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Soliton Wave for the Magnetic Electron

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Abstract

A century ago, de Broglie discovered the wave associated to the motion of the electron. We present here the soliton solutions of a nonlinear relativistic wave equation. Two such solitons exist, corresponding to the two possible states of a particle with spin $j = 1/2$. The mystery of wave-particle dualism is solved: the electron is both a particle, a point which is a singularity, and a wave extended throughout the whole space.

Keywords

Quantum Mechanics, Nonlinear Wave Equation, Relativity, Wave-Particle Dualism

1. Introduction

Wave particle dualism began with Einstein's paper [1] explaining that light was both an electromagnetic wave and contained energy-momentum quanta (photons). This dualism was extended by L. de Broglie to any matter particle [2]. A major controversy followed Schrödinger's discovery of a non-relativistic wave equation for de Broglie's wave: Bohr thought that matter was sometimes wave and sometimes particle. Schrödinger thought matter was only wave. Like Einstein, he was seeking a solution to the dualism in Einstein's gravitational theory: General Relativity. De Broglie continued to think of matter as simultaneously wave and particle. He first tried the idea of a wave guiding his particle, later a double wave following the same equation. He also studied the relativistic wave equation formulated by Dirac [3], in the 30s and again in the 50s [4] [5]. Then Einstein succeeded in linking the motion of a matter particle, which is a singularity in the gravitational field, to the equation of this gravitational field. The soliton solutions that we present here are thus the result of a very long quest, mainly developed by Einstein, de Broglie and their relatively few followers.

2. The Wave Equation

The wave equation is best suited to the use of the Pauli algebra, which is also the Clifford algebra Cl_3 of three-dimensional physical space (see for instance [6], Chapter A). The wave is a function ϕ of space and time, with value in Cl_3. Our wave equation reads:

$$\nabla\hat{\phi}i\sigma_3 = qA\hat{\phi} + \mathrm{e}^{-i\beta}\phi\mathbf{m}, \tag{1}$$

$$\phi := \sqrt{2}\begin{pmatrix}\xi & \hat{\eta}\end{pmatrix} = \sqrt{2}\begin{pmatrix}\xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^*\end{pmatrix}; \hat{\phi} = \sqrt{2}\begin{pmatrix}\eta & \hat{\xi}\end{pmatrix} = \sqrt{2}\begin{pmatrix}\eta_1 & -\xi_2^* \\ \eta_2 & \xi_1^*\end{pmatrix}, \tag{2}$$

$$\nabla := \sigma^\mu\partial_\mu; \hat{\nabla} := \hat{\sigma}^\mu\partial_\mu; A := \sigma^\mu A_\mu; \hat{A} := \hat{\sigma}^\mu A_\mu, \tag{3}$$

$$\rho\mathrm{e}^{i\beta} := \det(\phi) = \eta^\dagger\xi; \mathbf{m} := \begin{pmatrix}\mathbf{l} & 0 \\ 0 & \mathbf{r}\end{pmatrix} \tag{4}$$

where β is the Yvon-Takabayasi angle, $\mathbf{l}$ and $\mathbf{r}$ are two masses which replace the unique mass of the Dirac wave equation. Using the Planck length l_P, our wave equation may be obtained from the following Lagrangian density:

$$0 = k\frac{\mathcal{L}}{m} = \frac{\mathcal{L}_L}{\mathbf{l}} + \frac{\mathcal{L}_R}{\mathbf{r}}; \mathcal{L}_L = \Re\left[\eta^\dagger\left(-i\nabla + qA + \mathbf{l}\mathrm{v}\right)\eta\right], \tag{5}$$
$$l_P^3 = k\hbar c; \mathcal{L}_R = \Re\left[\xi^\dagger\left(-i\hat{\nabla} + q\hat{A} + \mathbf{r}\hat{\mathrm{v}}\right)\xi\right],$$

and this density may also be obtained from the wave equation, since it is equivalent to the system:

$$0 = \left(-i\nabla + qA + \mathbf{l}\mathrm{v}\right)\eta,$$
$$0 = \left(-i\hat{\nabla} + q\hat{A} + \mathbf{r}\hat{\mathrm{v}}\right)\xi \tag{6}$$

$$\mathrm{v} := \frac{\mathrm{J}}{\rho}; \mathrm{J} := \phi\phi^\dagger; \rho := \left|\det(\phi)\right|. \tag{7}$$

3. The Soliton Wave of the Electron

To solve our wave equation and to obtain a soliton wave, in the case of a wave equation without exterior electromagnetic potential A, we use the separation of variables in spherical coordinates discovered by H. Krüger [7] [8], letting (see also [6] Chapter C):

$$x^1 := r\sin\theta\cos\varphi \; ; \; x^2 := r\sin\theta\sin\varphi \; ; \; x^3 := r\cos\theta. \tag{8}$$

The following notations are used:

$$i_1 := \sigma_{23} = i\sigma_1 \, ; \; i_2 := \sigma_{31} = i\sigma_2 \, ; \; i_3 := \sigma_{12} = i\sigma_3, \tag{9}$$

$$S := \exp\left(-\frac{\varphi}{2}i_3\right)\exp\left(-\frac{\theta}{2}i_2\right); \; \Omega = \hat{\Omega} := r^{-1}(\sin\theta)^{-\frac{1}{2}}S, \tag{10}$$

$$\vec{\partial}' := \sigma_3\partial_r + \frac{1}{r}\sigma_1\partial_\theta + \frac{1}{r\sin\theta}\sigma_2\partial_\varphi.$$

H. Krüger obtained the remarkable identity [7]:

$$\vec{\partial}=\Omega\vec{\partial}'\Omega^{-1}, \tag{11}$$

The wave equation to be solved reads:

$$0=\nabla\hat{\phi}\sigma_{21}+qA\hat{\phi}+\mathrm{e}^{-i\beta}\phi\mathbf{m}, \tag{12}$$

$$\left(\partial_0-\vec{\partial}\right)\hat{\phi}i_3=qA\hat{\phi}+\mathrm{e}^{-i\beta}\phi\mathbf{m}. \tag{13}$$

It's possible to separate at one go the t and φ variables from the r and θ variables by using:

$$\phi=:\Omega X\mathrm{e}^{\left(\lambda\varphi-Ex^0\right)i_3};\ \hat{\phi}=\Omega\hat{X}\mathrm{e}^{\left(\lambda\varphi-Ex^0\right)i_3}, \tag{14}$$

$$X:=\begin{pmatrix} AU & -B^*V \\ CV & D^*U \end{pmatrix};\ \hat{X}=\begin{pmatrix} DU & -C^*V \\ BV & A^*U \end{pmatrix},$$

where A, B, C and D are functions, with value in $\mathbb{C}$, of the radial variable r; U and V are real functions of the angular variable θ; $\hbar cE$ is the electron energy; and λ is a real constant referred to as the magnetic quantum number in the case of the electron in an hydrogen atom. We have:

$$\rho\mathrm{e}^{i\beta}=\eta^\dagger\xi=\phi\overline{\phi}=\det(\phi)=\frac{\det(X)}{r^2\sin\theta}, \tag{15}$$

$$\rho_X\mathrm{e}^{i\beta}:=\det(X)=AD^*U^2+CB^*V^2. \tag{16}$$

The wave Equation (12) uses:

$$\begin{aligned}\vec{\partial}'\left(\hat{X}\mathrm{e}^{\lambda\varphi i_3}\right)&=\begin{pmatrix} \partial_r & \frac{1}{r}\left(\partial_\theta-\frac{i}{\sin\theta}\partial_\varphi\right) \\ \frac{1}{r}\left(\partial_\theta+\frac{i}{\sin\theta}\partial_\varphi\right) & -\partial_r \end{pmatrix}\begin{pmatrix} DU\mathrm{e}^{i\lambda\varphi} & -C^*V\mathrm{e}^{-i\lambda\varphi} \\ BV\mathrm{e}^{i\lambda\varphi} & A^*U\mathrm{e}^{-i\lambda\varphi} \end{pmatrix} \\ &=\begin{pmatrix} D'U+\frac{B}{r}\left(V'+\frac{\lambda V}{\sin\theta}\right) & -C'^*+\frac{A^*}{r}\left(U'-\frac{\lambda U}{\sin\theta}\right) \\ \frac{A}{r}\left(U'-\frac{\lambda U}{\sin\theta}\right)-B'V & -\frac{C^*}{r}\left(V'+\frac{\lambda V}{\sin\theta}\right)-A'^*U \end{pmatrix}\mathrm{e}^{\lambda\varphi i_3}.\end{aligned} \tag{17}$$

If U and V are solutions of the following system:

$$U'-\frac{\lambda U}{\sin\theta}=-\kappa V;\ V'+\frac{\lambda V}{\sin\theta}=\kappa U, \tag{18}$$

it implies that:

$$\vec{\partial}'\left(\hat{X}\mathrm{e}^{\lambda\varphi i_3}\right)=\begin{pmatrix} \left(D'+\frac{\kappa}{r}B\right)U & \left(-C'^*-\frac{\kappa}{r}A^*\right)V \\ \left(-B'-\frac{\kappa}{r}D\right)V & \left(-A'^*-\frac{\kappa}{r}C^*\right)U \end{pmatrix}\mathrm{e}^{\lambda\varphi i_3}. \tag{19}$$

The Equation (12) gives, if we suppose the existence of an interior potential $qA=-u/r$:

$$0=\left(E+\frac{u}{r}\right)\hat{X}\mathrm{e}^{\lambda\varphi i_3}-\vec{\partial}'\left(\hat{X}\mathrm{e}^{\lambda\varphi i_3}\right)i_3-\mathrm{e}^{-i\beta}X\mathbf{m}\mathrm{e}^{\lambda\varphi i_3}. \tag{20}$$

The following system is hence obtained:

$$0=\left[\left(E+\frac{u}{r}\right)D-i\left(D'+\frac{\kappa}{r}B\right)-\mathrm{e}^{-i\beta}\mathbf{l}A\right]U, \tag{21}$$

$$0=\left[\left(E+\frac{u}{r}\right)B+i\left(B'+\frac{\kappa}{r}D\right)-\mathrm{e}^{-i\beta}\mathbf{l}C\right]V, \tag{22}$$

$$0=\left[\left(E+\frac{u}{r}\right)C^*+i\left(C'^*+\frac{\kappa}{r}A^*\right)-\mathrm{e}^{-i\beta}\mathbf{r}B^*\right]V, \tag{23}$$

$$0=\left[\left(E+\frac{u}{r}\right)A^*-i\left(A'^*+\frac{\kappa}{r}C^*\right)-\mathrm{e}^{-i\beta}\mathbf{r}D^*\right]U. \tag{24}$$

A similar system was solved in Chapter C of [6]. We will thus use the same method. The κ constant is an integer number which cannot be zero. It is linked to the total angular number j by $|\kappa|=j+1/2$. In the case $\kappa=1$ and $\lambda=1/2$, we obtain:

$$U=-\sqrt{\sin\theta}\cos\left(\frac{\theta}{2}\right);V=-\sqrt{\sin\theta}\sin\left(\frac{\theta}{2}\right), \tag{25}$$

$$U^2+V^2=\sin\theta;U^2-V^2=\sin\theta\cos\theta;2UV=\sin^2\theta. \tag{26}$$

Similarly with $\kappa=1$ and $\lambda=-1/2$, we obtain:

$$U=-\sqrt{\sin\theta}\sin\left(\frac{\theta}{2}\right);V=\sqrt{\sin\theta}\cos\left(\frac{\theta}{2}\right), \tag{27}$$

$$U^2+V^2=\sin\theta;U^2-V^2=-\sin\theta\cos\theta;2UV=-\sin^2\theta. \tag{28}$$

It is easy to prove that, in all cases, U^2-V^2 has a $\cos\theta$ factor and is hence null in the equatorial plane $\theta=\pi/2$. The radial system of our wave equation is thus close to the radial system of the Dirac equation.

3.1. Resolution of the Radial System

Conjugating the two last equations, the radial system becomes:

$$0=\left(E+\frac{u}{r}\right)D-i\left(D'+\frac{\kappa}{r}B\right)-\mathrm{e}^{-i\beta}\mathbf{l}A, \tag{29}$$

$$0=\left(E+\frac{u}{r}\right)B+i\left(B'+\frac{\kappa}{r}D\right)-\mathrm{e}^{-i\beta}\mathbf{l}C, \tag{30}$$

$$0=\left(E+\frac{u}{r}\right)C-i\left(C'+\frac{\kappa}{r}A\right)-\mathrm{e}^{i\beta}\mathbf{r}B, \tag{31}$$

$$0=\left(E+\frac{u}{r}\right)A+i\left(A'+\frac{\kappa}{r}C\right)-\mathrm{e}^{i\beta}\mathbf{r}D. \tag{32}$$

The mass-energy is supposed to be the harmonic mean m of the two masses ($2/m=1/\mathbf{l}+1/\mathbf{r}$), and the case $\kappa=1$, $\lambda=1/2$ is chosen. We now let:

$$A=ar^s\mathrm{e}^{-\Lambda mr};B=br^s\mathrm{e}^{-\Lambda mr};C=cr^s\mathrm{e}^{-\Lambda mr};D=dr^s\mathrm{e}^{-\Lambda mr}. \tag{33}$$

where Λ is a positive real constant and a, b, c and d are complex constants. Dividing by $r^s\mathrm{e}^{-\Lambda mr}$ the radial system becomes:

$$0=\left(m+\frac{u}{r}\right)d-i\left(\frac{s}{r}-\Lambda m\right)d-i\frac{\kappa}{r}b-\mathrm{e}^{-i\beta}\mathbf{l}a, \tag{34}$$

$$0=\left(m+\frac{u}{r}\right)b+i\left(\frac{s}{r}-\Lambda m\right)b+i\frac{\kappa}{r}d-\mathrm{e}^{-i\beta}\mathbf{l}c, \tag{35}$$

$$0=\left(m+\frac{u}{r}\right)c-i\left(\frac{s}{r}-\Lambda m\right)c-i\frac{\kappa}{r}a-\mathrm{e}^{i\beta}\mathbf{r}b, \tag{36}$$

$$0=\left(m+\frac{u}{r}\right)a+i\left(\frac{s}{r}-\Lambda m\right)a+i\frac{\kappa}{r}c-\mathrm{e}^{i\beta}\mathbf{r}d. \tag{37}$$

This system, with any r value, is equivalent to the following system:

$$0=m(1+i\Lambda)d-\mathrm{e}^{-i\beta}\mathbf{l}a;\ 0=(u-is)d-i\kappa b, \tag{38}$$

$$0=m(1-i\Lambda)a-\mathrm{e}^{i\beta}\mathbf{r}d;\ 0=(u+is)b+i\kappa d, \tag{39}$$

$$0=m(1-i\Lambda)b-\mathrm{e}^{-i\beta}\mathbf{l}c;\ 0=(u-is)c-i\kappa a, \tag{40}$$

$$0=m(1+i\Lambda)c-\mathrm{e}^{i\beta}\mathbf{r}b;\ 0=(u+is)a+i\kappa c. \tag{41}$$

A non-zero solution exists only if the determinant of each system of two linear equations is zero. And we obtain actually only two conditions:

$$0=1+\Lambda^2-\frac{\mathbf{lr}}{m^2}, \tag{42}$$

$$0=u^2+s^2-\kappa^2. \tag{43}$$

We let:

$$1+i\Lambda=:\sqrt{1+\Lambda^2}\mathrm{e}^{i\delta};\ s+iu=:\mathrm{e}^{i\gamma}\kappa. \tag{44}$$

Since we supposed that the mass-energy is the harmonic mean m, not the geometric mean $m_g=\sqrt{\mathbf{lr}}$, we can have:

$$\Lambda^2=\frac{\mathbf{lr}}{m^2}-1=\frac{m_g^2}{m^2}-1;\ \Lambda=\sqrt{\frac{\mathbf{lr}}{m^2}-1}. \tag{45}$$

Moreover, we have:

$$1+i\Lambda=\sqrt{1+\Lambda^2}\mathrm{e}^{i\delta}=\frac{m_g}{m}\mathrm{e}^{i\delta}, \tag{46}$$

$$s-iu=\mathrm{e}^{-i\gamma};1-i\Lambda=\frac{m_g}{m}\mathrm{e}^{-i\delta} \tag{47}$$

with those conditions, the radial system (34)-(37) is equivalent to the four independent relations:

$$d=\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}\mathrm{e}^{-\mathrm{i}(\delta+\beta)}a;\ b=\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}\mathrm{e}^{\mathrm{i}(\delta-\beta)}c, \tag{48}$$

$$b=-\mathrm{e}^{\mathrm{i}\gamma}d;\ c=-\mathrm{e}^{-\mathrm{i}\gamma}a. \tag{49}$$

And those relations imply:

$$b=-\mathrm{e}^{\mathrm{i}\gamma}d=-\mathrm{e}^{\mathrm{i}\gamma}\sqrt{\frac{l}{r}}\mathrm{e}^{-\mathrm{i}(\delta+\beta)}a=-\sqrt{\frac{l}{r}}\mathrm{e}^{\mathrm{i}(-\delta-\beta+\gamma)}a, \tag{50}$$

$$b=\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}\mathrm{e}^{i(\delta-\beta)}c=\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}\mathrm{e}^{i(\delta-\beta)}\left(-\mathrm{e}^{-i\gamma}a\right)=-\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}\mathrm{e}^{i(\delta-\beta-\gamma)}a. \tag{51}$$

Hence we obtain:

$$-\delta-\beta+\gamma=\delta-\beta-\gamma \bmod 2\pi;\ \gamma=\delta \bmod \pi,$$
$$\mathrm{e}^{i\gamma}=\pm\mathrm{e}^{i\delta} \tag{52}$$

To obtain a probability density, it is necessary to choose $s>0$. We thus suppose, with $\kappa=1$:

$$\gamma=\delta;\ s+iu=\mathrm{e}^{i\delta}=\frac{1+i\Lambda}{\sqrt{1+\Lambda^2}}$$
$$s=\frac{1}{\sqrt{1+\Lambda^2}}=\frac{m}{m_g};\ u=\frac{\Lambda}{\sqrt{1+\Lambda^2}}, \tag{53}$$

$$b=-\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}\mathrm{e}^{-i\beta}a;\ c=-\mathrm{e}^{-i\delta}a;\ d=\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}\mathrm{e}^{-i(\delta+\beta)}a. \tag{54}$$

If the anomalous gyromagnetic ratio comes from the difference between the two means m and m_g (see [6] 1.5.7) we have:

$$\frac{m_g}{m}=\frac{m_a}{m_g}=\frac{g}{2}=1.00115965218091(26);\ \Lambda=\sqrt{\frac{g^2-4}{4}}\approx 0.048 \tag{55}$$

Now by letting:

$$a_1:=|a|;\ a=:a_1\mathrm{e}^{ia_2};\ \zeta:=\ln(\mathbf{l})-\ln(\mathbf{r}),$$
$$f_\delta\begin{pmatrix} a & b \\ c & d \end{pmatrix}:=\begin{pmatrix} a & \mathrm{e}^{i\delta}b \\ \mathrm{e}^{i\delta}c & d \end{pmatrix}, \tag{56}$$

the wave becomes:

$$\hat{\phi}=-a_1r^{s-1}\mathrm{e}^{\frac{\zeta}{4}-\Lambda mr}\mathrm{e}^{i\left(-\frac{\delta}{2}-\frac{\beta}{2}\right)}\mathrm{e}^{-\frac{\varphi}{2}i_3}\mathrm{e}^{-\frac{\theta}{2}i_2}f_\delta\left(\mathrm{e}^{\frac{\theta}{2}i_2}\right)\mathrm{e}^{\left(a_2-i\frac{\zeta}{4}-\frac{\delta}{2}-\frac{\beta}{2}+\frac{\varphi}{2}-Ex^0\right)i_3}. \tag{57}$$

Since $s<1$, the center is a singularity of the electron wave, which appears both as a point particle and as extended throughout the whole space. Moreover we have:

$$\mathrm{e}^{-\frac{\varphi}{2}i_3}\mathrm{e}^{-\frac{\theta}{2}i_2}f_\delta\left(\mathrm{e}^{\frac{\theta}{2}i_2}\right)\mathrm{e}^{\frac{\varphi}{2}i_3}=\frac{1+\mathrm{e}^{i\delta}}{2}+\frac{1-\mathrm{e}^{i\delta}}{2}\frac{\vec{\mathrm{x}}}{r}\sigma_3, \tag{58}$$

$$\hat{\phi}=-a_1r^{s-1}\mathrm{e}^{\frac{\zeta}{4}-\Lambda mr}\mathrm{e}^{i\left(-\frac{\delta}{2}-\frac{\beta}{2}\right)}\left[\frac{1+\mathrm{e}^{i\delta}}{2}+\frac{1-\mathrm{e}^{i\delta}}{2}\frac{\vec{\mathrm{x}}}{r}\sigma_3\right]\mathrm{e}^{\left(a_2-i\frac{\zeta}{4}-\frac{\delta}{2}-\frac{\beta}{2}-Ex^0\right)i_3}, \tag{59}$$

$$\phi=-a_1r^{s-1}\mathrm{e}^{\frac{\zeta}{4}-\Lambda mr}\mathrm{e}^{i\left(\frac{\delta}{2}+\frac{\beta}{2}\right)}\left[\frac{1+e^{-i\delta}}{2}+\frac{1-e^{-i\delta}}{2}\frac{\vec{\mathrm{x}}}{r}\sigma_3\right]\mathrm{e}^{\left(a_2+i\frac{\zeta}{4}-\frac{\delta}{2}-\frac{\beta}{2}-Ex^0\right)i_3}. \tag{60}$$

3.2. Case $\kappa=1$ and $\lambda=-1/2$

The wave Equation (12) now uses:

$$\vec{\partial}'\left(\hat{X}\mathrm{e}^{-\frac{\varphi}{2}i_3}\right)=\begin{pmatrix}\partial_r & \frac{1}{r}\left(\partial_\theta-\frac{i}{\sin\theta}\partial_\varphi\right)\\ \frac{1}{r}\left(\partial_\theta+\frac{i}{\sin\theta}\partial_\varphi\right) & -\partial_r\end{pmatrix}\begin{pmatrix}DU\mathrm{e}^{-i\frac{\varphi}{2}} & -C^*V\mathrm{e}^{i\frac{\varphi}{2}}\\ BV\mathrm{e}^{-i\frac{\varphi}{2}} & A^*U\mathrm{e}^{i\frac{\varphi}{2}}\end{pmatrix}$$
$$=\begin{pmatrix}D'U+\frac{B}{r}\left(V'-\frac{V}{2\sin\theta}\right) & -C'^*+\frac{A^*}{r}\left(U'+\frac{U}{2\sin\theta}\right)\\ \frac{A}{r}\left(U'+\frac{U}{2\sin\theta}\right)-B'V & -\frac{C^*}{r}\left(V'-\frac{V}{2\sin\theta}\right)-A'^*U\end{pmatrix}\mathrm{e}^{-\frac{\varphi}{2}i_3}. \quad (61)$$

Since κ is still +1, we obtain the same radial system. Next we have:

$$\hat{X}\mathrm{e}^{-\frac{\varphi}{2}i_3}=\begin{pmatrix}DU\mathrm{e}^{-i\frac{\varphi}{2}} & -C^*V\mathrm{e}^{i\frac{\varphi}{2}}\\ BV\mathrm{e}^{-i\frac{\varphi}{2}} & A^*U\mathrm{e}^{i\frac{\varphi}{2}}\end{pmatrix}$$
$$=\sqrt{\sin\theta}\,r^s\mathrm{e}^{-\Lambda mr}\begin{pmatrix}-d\sin\frac{\theta}{2} & -c^*\cos\frac{\theta}{2}\\ b\cos\frac{\theta}{2} & -a^*\sin\frac{\theta}{2}\end{pmatrix} \quad (62)$$

Next we use (54):

$$\begin{pmatrix}-d\sin\frac{\theta}{2} & -c^*\cos\frac{\theta}{2}\\ b\cos\frac{\theta}{2} & -a^*\sin\frac{\theta}{2}\end{pmatrix}=\begin{pmatrix}-\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}\mathrm{e}^{i(-\delta-\beta)}a\sin\frac{\theta}{2} & \mathrm{e}^{i\delta}a^*\cos\frac{\theta}{2}\\ -\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}\mathrm{e}^{-i\beta}a\cos\frac{\theta}{2} & -a^*\sin\frac{\theta}{2}\end{pmatrix} \quad (63)$$

$$=a_1\mathrm{e}^{\frac{\zeta}{4}}\mathrm{e}^{i\left(-\frac{\delta}{2}-\frac{\beta}{2}\right)}i_2\begin{pmatrix}\mathrm{e}^{i\delta}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}\\ -\sin\frac{\theta}{2} & \mathrm{e}^{i\delta}\cos\frac{\theta}{2}\end{pmatrix}\mathrm{e}^{\left(a_2-i\frac{\zeta}{4}-\frac{\delta}{2}-\frac{\beta}{2}\right)i_3}, \quad (64)$$

$$\mathrm{e}^{-\frac{\varphi}{2}i_3}\mathrm{e}^{-\frac{\theta}{2}i_2}i_2\begin{pmatrix}\mathrm{e}^{i\delta}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}\\ -\sin\frac{\theta}{2} & \mathrm{e}^{i\delta}\cos\frac{\theta}{2}\end{pmatrix}\mathrm{e}^{-\frac{\varphi}{2}i_3}=i_2\left[\frac{1+\mathrm{e}^{i\delta}}{2}+\frac{\mathrm{e}^{i\delta}-1}{2}\frac{\vec{\mathrm{x}}}{r}\sigma_3\right]. \quad (65)$$

Here also the center is a singularity of the electron wave, which appears both as a point particle and as extended to the whole space. We have:

$$\hat{\phi}=a_1r^{s-1}\mathrm{e}^{\frac{\zeta}{4}-\Lambda mr}\mathrm{e}^{i\left(-\frac{\delta}{2}-\frac{\beta}{2}\right)}\times i_2\left[\frac{1+\mathrm{e}^{i\delta}}{2}+\frac{\mathrm{e}^{i\delta}-1}{2}\frac{\vec{\mathrm{x}}}{r}\sigma_3\right]\mathrm{e}^{\left(a_2-i\frac{\zeta}{4}-\frac{\delta}{2}-\frac{\beta}{2}-Ex^0\right)i_3}, \quad (66)$$

$$\phi=a_1r^{s-1}\mathrm{e}^{\frac{\zeta}{4}-\Lambda mr}\mathrm{e}^{i\left(\frac{\delta}{2}+\frac{\beta}{2}\right)}\times i_2\left[\frac{1+\mathrm{e}^{-i\delta}}{2}+\frac{\mathrm{e}^{-i\delta}-1}{2}\frac{\vec{\mathrm{x}}}{r}\sigma_3\right]\mathrm{e}^{\left(a_2+i\frac{\zeta}{4}-\frac{\delta}{2}-\frac{\beta}{2}-Ex^0\right)i_3}. \quad (67)$$

3.3. Normalization

The normalization of the wave is a necessary consequence of the equivalence principle, the equality between inertial mass and gravitational mass linked to the frequency of de Broglie's clock (see [6] 1.5.5), which means that $E=\iiint \mathrm{d}vT_0^0$. This equality is indeed equivalent, for any stationary solution of the wave equation, to:

$$1=\iiint \mathrm{d}v\frac{\mathbf{J}^0}{\hbar c};\ \mathbf{J}:=\frac{m}{k\mathbf{l}}\mathrm{D}_L+\frac{m}{k\mathbf{r}}\mathrm{D}_R, \tag{68}$$

$$\frac{m}{\mathbf{l}}\mathrm{D}_L+\frac{m}{\mathbf{r}}\mathrm{D}_R=\frac{m}{\mathbf{l}}\phi\frac{1-\sigma_3}{2}\phi^\dagger+\frac{m}{\mathbf{r}}\phi\frac{1+\sigma_3}{2}\phi^\dagger=\phi\left(1+\frac{\mathbf{l}-\mathbf{r}}{\mathbf{l}+\mathbf{r}}\sigma_3\right)\phi^\dagger. \tag{69}$$

In the case $\lambda=1/2$, we obtain:

$$\phi^\dagger=-a_1r^{s-1}\mathrm{e}^{\frac{\zeta}{4}-\Lambda mr}\mathrm{e}^{-i\left(\frac{\delta}{2}+\frac{\beta}{2}\right)}\mathrm{e}^{-\left(a_2-i\frac{\zeta}{4}-\frac{\delta}{2}-\frac{\beta}{2}-Ex^0\right)i_3}\left[\frac{1+\mathrm{e}^{i\delta}}{2}+\frac{1-\mathrm{e}^{i\delta}}{2}\sigma_3\frac{\vec{\mathrm{x}}}{r}\right] \tag{70}$$

$$\begin{aligned}\phi\left(1+\frac{\mathbf{l}-\mathbf{r}}{\mathbf{l}+\mathbf{r}}\sigma_3\right)\phi^\dagger&=a_1^2r^{2s-2}\mathrm{e}^{-2\Lambda mr}\left[\frac{1+\mathrm{e}^{-i\delta}}{2}+\frac{1-\mathrm{e}^{-i\delta}}{2}\frac{\vec{\mathrm{x}}}{r}\sigma_3\right]\\&\times\left[1+\frac{\mathbf{l}-\mathbf{r}}{\mathbf{l}+\mathbf{r}}\sigma_3\right]\mathrm{e}^{\frac{\zeta}{2}}\mathrm{e}^{-\frac{\zeta}{2}\sigma_3}\left[\frac{1+\mathrm{e}^{i\delta}}{2}+\frac{1-\mathrm{e}^{i\delta}}{2}\sigma_3\frac{\vec{\mathrm{x}}}{r}\right].\end{aligned} \tag{71}$$

This gives:

$$\mathbf{J}=\frac{m}{k}\left(\frac{\mathrm{D}_L}{\mathbf{l}}+\frac{\mathrm{D}_R}{\mathbf{r}}\right)=\frac{a_1^2}{k}\frac{2\mathbf{l}}{\mathbf{l}+\mathbf{r}}r^{2s-2}\mathrm{e}^{-2\Lambda mr}\left[1+\frac{\sin\delta}{r}\left(\sigma_3\times\vec{\mathrm{x}}\right)\right], \tag{72}$$

$$\mathbf{J}^0=\frac{a_1^2}{k}\frac{2\mathbf{l}}{\mathbf{l}+\mathbf{r}}r^{2s-2}\mathrm{e}^{-2\Lambda mr}. \tag{73}$$

In the proper frame of the electron, its probability density has indeed the spherical symmetry. But this density—contrary to non-relativistic quantum mechanics—is not at all static, but rather turns about the third axis: the electron is a particle with spin, and thus a magnet, as explained by de Broglie [4].

With $\lambda=-1/2$ we have:

$$\phi^\dagger=a_1r^{s-1}\mathrm{e}^{\frac{\zeta}{4}-\Lambda mr}\mathrm{e}^{-i\left(\frac{\delta}{2}+\frac{\beta}{2}\right)}\mathrm{e}^{-\left(a_2-i\frac{\zeta}{4}-\frac{\delta}{2}-\frac{\beta}{2}-Ex^0\right)i_3}\left[\frac{1+\mathrm{e}^{i\delta}}{2}-\frac{1-\mathrm{e}^{i\delta}}{2}\sigma_3\frac{\vec{\mathrm{x}}}{r}\right](-i_2) \tag{74}$$

$$\begin{aligned}\phi\left(1+\frac{\mathbf{l}-\mathbf{r}}{\mathbf{l}+\mathbf{r}}\sigma_3\right)\phi^\dagger&=a_1^2r^{2s-2}\mathrm{e}^{-2\Lambda mr}i_2\left[\frac{1+\mathrm{e}^{-i\delta}}{2}-\frac{1-\mathrm{e}^{-i\delta}}{2}\frac{\vec{\mathrm{x}}}{r}\sigma_3\right]\\&\times\mathrm{e}^{\frac{\zeta}{4}}\mathrm{e}^{\frac{\zeta}{4}\sigma_3}\left[1+\frac{\mathbf{l}-\mathbf{r}}{\mathbf{l}+\mathbf{r}}\sigma_3\right]\mathrm{e}^{\frac{\zeta}{4}}\mathrm{e}^{\frac{\zeta}{4}\sigma_3}\left[\frac{1+\mathrm{e}^{i\delta}}{2}-\frac{1-\mathrm{e}^{i\delta}}{2}\sigma_3\frac{\vec{\mathrm{x}}}{r}\right](-i_2)\\&=a_1^2\frac{2\mathbf{l}}{\mathbf{l}+\mathbf{r}}r^{2s-2}\mathrm{e}^{-2\Lambda mr}\left[1-\frac{\sin\delta}{r}\left(\sigma_3\times\vec{\mathrm{x}}^*\right)\right],\end{aligned} \tag{75}$$

where a plane symmetry is used, with:

$$\vec{\mathrm{x}}^*:=i_2\vec{\mathrm{x}}i_2=\mathrm{x}^1\sigma_1-\mathrm{x}^2\sigma_2+\mathrm{x}^3\sigma_3 \tag{76}$$

A symmetry thus appears, which was completely overlooked by nonrelativistic quantum mechanics: when we go from the $\lambda=1/2$ to the $\lambda=-1/2$, the current density not only changes sign, but the space itself is changed by a symmetry. And we obtain:

$$\mathbf{J}=\frac{m}{k}\left(\frac{\mathrm{D}_L}{\mathbf{l}}+\frac{\mathrm{D}_R}{\mathbf{r}}\right)=\frac{a_1^2}{k}\frac{2\mathbf{l}}{\mathbf{l}+\mathbf{r}}r^{2s-2}\mathrm{e}^{-2\Lambda mr}\left[1-\frac{\sin\delta}{r}\left(\sigma_3\times\vec{\mathrm{x}}^*\right)\right], \tag{77}$$

$$\mathbf{J}^0=\frac{a_1^2}{k}\frac{2\mathbf{l}}{\mathbf{l}+\mathbf{r}}r^{2s-2}\mathrm{e}^{-2\Lambda mr}. \tag{78}$$

Only the probability density is unchanged; the normalization of the wave is the same and we obtain:

$$1=\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{\mathbf{J}^0}{\hbar c}r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi, \tag{79}$$

$$a_1=\frac{l_P^{3/2}\sqrt{\mathbf{r}+\mathbf{l}}\left(2\Lambda m\right)^{s+1/2}}{\sqrt{8\pi\mathbf{l}\Gamma\left(2s+1\right)}}. \tag{80}$$

4. Discussion

The electron without interaction is nevertheless a complicated object: the center of the wave is a singularity and the wave is present in the whole space. The electron wave is stationary in its proper frame and indefinitely stable. Two spin states are possible, the magnetic quantum number is $\lambda=+1/2$ or $\lambda=-1/2$. The stationary wave is seen by any observer moving relative to the electron as a progressive wave, as a consequence of the transformation $R:\mathrm{x}\mapsto\mathrm{x}'=M\mathrm{x}M^{\dagger}$ which acts also on the time, thus on the phase of the wave, seen as propagating in space.

The phase used in non-relativistic quantum mechanics is the angle of a rotation of the probability current in the "spin plane" as understood by de Broglie's followers, as explained by R. Boudet [9]. The particle is the "small clock" of de Broglie's thesis [2]. Any clock uses a periodic movement, which is here the movement of rotation of the probability current about the third axis.

The soliton wave of the electron was imagined for a very long time, and is similar to Descartes' vortex; many physicists followed this idea, but two reasons stopped them: first the rapid dispersion of the quantum wave, and the non-physical nature of the wave as only a mathematical tool to calculate a probability density. Our soliton wave is a physical vortex, with a proper momentum with value $\hbar/2$ (see [6] 2.5). This momentum is quantized, it is impossible to change its value; there is no dispersion, it is linked both to the non-linearity of our wave equation and to the conservation of the kinetic momentum. Next the probability density is a physical necessity, arising from the equivalence principle, and actually issues from an energy-momentum density. And there are not only one but two energy-momentum tensors, because the electron wave has not only one but two parts, left η and right ξ. The necessity of the Lagrangian formalism is also part of the wave properties (see [6] 2.3.4), and determines the value of the energy-momentum and kinetic momentum tensors.

Our soliton wave also bypasses the difficulty of the infinite proper energy of Lorentz's attempt for a model of electron-particle: the mass-energy of the electron is not the energy of the electromagnetic field, but the energy-momentum of the wave. At the center of the soliton wave the energy density is indeed infinite, but the sum over all space of this energy density is finite, and this allows the normalization of the wave. Moreover the electromagnetic field associated to this wave directly combines the components of the energy-momentum tensor (see [6] 1.10).

The electron wave, as calculated above, cannot be obtained without a nonzero difference between the left and right masses (otherwise the exponential term cancels in the radial functions). This explains why this calculation was not previously made. It was indeed necessary, before:

1) to avoid the non-relativistic framework of the Schrödinger equation (quantum field theory was not able to realize that change, because this theory cannot work out the domain of the unique phase of the non-relativistic wave),

2) to discover the importance of chirality in the quantum domain, which was discovered only in 1956,

3) to obtain the true nonlinear wave equation, from the relativistic wave equation of the electron and from the wave equation of Lochak's magnetic monopole [10] [11],

4) to understand the yield brought by the strict constraints of Cl_3^*, a mathematical framework where chirality and relativistic invariance are naturally highlighted,

5) to observe the tracks of magnetic monopoles [12] and to see there the possibility of two different masses for a single fermion wave: physics is first an experimental science,

6) to transpose this hypothesis to the electron wave, and to state that electron physics is compatible with the existence of two masses,

7) It was finally necessary to understand how the above soliton waves may be obtained, particularly to abandon the idea that the electromagnetic potential u/r is exterior to the wave.

The soliton wave obtained for the electron may be generalized to the other fermions (positron, muon, quarks, neutrinos…), and this was already described in [6].

5. Conclusion

We thus may consider as finally solved that difficult question of wave-particle dualism for the fermions: a fermion is a soliton wave, following an equation with partial derivatives, strictly deterministic. But this wave is also compelled to adjust its internal mass-energy, that of the little clock imagined by de Broglie, to the energy-momentum density of the whole wave: this is a perfectly nonlocal condition.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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