) \leftrightarrow x = g(y)$, it remains to substitute the formulas $B_r^i = f_r^k \partial g^i / \partial y^k$ while introducing the finite Lie equations:

$$\Phi_{ij}^k \equiv g_u^k \left(f_{ij}^u + \gamma_{rs}^u \left(f(x)\right) f_i^r f_j^s\right) = \gamma_{ij}^k(x) + \delta_i^k a_j + \delta_j^k a_i - \omega_{ij} \omega^{kr} a_r$$

and setting $\gamma = 0$ in the conformal case, a local result leading to $\Phi_{ij}^k = \delta_i^k a_j + \delta_j^k a_i - \omega_{ij} \omega^{kr} a_r$ in $S_2 T^* \otimes T$ which only depends on the Minkowski metric ω and not on a conformal factor.

The novelty and most tricky point is to notice that we have now only n^2 components for $\left(\tau_{li,j}^k\right) \in T^* \otimes \hat{g}_2$ and no longer the $n^2\left(n^2-1\right)/12$ components of the Riemannian curvature.

As we have already used the $n(n-1)/2$ components $\varphi_{ij} = \tau_{ri,j}^r - \tau_{rj,i}^r = -\varphi_{ji}$ for describing EM in ([29]), we may choose the $n(n+1)/2$ symmetric components $\tau_{ij} = \frac{1}{2}\left(\tau_{ri,j}^r + \tau_{rj,i}^r\right) = \tau_{ji}$ that should involve the third order jets which are only vanishing in the linear case but do not vanish at all in the non-linear case. To avoid such a situation, we shall use the following key proposition that must be compared to the procedure used in classical GR for defining the $Ricci = S_2 T^*$, $Riemann = H_1^2(g_1)$ and $Weyl = H_1^2(\hat{g}_1)$ tensor bundles along the *Fundamental diagram II* that only depends on the Spencer δ-cohomology through the second order symbol $\hat{g}_2$ allowing to define the elations and has been provided for the first time in 1983 ([4]):

$$\begin{array}{ccccccccc}
 & & & & & & & 0 & \\
 & & & & & & & \downarrow & \\
 & & & & & 0 & & Ricci & \\
 & & & & & \downarrow & & \downarrow & \\
 & & & 0 & \to & Z_1^2(g_1) & \to & Riemann & \to 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \to & T^* \otimes \hat{g}_2 & \xrightarrow{\delta} & Z_1^2(\hat{g}_1) & \to & Weyl & \to 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & S_2 T^* & \xrightarrow{\delta} & T^* \otimes T^* & \xrightarrow{\delta} & \wedge^2 T^* & \to & 0 & \\
 & & & \downarrow & & \downarrow & & & \\
 & & & 0 & & 0 & & &
\end{array}$$

It is important to notice that *this diagram splits and does not depend on any* *conformal factor*.

PROPOSITION 3.2.4: Defining $\rho^k_{l,ij}=\tau^k_{li,j}-\tau^k_{lj,i}$ it is just sufficient to study $\rho_{i,j}=\rho^r_{i,rj}\neq\rho_{j,i}$ and $tr(\rho)=\omega^{ij}\rho_{i,j}$ or $\tau_{i,j}=\tau^r_{ri,j}$ and $tr(\tau)=\omega^{ij}\tau_{i,j}$. Setting:

$$\varphi_{ij}=\tau^r_{ri,j}-\tau^r_{rj,i},\ \tau_{ij}=\frac{1}{2}\left(\tau^r_{ri,j}+\tau^r_{rj,i}\right),\ \rho_{ij}=\frac{n-2}{n}\tau_{ij}+\frac{1}{n}\omega_{ij}tr(\tau)$$

in a way not depending on any conformal factor, we have the equivalences:

$$\tau^k_{li,j}=0\Leftrightarrow\rho^k_{l,ij}\Leftrightarrow\varphi_{ij}=0\oplus\tau_{ij}=0\Leftrightarrow\varphi_{ij}=0\oplus\rho_{ij}=0$$

Proof: As $\hat{g}_2\simeq T^*$, we have successively:

$$n\rho^k_{l,ij}=\delta^k_l\tau^r_{ri,j}+\delta^k_i\tau^r_{rl,j}-\omega_{li}\omega^{ks}\tau^r_{rs,j}-\delta^k_l\tau^r_{rj,i}-\delta^k_j\tau^r_{rl,i}+\omega_{lj}\omega^{ks}\tau^r_{rs,i}$$
$$\Rightarrow\rho^r_{r,ij}=\tau_{i,j}-\tau_{j,i}$$
$$n\rho_{i,j}=(n-1)\tau^r_{ri,j}-\tau^r_{rj,i}+\omega_{ij}\omega^{st}\tau^r_{rs,t}=(n-1)\tau_{i,j}-\tau_{j,i}+\omega_{ij}tr(\tau)$$
$$\Rightarrow\rho_{i,j}-\rho_{j,i}=\tau_{i,j}-\tau_{j,i}$$
$$ntr(\rho)=2(n-1)\omega^{ij}\tau^r_{ri,j}=2(n-1)tr(\tau)$$

When we suppose that there is no EM, that is:

$$\varphi_{ij}=0\Leftrightarrow\tau_{i,j}=\tau_{j,i}=\tau_{ij}=\tau_{ji}\Leftrightarrow\rho_{i,j}=\rho_{j,i}=\rho_{ij}=\rho_{ji}$$

the above formulas become simpler with:

$$n\rho_{ij}=(n-2)\tau_{ij}+\omega_{ij}tr(\tau)\Leftrightarrow\tau_{ij}=\frac{n}{n-2}\rho_{ij}-\frac{n}{2(n-1)(n-2)}\omega_{ij}tr(\rho)$$

In the general situation, we have:

$$n\rho_{i,j}=(n-1)\tau_{i,j}-\tau_{j,i}+\omega_{ij}tr(\tau),\ n\rho_{j,i}=(n-1)\tau_{j,i}-\tau_{i,j}+\omega_{ij}tr(\tau)$$

Surprisingly while summing, we discover that the same formula is still valid.

We may thus express $\rho^k_{l,ij}$ by means of $\rho_{i,j}$ or by means of $\tau_{i,j}$ while using the relations $\varphi_{ij}=\rho^r_{r,ij}=\tau_{i,j}-\tau_{j,i}=\rho_{i,j}-\rho_{j,i}$. As $\hat{g}_2$ is 2-acyclic when $n\geq 4$ in the conformal case ([17]), we have the short exact sequence:

$$0\rightarrow\hat{g}_3\xrightarrow{\delta}T^*\otimes\hat{g}_2\xrightarrow{\delta}\delta\left(T^*\otimes\hat{g}_2\right)\rightarrow 0$$

Moreover, as $\hat{g}_3=0$ when $n\geq 3$, we have an isomorphism $T^*\otimes\hat{g}_2\simeq\delta\left(T^*\otimes\hat{g}_2\right)$, both vector bundles having the same fiber dimension $n^2=\frac{n(n-1)}{2}+\frac{n(n+1)}{2}$ when $n\geq 4$ and thus $\tau^k_{li,j}=0\Leftrightarrow\rho^k_{l,ij}=0$.

When there is no EM, that is when $\varphi=0$, then one can express $\rho^k_{l,ij}$ by means of $\rho_{ij}=\rho_{i,j}=\rho_{j,i}=\rho_{ji}$ but there is no longer the Levi-Civita isomorphism $(\omega,\gamma)\simeq j_1(\omega)$ in the Spencer sequence and the above proposition is quite different from the concept of curvature in GR as it just amounts to the vanishing of the Weyl tensor. □

We notice that no one of the preceding results could be obtained by classical methods because they crucially depend on the Spencer δ-cohomology. As a by-product, the same formulas provide:

COROLLARY 3.2.5: The corresponding Weyl tensor vanishes.

Supposing again that there is no EM and looking for the derivatives of the second order jets, contracting in k and r while replacing l by i and s by j, we get with $a_i = f_i^k b_k$:

$$\begin{aligned}\rho_{ij} = \rho_{ji} &= \tau^r_{ri,j} - \tau^r_{ij,r} = B^t_j \partial_t \left(g^r_u f^u_{ri} \right) - B^t_r \partial_t \left(g^r_u f^u_{ij} \right) + \cdots \\ &= n B^t_j \partial_t a_i - B^t_r \partial_t \left(\delta^r_i a_j + \delta^r_j a_i - \omega_{ij} \omega^{rs} a_s \right) + \cdots \\ &= n f^l_j \frac{\partial a_i}{\partial y^l} - f^k_i \frac{\partial a_j}{\partial y^k} - f^l_j \frac{\partial a_i}{\partial y^l} + \omega_{ij} \omega^{rs} f^k_r \frac{\partial a_s}{\partial y^k} + \cdots \\ &= f^k_i f^l_j \left[(n-2) \frac{\partial b_k}{\partial y^l} + \omega_{kl}(y) \omega^{rs}(y) \frac{\partial b_s}{\partial y^r} \right] + \cdots\end{aligned}$$

with a bracket symmetric under the exchange of k and l that has been obtained by using the fact that we have for example:

$$f^l_j \frac{\partial a_i}{\partial y^l} = f^l_j \frac{\partial \left(f^k_i b_k \right)}{\partial y^l} = f^k_i f^l_j \frac{\partial b_k}{\partial y^l} + f^l_j \frac{\partial f^k_i}{\partial y^l} b_k$$

We have thus to take into account the following terms linear in the b_k, left aside in the derivations:

$$\begin{aligned}&\left[(n-1) f^l_j \frac{\partial f^k_i}{\partial y^l} + f^l_i \frac{\partial f^k_j}{\partial y^l} + \omega_{ij} \omega^{rs} f^l_r \frac{\partial f^k_s}{\partial y^l} \right] b_k \\ &= \left[(n-1) B^t_j \partial_t f^k_i + B^t_i \partial_t f^k_j + \omega_{ij} \omega^{rs} f^l_r \partial_t f^k_s \right] b_k\end{aligned}$$

Under the same assumption, let us work out the quadratic terms in b_k as follows:

$$\left(\tau^t_{l,r} \tau^k_{t,s} - \tau^t_{l,s} \tau^k_{t,r} \right) = \left(g^t_u f^u_{rl} \right) \left(g^k_v f^v_{st} \right) - \left(g^t_u f^u_{sl} \right) \left(g^k_v f^v_{rt} \right)$$

Contracting in k and r as above while replacing l by i and s by j, we get:

$$\left(\tau^t_{i,r} \tau^r_{t,j} - \tau^t_{i,j} \tau^r_{t,r} \right) = \left(g^t_u f^u_{ri} \right) \left(g^r_v f^v_{jt} \right) - \left(g^t_u f^u_{ij} \right) \left(g^r_v f^v_{rt} \right)$$

that is:

$$\left(\delta^t_r a_i + \delta^t_i a_r - \omega_{ri} \omega^{st} a_s \right) \left(\delta^r_j a_t + \delta^r_t a_j - \omega_{jt} \omega^{rs} a_s \right) - n \left(\delta^t_i a_j + \delta^t_j a_i - \omega_{ij} \omega^{st} a_s \right) a_t$$

Effecting all the contractions, we get:

$$\left(n a_i a_j \right) + \left(2 a_i a_j - \omega_{ij} \omega^{rs} a_r a_s \right) - \left(\omega_{ij} \omega^{rs} a_r a_s \right) - n \left(2 a_i a_j - \omega_{ij} \omega^{rs} a_r a_s \right)$$

and obtain the unexpected very simple formula:

$$\begin{aligned}&n a_i a_j + 2 a_i a_j - 2 \omega_{ij} \omega^{rs} a_r a_s - 2n a_i a_j + n \omega_{ij} \omega^{rs} a_r a_s \\ &= (2-n) a_i a_j + (n-2) \omega_{ij} \omega^{rs} a_r a_s\end{aligned}$$

or, equivalently $f^k_i f^l_j \left[(2-n) b_k b_l + (n-2) \omega_{kl}(y) \omega^{rs} b_r b_s \right]$. Collecting these results, we finally get:

THEOREM 3.2.6: When there is no EM, we have *over the target* the formulas:

$$\boxed{\rho_{ij} = f^k_i f^l_j \left[(n-2) \frac{\partial b_k}{\partial y^l} + \omega_{kl}(y) \omega^{rs}(y) \frac{\partial b_s}{\partial y^r} + (n-2) b_k b_l - (n-2) \omega_{kl}(y) \omega^{rs} b_r b_s \right]}$$

$$\boxed{\tau_{ij} = nf_i^k f_j^l \left[\frac{\partial b_k}{\partial y^l} + b_k b_l - \frac{1}{2}\omega_{kl}(y)\omega^{rs}(y) b_r b_s \right]}$$

that *do not depend on any conformal factor for* ω and thus simply:

$$\boxed{\tau = \frac{n}{\Theta^2}\left[\omega^{kl}(y)\frac{\partial b_k}{\partial y^l} - \frac{n-2}{2}\omega^{kl}(y) b_k b_l\right] = n\left[\bar{\omega}^{kl}(y)\frac{\partial b_k}{\partial y^l} - \frac{n-2}{2}\bar{\omega}^{kl}(y) b_k b_l\right]}$$

that only depends on the new metric $\bar{\omega} = \Theta^2 \omega$ defined over the target.

Proof: We have to prove the following technical result which is indeed the hardest step of this paper, namely that ρ_{ij} does not contain terms linear in b_k *over the target*. The main problem is that, if we have any derivative of the second order jets *over the source*, like $\partial_r a_i$, we obtain therefore a term like $\partial_r\left(f_i^k b_k\right) = f_i^k \partial_r b_k + \left(\partial_r f_i^k\right) b_k$ which is bringing a term linear in the b_k and we have to prove that *such terms may not exist if we work only over the target*.

For this, let us set over the source when ω is the Minkowski metric with $\gamma = 0$:

$$\boxed{\tau_{s,r}^k = T_{s,r}^k - \Phi_{rs}^k,\ T_{s,r}^k = g_u^k B_r^i \partial_i f_s^u \neq T_{r,s}^k,\ \Phi_{rs}^k = \delta_r^k a_s + \delta_s^k a_r - \omega_{rs}\omega^{kt} a_t = \Gamma_{sr}^k}$$

Looking for the derivatives of the second order jets, we already saw in (13) that they can only appear through the terms:

$$B_s^i \partial_i \Phi_{rl}^k - B_r^i \partial_i \Phi_{sl}^k = f_s^v \frac{\partial \Phi_{rl}^k}{\partial y^v} - f_r^u \frac{\partial \Phi_{sl}^k}{\partial y^u}$$

Contracting in k and r, we get when there is no EM:

$$\begin{aligned} f_s^v \frac{\partial \Phi_{rl}^r}{\partial y^v} - f_r^u \frac{\partial \Phi_{sl}^r}{\partial y^u} &= f_s^v \frac{\partial}{\partial y^v}(n a_l) - f_r^u \frac{\partial}{\partial y^u}\left(\delta_s^r a_l + \delta_l^r a_s - \omega_{sl}\omega^{rt} a_t\right) \\ &= (n-1) f_s^v \frac{\partial\left(f_l^u b_u\right)}{\partial y^v} - f_l^u \frac{\partial\left(f_s^v b_v\right)}{\partial y^u} + \omega_{sl}\omega^{rt} f_r^u \frac{\partial\left(f_t^v b_v\right)}{\partial y^u} \\ &= (n-2) f_l^u f_s^v \frac{\partial b_v}{\partial y^u} + \omega_{sl}\omega^{rt} f_r^u f_t^v \frac{\partial b_v}{\partial y^u} + \cdots \end{aligned}$$

but we have to take into account the linear terms produced by an integration by parts:

$$(n-1) f_s^v \frac{\partial f_l^u}{\partial y^v} b_u - f_l^u \frac{\partial f_s^v}{\partial y^u} b_v + \omega_{ls}\omega^{rt} f_r^u \frac{\partial f_t^v}{\partial y^u} b_v$$

that is to say, we have to subtract:

$$\begin{aligned} &(n-1) g_u^t B_s^i \partial_i f_l^u a_t - g_v^t B_l^i \partial_i f_s^v a_t + \omega_{ls}\omega^{rt} g_v^u B_r^i \partial_i f_t^v a_u \\ &= (n-1) T_{l,s}^t a_t - T_{s,l}^t a_t - \omega_{ls}\omega^{rt} T_{t,r}^u a_u \end{aligned}$$

Meanwhile, as we already saw in (12), we have to compute:

$$\left(T_{r,s}^t - T_{s,r}^t\right)\Phi_{lt}^k - \left(T_{l,r}^t \Phi_{st}^k + T_{t,s}^k \Phi_{lr}^t\right) + \left(T_{l,s}^t \Phi_{rt}^k + T_{t,r}^k \Phi_{ls}^t\right)$$

and to contract in k and r in order to get:

$$\left(T_{r,s}^t - T_{s,r}^t\right)\Phi_{lt}^r - \left(T_{l,r}^t \Phi_{st}^r + T_{t,s}^r \Phi_{lr}^t\right) + \left(T_{l,s}^t \Phi_{rt}^r + T_{t,r}^r \Phi_{ls}^t\right)$$

However, *two terms are disappearing* and we are left with:

$$-T^t_{s,r}\Phi^r_{lt}-T^t_{l,r}\Phi^r_{st}+\left(T^t_{l,s}\Phi^r_{rt}+T^r_{t,r}\Phi^t_{ls}\right)$$

that is to say:

$$\begin{aligned}&-T^t_{s,r}\left(\delta^r_l a_t+\delta^r_t a_l-\omega_{lt}\omega^{ru}a_u\right)-T^t_{l,r}\left(\delta^r_s a_t+\delta^r_t a_s-\omega_{st}\omega^{ru}a_u\right)\\&+nT^t_{l,s}a_t+T^r_{t,r}\left(\delta^t_l a_s+\delta^t_s a_l-\omega_{ls}\omega^{tu}a_u\right)\end{aligned}$$

and thus:

$$\begin{aligned}&-T^t_{s,l}-T^r_{s,r}a_l+\omega_{lt}\omega^{ru}T^t_{r,s}a_u-T^t_{l,s}a_t-T^t_{l,t}a_s+\omega_{st}\omega^{ru}T^t_{l,r}a_u\\&+nT^t_{l,s}a_t+T^r_{l,r}a_s+T^r_{s,r}a_l-\omega_{ls}\omega^{tu}T^r_{t,r}a_u\end{aligned}$$

The four terms containing a_l and a_s are disappearing and we are left with:

$$(n-1)T^t_{l,s}a_t-T^t_{s,l}a_t\omega_{lt}\omega^{ru}T^t_{r,s}a_u+\omega_{st}\omega^{ru}T^t_{l,r}a_u-\omega_{ls}\omega^{tu}T^r_{t,r}a_u$$

Taking into account *twice successively* the conformal Killing equations, we obtain:

$$\begin{aligned}&(n-1)T^t_{l,s}a_t-T^t_{s,l}a_t+\frac{2}{n}\omega_{ls}\omega^{ru}T^t_{t,r}a_u-\omega_{ls}\omega^{tu}T^r_{t,r}a_u\\&=(n-1)T^t_{l,s}a_t-T^t_{s,l}a_t-\omega_{ls}\omega^{rt}T^u_{t,r}a_u\end{aligned}$$

that is *exactly the terms we had to substract* and there is thus no term linear in a_i in the Ricci tensor over the target, *a quite difficult result indeed because no concept of classical Riemannian geometry could be used.*

We finally obtain from the definition of Θ while taking inverse matrices:

$$\begin{aligned}&\Theta^2\omega_{kl}(y)f^k_i f^l_j=\omega_{ij}(x)\Rightarrow\Theta^{-2}\omega^{kl}(y)g^i_k g^j_l=\omega^{ij}(x)\\&\Rightarrow\Theta^{-2}\omega^{kl}(y)=\omega^{ij}(x)f^k_i f^l_j\end{aligned}$$

and just need to set $\tau=\omega^{ij}\tau_{ij}$ in order to get the last formula. □

REMARK 3.2.7: When $A^r_i=\delta^r_i$, we get $\rho^k_{l,ij}=\partial_i\chi^k_{l,j}-\partial_j\chi^k_{l,i}-\chi^r_{l,i}\chi^k_{r,j}+\chi^r_{l,j}\chi^k_{r,i}$ with $\chi^k_{j,i}=g^k_u\partial_i f^u_j-g^k_u f^u_{ij}$. However, in such a situation, we have:

$$\begin{aligned}&\omega_{kl}(f(x))f^k_i f^l_j=e^{2a(x)}\omega_{ij}(x)\\&\Rightarrow\omega_{kl}(f(x))\partial_i f^k(x)\partial_j f^l(x)=e^{2a(x)}\omega_{ij}(x)=\bar{\omega}_{ij}(x)\end{aligned}$$

Using the Minkowski metric ω which is locally constant and thus flat, it follows from the Vessiot structure equations that $\bar{\omega}$ *must* also be flat but we may have $f_2\neq j_2(f)$ even though $f_1=j_1(f)$. As $\bar{\omega}$ is conformally equivalent to ω, then both metric have vanishing Weyl tensor and the integrability condition for $\bar{\omega}$ is thus to have a vanishing Ricci tensor, that is to say, prolonging once the system $j_1(f)^{-1}(\omega)=\bar{\omega}$, we get $j_2(f)^{-1}(\gamma)=\bar{\gamma}$ and obtain:

$$\gamma=0\Rightarrow\bar{\gamma}^k_{ij}=\delta^k_i\partial_j a+\delta^k_j\partial_i a-\omega_{ij}\omega^{kr}\partial_r a$$

$$(n-2)\partial_{ij}a+\omega_{ij}\omega^{rs}\partial_{rs}a+(n-2)\partial_i a\partial_j a-(n-2)\omega_{ij}\omega^{rs}\partial_r a\partial_s a=0$$

This is a very striking result showing out for the first time that there may be links between the non-linear Spencer sequence and classical conformal geometry as the above result is just the variation of the classical Ricci tensor under a conformal change of the metric and the reason for which we introduced exponentials for describing conformal factors.

THEOREM 3.2.8: We have the variation over the source:

$$\delta\tau_{j,i} = B_i^r \partial_r \xi_{sj}^s + \xi^r \partial_r \tau_{j,i} + \tau_{j,r}\xi_i^r + \tau_{r,i}\xi_j^r - \tau_{j,i}^r \xi_{sr}^s$$

Proof: Using the general variational formulas one obtains:

$$\delta\chi_{lj,i}^k = \left(\partial_i \xi_{lj}^k - \xi_{lij}^k\right) + \xi^r \partial_r \chi_{lj,i}^k + \chi_{lj,r}^k \partial_i \xi^r + \chi_{lr,i}^k \xi_j^r$$
$$+\left(\chi_{rj,i}^k \xi_l^r - \chi_{lj,i}^r \xi_r^k\right) + \chi_{r,i}^k \xi_{lj}^r - \chi_{l,i}^r \xi_{rj}^k - \chi_{j,i}^r \xi_{lr}^k - \chi_{,i}^r \xi_{lrj}^k$$

where one must take into account that the third order jets of conformal vector fields vanish, that is to say $\xi_{lrj}^k = 0$. Contracting in k and l, we get:

$$\delta\chi_{sj,i}^s = \partial_i \xi_{sj}^s + \xi^r \partial_r \chi_{sj,i}^s + \chi_{sj,r}^s \partial_i \xi^r + \chi_{sr,i}^s \xi_j^r - \chi_{j,i}^r \xi_{sr}^s - \chi_{,i}^r \xi_{srj}^s$$

$$\chi_{lj,i}^k = A_i^r \tau_{lj,r}^k \Rightarrow \delta\chi_{lj,i}^k = A_i^r \delta\tau_{lj,r}^k + \tau_{lj,r}^k \delta A_i^r \Rightarrow A_i^r \delta\tau_{sj,r}^s = \delta\chi_{sj,i}^s - \tau_{j,r}\delta A_i^r$$

$$A_i^r \delta\tau_{j,r} = \partial_i \xi_{sj}^s + \xi^r \partial_r \chi_{sj,i}^s + \chi_{sj,r}^s \partial_i \xi^r + \chi_{sr,i}^s \xi_j^r - \chi_{j,i}^r \xi_{sr}^s - \tau_{j,r}\left(\xi^s \partial_s A_i^r + A_s^r \partial_i \xi^s - A_i^s \xi_s^r\right)$$

$$\delta\tau_{j,i} = B_i^r \partial_r \xi_{sj}^s + \xi^r \partial_r \tau_{j,i} + \left(\tau_{j,r}\xi_i^r + \tau_{r,i}\xi_j^r\right) + \tau_{j,i}^r \xi_{sr}^s \qquad \square$$

Using the fact that ω is locally constant and not varied (*care*), we have at once:

$$\delta\tau = \omega^{ij}\left(B_i^r \partial_r \xi_{sj}^s\right) + \xi^r \partial_r \tau + \omega^{ij}\left(\tau_{j,r}\xi_i^r + \tau_{r,i}\xi_j^r\right) - \omega^{ij}\tau_{j,i}^r \xi_{sr}^s$$
$$= \omega^{ij}\left(B_i^r \partial_r \xi_{sj}^s\right) + \xi^r \partial_r \tau + \tau_{r,s}\left(\omega^{is}\xi_i^r + \omega^{js}\xi_j^r\right) - \omega^{ij}\tau_{j,i}^r \xi_{sr}^s$$
$$= \omega^{ij}\left(B_i^r \partial_r \xi_{sj}^s\right) + \xi^r \partial_r \tau + \frac{2}{n}\omega^{rs}\tau_{r,s}\xi_t^t - \omega^{ij}\tau_{j,i}^r \xi_{sr}^s$$

and thus:

COROLLARY 3.2.9: $\boxed{\delta\tau_{j,i} = B_i^r \partial_r \xi_{sj}^s + \xi^r \partial_r \tau_{j,i} + \left(\tau_{j,r}\xi_i^r + \tau_{r,i}\xi_j^r\right) - \tau_{j,i}^r \xi_{sr}^s}$

Combining this result with the three preceding Lemmas, we finally obtain:

COROLLARY 3.2.10: The action variation over the source is:

$$\boxed{\begin{aligned}\delta\left(\tau det(A)\right) = \partial_r\left(\xi^r \tau det(A) + \omega^{ij}(x) B_i^r det(A)\xi_{sj}^s\right) - \frac{n-2}{n}\tau det(A)\xi_r^r \\ + \frac{n-2}{n}\omega^{ij}(x)\tau_{r,i}^r det(A)\xi_{sj}^s\end{aligned}}$$

Proof: According to Lemma 5.C.3, we have:

$$\delta\left(\tau det(A)\right) = (\delta\tau) det(A) + \tau\delta det(A)$$
$$= \omega^{ij} B_i^r det(A)\partial_r \xi_{sj}^s + \partial_r\left(\xi^r \tau det(A)\right) - \frac{n-2}{n}\tau det(A)\xi_r^r$$
$$- \omega^{ij}\tau_{j,i}^r det(A)\xi_{sj}^s$$
$$= \partial_r\left(\xi^r \tau det(A) + \omega^{ij}(x) B_i^r det(A)\xi_{sj}^s\right) - \frac{n-2}{n}\tau det(A)\xi_r^r$$
$$- \omega^{ij}\partial_r\left(B_i^r det(A)\right)\xi_{sj}^s - \omega^{ij}\tau_{j,i}^r det(A)\xi_{sr}^s$$

and we just need to use Lemma 3.2.1. □

THEOREM 3.2.11: We have the following Euler-Lagrange equations when $n=4$ *only*:

$$\begin{cases} \xi^r_{ri} \rightarrow \exists \text{ gravitational potential} \\ \xi^r_r \rightarrow \exists \text{ Poisson equation} \\ \xi^r \rightarrow \exists \text{ Newton law} \end{cases}$$

In particular $\tau^r_{r,i}=0 \Leftrightarrow \chi^r_{r,i}=0 \Leftrightarrow \alpha_i=0 \Leftrightarrow b_k=-\frac{1}{\Theta}\frac{\partial\Theta}{\partial y^k}$.

Proof: For n arbitrary, we have:

$$\begin{aligned}\tau det(A) &= n\Theta^{(n-2)}\Delta\left(\omega^{kl}\frac{\partial b_k}{\partial y^l}-\frac{n-2}{2}\omega^{kl}b_k b_l\right)\\ &= -n\Theta^{(n-2)}\Delta\left(\Theta^{-1}\omega^{kl}\frac{\partial^2\Theta}{\partial y^k\partial y^l}+\frac{n-4}{2}\Theta^{-2}\omega^{kl}\frac{\partial\Theta}{\partial y^k}\frac{\partial\Theta}{\partial y^l}\right)\end{aligned}$$

Hence, *for* $n=4$ *only*, we have $\tau det(A)=-4\Delta\Theta\omega^{kl}\frac{\partial^2\Theta}{\partial y^k\partial y^l}$. In the static case the gravity vector *must* be in first approximation $g^k \simeq -\bar{\gamma}^k_{44}=\omega_{44}\omega^{kl}b_l=-b_k<0 \Leftrightarrow b_k>0$, $\forall k=1,2,3$ (care to the minus sign coming from the inversion of the elations). If we introduce the gravitational potential $\phi=\frac{GM}{r}$ where r is the distance at the central attractive mass M and G is the gravitational constant, then we have $\frac{\phi}{c^2}\ll 1$ as a dimensionless number and $\Theta=1$ when there is no gravity. When there is static gravity, the conformal factor Θ must be therefore close to 1 with vanishing Laplacian and $\frac{\partial\Theta}{\partial y}<0$. The only coherent possibility is to set $\Theta=1+\frac{\phi}{c^2}$ in order to correct the value $\Theta=1-\frac{\phi}{c^2}$ we found in ([7], p. 450) and we have already explained the confusion we made on the physical meaning of *source* and *target*. Hence, *gravity in vacuum only depends on the conformal isotropy groupoid through the conformal factor* but this new approach is quite different from the ideas of G. Nordström, H. Weyl or even Einstein-Fokker. Indeed, it has only to do with the nonlinear Spencer sequence and *not at all* with the nonlinear Janet sequence, contrary to all these theories, as we just said, and the conformal factor Θ is now well defined everywhere apart from the origin of coordinates where is the central attractive mass. We have thus no longer any need to introduce the so-called *horizon* $r=GM/c^2$ and gravitation only depends on the structure of the conformal group theory like electromagnetism, with the only experimental need to fix the gravitational constant. Such a "*philosophy*" has been first proposed by the Cosserat brothers in ([10] [13] [14]) for elasticity with the only experimental need to measure the elastic constants and extended to electromagnetism in the last section with the same comments. An additional dynamical term must be added for the Newton law but this rather physical question will be studied in another paper as we already said in the Introduction. □

With $\phi=\frac{GM}{r}$ and thus $\frac{\phi}{c^2}\ll 1$, we have thus been able to replace $1-\frac{\phi}{c^2}$ by $1+\frac{\phi}{c^2}$, suppressing therefore the *horizon* $r=GM/c^2$ when G is the gravitational constant and M the central mass, along the following scheme based on the fact that *inversion exchanges source with target*:

$$\boxed{\text{ATTRACTION}\xleftrightarrow{\text{inversion}}\text{REPULSION}}$$

With more details, the inversion rule for the second order jets is $f_{ij}^k=-f_t^k f_i^r f_j^s g_{rs}^t$ or, equivalently, $g_{kl}^u=-g_r^u g_k^i g_l^j f_{ij}^r$. In the case of the conformal Lie groupoid, we may set $y=x$, $f_i^k=\Theta^{-1}\delta_i^k$ and obtain therefore $g_{44}^k=-\Theta^3 f_{44}^k$ for $k=1,2,3$ but *such a procedure could not be even imagined in any classical framework dealing with Lie groups of transformations*.

REMARK 3.2.12: We shall find back the same Euler-Lagrange variational equations by using the variation over the target. With $dy=\Delta dx$ by definition, we have indeed for n arbitrary:

$$\int \tau det(A)dx=\int n\Theta^{(n-2)}\left[\omega^{kl}(y)\frac{\partial b_k}{\partial y^l}-\frac{n-2}{2}\omega^{kl}(y)b_k b_l\right]dy$$

If we are only interested by the variation of the second order jets, we may equivalently vary the b_k alone and get after integration by parts:

$$\delta b_l\to(n-2)\Theta^{(n-3)}\omega^{kl}\frac{\partial\Theta}{\partial y^k}+(n-2)\Theta^{(n-2)}\omega^{kl}b_k=0\Rightarrow\boxed{b_k=-\frac{1}{\Theta}\frac{\partial\Theta}{\partial y^k}}$$

Now, with $dx=dx^1\wedge\cdots\wedge dx^n$ and $dy=dy^1\wedge\cdots\wedge dy^n$, we have:

$$\begin{aligned}\int\tau det(A)dx&=-\int\left[n\Theta^{(n-3)}\omega^{kl}(y)\frac{\partial^2\Theta}{\partial y^k\partial y^l}+\frac{n(n-4)}{2}\Theta^{(n-4)}\omega^{kl}(y)\frac{\partial\Theta}{\partial y^k}\frac{\partial\Theta}{\partial y^l}\right]dy\\&=-\int\frac{\partial}{\partial y^l}\left(n\Theta^{(n-3)}\omega^{kl}(y)\frac{\partial\Theta}{\partial y^k}\right)dy-\int\frac{n(n-2)}{2}\Theta^{(n-4)}\omega^{kl}(y)\frac{\partial\Theta}{\partial y^k}\frac{\partial\Theta}{\partial y^l}dy\end{aligned}$$

If we only vary the section $y=f(x)$ of $X\times Y$ over X, we have $dy=\Delta dx$, $\delta\Delta=\Delta\frac{\partial\eta^u}{\partial y^u}$ and:

$$\begin{aligned}&\Theta^n det\left(f_i^k(x)\right)=1\Rightarrow 0=\delta(\partial_i\Theta)=\delta\left(\frac{\partial\Theta}{\partial y^k}\right)\partial_i f^k+\frac{\partial\Theta}{\partial y^u}\frac{\partial\eta^u}{\partial y^k}\partial_i f^k\\&\Rightarrow\delta\left(\frac{\partial\Theta}{\partial y^k}\right)=-\frac{\partial\Theta}{\partial y^u}\frac{\partial\eta^u}{\partial y^k}\end{aligned}$$

It follows that the variation of the last integral is:

$$-\int n(n-2)\Theta^{(n-4)}\omega^{kl}(y)\left(\frac{\partial\Theta}{\partial y^l}\frac{\partial\Theta}{\partial y^u}\frac{\partial\eta^u}{\partial y^k}-\frac{1}{2}\frac{\partial\Theta}{\partial y^k}\frac{\partial\Theta}{\partial y^l}\frac{\partial\eta^u}{\partial y^u}\right)dy$$

After integration by parts, we get, up to a divergence:

$$-n(n-2)\int\frac{\partial}{\partial y^k}\left[\Theta^{(n-4)}\left(\omega^{rk}(y)\frac{\partial\Theta}{\partial y^r}\frac{\partial\Theta}{\partial y^u}-\frac{1}{2}\delta_u^k\omega^{rs}(y)\frac{\partial\Theta}{\partial y^r}\frac{\partial\Theta}{\partial y^s}\right)\right]\eta^u dy$$

When $n=4$, the direct computation becomes simpler because a part of the integral disappears. We are left with $\tau det(A)=-4\Theta\Box\Theta$ and we recognize the well known *Abraham tensor* in the bracket, without any other assumption. Accordingly, we may finally say as in the previous section that *the whole gravitational scheme only depends on the structure of the conformal group*.

REMARK 3.2.13: Proceeding as in GR, we may consider the variation:

$$\delta\int \mathfrak{g}^{kl}(y)\left[\frac{\partial b_l}{\partial y^k}+b_kb_l-\frac{1}{2}\omega_{kl}(y)\omega^{rs}(y)b_rb_s\right]dy=0$$

Varying only the second order jets b_k, we get equivalently through an integration by parts:

$$\left(2\mathfrak{g}^{kl}-\omega^{kl}\omega_{rs}\mathfrak{g}^{rs}\right)b_l=\frac{\partial\mathfrak{g}^{kl}}{\partial y^l}$$

If we set $b(f(x))=a(x)$ and $\Theta(y)=e^{-b(y)}$, then $\Theta^2(y)=e^{-2b(y)}$ and we have successively:

$$\omega_{kl}(y)f_i^kf_j^l=e^{2a(x)}\omega_{ij}(x)\Leftrightarrow e^{-2b(y)}\omega_{kl}(y)f_i^kf_j^l=\omega_{ij}(x)$$
$$\Leftrightarrow\Theta^2(y)\omega_{kl}(y)f_i^kf_j^l=\omega_{ij}(x)$$

Inverting the matrices, we obtain equivalently:

$$\Theta^{-2}(y)\omega^{kl}(y)g_k^ig_l^j=\omega^{ij}(x)\Leftrightarrow\Theta^{-2}(y)\omega^{kl}(y)=\omega^{ij}(x)f_i^kf_j^l$$

and thus:

$$\Theta^{2n}det\left(f_i^k\right)^2=1\Rightarrow\Theta^ndet\left(f_i^k\right)=1\Rightarrow det(A)=det\left(\partial_if^k\right)/det\left(f_i^k\right)=\Theta^n\Delta$$

Hence, if we set $\mathfrak{g}^{kl}(y)=\Theta^{(n-2)}(y)\omega^{kl}(y)$, we finally obtain:

$$(n-2)\Theta^{(n-2)}b_k=-(n-2)\Theta^{(n-3)}\frac{\partial\Theta}{\partial y^k}\Rightarrow\boxed{b_k=-\frac{1}{\Theta}\frac{\partial\Theta}{\partial y^k}}$$

in a coherent way with the logarithmic derivatives:

$$\beta_k=0\Leftrightarrow\frac{\partial b}{\partial y^k}=-\frac{1}{\Theta}\frac{\partial\Theta}{\partial y^k}=b_k$$
$$\Leftrightarrow\partial_ia=-\frac{1}{\Theta}\frac{\partial\Theta}{\partial y^k}\partial_if^k=b_k\partial_if^k=a_rg_k^r\partial_if^k=A_i^ra_r\Leftrightarrow\alpha_i=0$$

4. Conclusion

This paper is the achievement of a lifetime research work on the common conformal origin of electromagnetism and gravitation. Roughly speaking, the Cosserat brothers have only been dealing with the 3 translations and 3 rotations of the group of rigid motions of space with 6 parameters ([10], p. 137) or with the Poincaré group of space-time with 10 parameters ([10], p. 167) while Weyl has only been dealing with the dilatation and the 4 elations of the conformal group of space-time with now $4+6+1+4=15$ parameters ([3]). Among the most striking results obtained from this conformal extension obtained by adding the elations, we successively notice:

- We have revisited the mathematical foundations of Special Relativity without appealing to any "*Gedanken experiment*" using specific signals. As a byproduct, the two sets of Maxwell equations are separately invariant by any diffeomorphism, but the conformal group is the biggest group of invariance of the Minkowski constitutive law between field and induction in vacuum that only depends on one constant. Contrary to classical Gauge Theory, the group $U(1)$ never appears.
- The generating linear *first order* compatibility conditions (CC) for the Cosserat fields are *exactly* described by the *first order* nonlinear second Spencer operator D_2. Accordingly, there is no conceptual difference between these linear CC and the first set $d:\wedge^2 T^* \to \wedge^3 T^*$ of Maxwell equations where d is the exterior derivative. However, the classical CC of elasticity are described by the linear *second order Riemann* operator existing in the linear Janet sequence but this different canonical linear differential sequence could not explain the existence of field-matter couplings like piezzoelectricity or photoelasticity ([5] [36]). On the contrary, in the conformal approach, it is essential to notice that *the elastic and electromagnetic fields are both specific sections of* $\hat{C}_1 \simeq T^* \otimes \hat{R}_2$ *killed by* D_2. They can thus be coupled in a natural way but *cannot be associated to the concept of curvature* described by $\hat{C}_2$. *This shift by one step to the left, even in the nonlinear framework, can be considered as the main novelty of this paper.*
- The linear Cosserat equations are *exactly* described by the (formal) adjoint $ad(D_1)$ of the *linear* first Spencer operator $D_1:\hat{C}_0 \to \hat{C}_1$ which is a first order operator ([13]). Accordingly, there is no conceptual difference between these equations and the second set $ad(d)$ of Maxwell equations where $d:T^* \to \wedge^2 T^*$. This result explains why the *Cosserat* equations are quite different from the *Cauchy* equations which are described by the formal adjoint of the *Killing* operator in the Janet sequence used in classical elasticity, that is $Cauchy = ad(Killing)$ in the language of operators. It follows that the elastic and electromagnetic inductions are both specific sections of $\wedge^4 T^* \otimes \hat{C}_1^* \simeq \wedge^3 T^* \otimes \hat{R}_2^*$, *independently of any constitutive relation.*
- Combining the two previous comments, respectively related to "*geometry*" and "*physics*" according to H. Poincaré ([16]), there is no conceptual difference between the elastic constitutive constants of elasticity and the magnetic constant μ or rather $1/\mu$ of electromagnetism in the case of homogeneous isotropic materials on one side (*space*) or between the mass per unit volume and the dielectric constant ε on the other side (*time*), a result confirmed by the speeds of the various elastic or electromagnetic existing waves ([4] [8] [29] [36]). In general, one has $\varepsilon\mu c^2 = n^2$ where n is the index of refraction but in vacuum we have $\varepsilon_0\mu_0 c^2 = 1$ and we have thus only one electromagnetic constant involved in the corresponding Minkowski constitutive law of vacuum ([2]).
- We have pointed out the fact that, in the framework of Lie groupoids consi-

dered by Spencer, all the fibered and vector bundles are constructed over the "source" x and that the "target" y is just used as a kind of "hidden variable". Like in classical fluid dynamics, this has been the reason for calling x the Euler variable and y the Lagrange variable, though using transformations $y = f(x)$.

- As for gravitation and the possibility to exhibit a conformal factor defined everywhere but at the origin, we may simply say that we needed 25 years in order to correct the result we already obtained in 1994 at the end of ([7]). Such a possibility highly depends on the new mathematical tools involved in the construction of the Janet or Spencer linear/nonlinear differential sequences for various groups, in particular for the conformal group of space-time. Indeed, *in this case only*, the Spencer δ-cohomology has very specific properties for the dimension $n = 4$ *only* (see [28] for a computer algebra checking by our former PhD student A. Quadrat, INRIA).

This paper proves that, what was surely true for electromagnetism a few years ago, must also become true for gravitation in the future. For the moment, in General Relativity, the usual way is to shrink down the group of invariance of the underlying metric as we have indeed 10 parameters for the Minkowski metric, 4 parameters for the Schwarzschild metric and only 2 parameters for the Kerr metric ([37]). On the contrary, following the Cosserat brothers and Weyl, group theory and differential sequences must define the guiding lines by enlarging the Poincaré group of space-time (10 parameters) to the Weyl group of space-time (11 parameters) by adding 1 dilatation and then to the conformal group of space-time (15 parameters) by adding the 4 elations. We claim that the most striking results of this paper are coming from the *Fundamental Diagram I* (1978) and from the *Fundamental Diagram II* (1983) that only depends on these elations. However, the Spencer operator and its δ restriction have yet never been used in mathematical physics, a fact explaining why we have not been able to provide many other references.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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