Journal of Modern Physics, 2023, 14, 1272-1285
https://www.scirp.org/journal/jmp
ISSN Online: 2153-120X
ISSN Print: 2153-1196

Further Investigations of the Aspect/Bell Error: Maximum Entropy Assessment

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How to cite this paper: Lad, F. (2023) Further Investigations of the Aspect/Bell Error: Maximum Entropy Assessment. *Journal of Modern Physics*, **14**, 1272-1285.
https://doi.org/10.4236/jmp.2023.148072

Received: April 27, 2023
Accepted: July 28, 2023
Published: July 31, 2023

Abstract

This article identifies the maximum entropy distribution among those in the polytope of probability distributions cohering with quantum theoretic prescriptions pertinent to Bell's inequality in the optical context. Perhaps surprisingly, the maxent distribution is not a uniform mixture of the extreme vertices of the convex hull of distributions agreeing with the theory. The expectation $E(s)$ it supports equals 1.1296, within the permitted coherent interval of $(1.1213, 2]$. The maxent mixture of the extreme agreeable vertices is compared herein with two other mixture distributions over the convex hull of those supported by quantum theory. One of these is a simple uniform mixture over the solution vectors to the appropriate linear programming problems that specify the polytope. The other is the mixture underlying simulated results of Aspect's experiments that have been shown to estimate $E(s)$ as 1.7678. Further computations provide examples of the types of claims that would be entailed in a unique distribution within the cohering convex hull such as maxent. These defy quantum theoretic adherence to the general uncertainty principle which proclaims an agnostic position with respect to imagined joint observation operators that do not commute. They also display questionable implications of the "many worlds" proposal which the author does not favour. The article raises questions that deserve to be discussed concerning the general proposal that the maximum entropy principle should be employed to make precise probabilistic assertions about equilibrium phenomena when specific physical theory prescribes only an interval.

Keywords

Bell Inequality, Maxent Distribution, Probability Bounds

1. Prelude and Outline of This Discussion

Well known since the insistent analysis of Fine [1], quantum theoretic considerations do not motivate a complete joint probability distribution for the polarization products constituting the linear combination that underlies Bell's inequality in a gedankenexperiment. The author of the present article has identified [2] precisely the four-dimensional convex hull of distributions that do cohere with the limited prescriptions of quantum theory in this regard. These do not specify a *unique* distribution for the gedanken quantity s which is defined as an unobservable combination of four polarization products on the same pair of photons. Neither does the theory motivate even a unique quantum expectation value $E(s)$ for this quantity. It surely does *not* specify the expectation as $2\sqrt{2}$ in Aspect's setup [3], which is widely promoted as the value defying Bell's inequality. Challenging several long-held assessments of the matter, the author's contribution has identified, rather, the interval $(1.1213, 2]$ as representing the limited implication of quantum theory for this expectation. The present article presumes some familiarity of the reader with these issues. It extends the analysis to expose some interesting implications of the suggestion that the maximum entropy distribution within the convex hull of distributions explicitly supported by quantum theory ought to be entertained as a unique distribution to represent the content of physical theory.

The article of Caticha [4] contains the latest review of extensive work on these matters. The proposition that the maxent distribution would be appropriate is motivated by its appending the minimum amount of information to an analysis in keeping with observation of a physical system in equilibrium. The equilibrium state of a pair of particles such as Aspect's pair of photons is typically characterized as changing smoothly in keeping with Schrödinger's equation. Some contestable interpretations of the Bell construct [5] [6] [7] see it as pertaining to the average polarization products of many paired particle emissions, which could constitute a system in equilibrium. At any rate, the author is not swayed by such a motivation for the maximum entropy reduction of the expectation assessment, preferring rather that the interval assessment of $E(s)$ remains unless or until further supplementary conditions of the optical experiment under consideration might be identified. Of course discussion of the matter would be welcome. The numerical results of this article will present some interesting implications relevant to the dialogue.

Section 1 begins directly with a presentation of computational results. These identify the maximum entropy distribution over eight conceivable observation vectors for polarization products induced by a pair of photons in an Aspect/Bell gedankenexperiment, agreeing both with the prescriptions of quantum theory and with the principle of local realism as it pertains to Bell. The experiment is framed in the CHSH form of Clauser *et al.* [8] as described in the review by Aspect [3]. The substantive implications of quantum theory for the complete results of the experiment are imprecise. However, the author [2] has identified precise restrictions on distributions allowed by the theory that still respect local realism. These leave four free dimensions of agreeable probability vectors for the imagined observation possibilities, and an identifiable convex hull of distributions within them. The maximum entropy selection procedure has been proposed to resolve this indeterminacy as a unique mixture distribution within the free solution space. The computed probability mass function this procedure supports is compared here with two other recognizable mass function vectors.

Section 2 then assesses these results relative to the general uncertainty principle which is honoured fastidiously within standard quantum theory. This specifies that when two non-commuting observation operators are defined on a state vector characterized within a Hilbert space, a joint measurement of the two considered aspects of the state is recognized to be impossible. Such impossibility is evident experimentally as well. Quantum theory specifies an agnostic position with respect to statements about the numerical values of unobservable recordings. Whatever its due merits may be, our deliberations lead us to question the motivation for relying on a maximum entropy resolution of the indeterminacy inhering in quantum theoretic prescriptions.

2. Computational Results for Specific Mixtures

The first four rows of Table 1 display 16 columns of possibilities for the paired polarization observations A and B in an Aspect/Bell thought experiment, wherein a single pair of photons is ejected simultaneously in opposite directions from a central site toward four pairs of polarizers. These are set in the (x, y) planes perpendicular to the direction of emissions at the relative angles $(\mathbf{a},\mathbf{b})$, $(\mathbf{a},\mathbf{b}')$, $(\mathbf{a}',\mathbf{b})$, and $(\mathbf{a}',\mathbf{b}')$. The bold notations of $\mathbf{a}$ or $\mathbf{a}'$ and $\mathbf{b}$ or $\mathbf{b}'$ identify alternate directions of the polarizers. The observation possibilities presume that the value achieved by $A(\mathbf{a})$ does not vary from its value in a paired observation with $B(\mathbf{b})$ when it is paired alternately with $B(\mathbf{b}')$. This accords with the principle of local realism whose relevance is studied via such an experiment. The second partitioned bank of four rows display the component-wise

Table 1. Maxent probabilities for vectors of possible gedanken observations.

$A(\mathbf{a})$	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1
$B(\mathbf{b})$	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1
$A(\mathbf{a}')$	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1
$B(\mathbf{b}')$	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1
*****																
$A(\mathbf{a})B(\mathbf{b})$	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1
$A(\mathbf{a})B(\mathbf{b}')$	1	−1	1	−1	−1	1	−1	1	1	−1	1	−1	−1	1	−1	1
$A(\mathbf{a}')B(\mathbf{b})$	1	1	−1	−1	1	1	−1	−1	−1	−1	1	1	−1	−1	1	1
$A(\mathbf{a}')B(\mathbf{b}')$	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1
*****																
maxentP	0.0978	0.0978	0.0272	0.0978	0.0978	0.0272	0.0272	0.0272	0.0272	0.0272	0.0272	0.0978	0.0978	0.0272	0.0978	0.0978

products of these various possibilities. The final partitioned row displays the maximum entropy probability mass function value among all those that honour the prescriptions of quantum theory assessed for each column possibility, whose generation we shall now discuss. All computations in this article pertain to the setting of relative angles between polarizer directions at stations A and B suspected by Aspect as stimulating the strongest defiance of Bell's inequality: $(\mathbf{a},\mathbf{b}) = -\pi/8$, $(\mathbf{a},\mathbf{b}') = -3\pi/8$, $(\mathbf{a}',\mathbf{b}) = \pi/8$, and $(\mathbf{a}',\mathbf{b}') = -\pi/8$.

While the first partitioned bank of polarization observations contains sixteen distinct columns, the second bank of polarization products contains only eight *distinct* columns. The final eight columns of the second bank repeat the first eight, merely reversed in order. The upshot of this feature of the thought experiment is that the realm matrix of possibilities for the four polarization products contains only eight columns rather than sixteen. These appear again on their own as follows, with each column appended by the value of Bell's quantity that it would entail,

$$s(\lambda) \equiv A(\mathbf{a})B(\mathbf{b}) - A(\mathbf{a})B(\mathbf{b}') + A(\mathbf{a}')B(\mathbf{b}) + A(\mathbf{a}')B(\mathbf{b}'):$$

$$\text{viz.,}\quad \mathbf{R}\begin{pmatrix} A(\mathbf{a})B(\mathbf{b}) \\ A(\mathbf{a})B(\mathbf{b}') \\ A(\mathbf{a}')B(\mathbf{b}) \\ A(\mathbf{a}')B(\mathbf{b}') \\ s(\lambda) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 2 & 2 & -2 & 2 & 2 & -2 & -2 & -2 \end{pmatrix} \tag{1}$$

Since the final row of $\mathbf{R}$ has components equal only to either 2 or −2, each one of the polarization products in the thought experiments appearing in the first four rows must equal a function value of the other three. For further reference, we shall identify this gedankenfunction as $G(\cdot,\cdot,\cdot)$. Specifically, notice that if any three of the first four elements in a column of $\mathbf{R}$ sum to +3 or to −1, then the fourth component must equal +1; whereas if the three sum to −3 or +1, then the fourth must equal −1. (In making sense of this, notice the negative sign on the second component of the linear combination defining $s(\lambda)$.) It is the neglect of these symmetric functional relations among the components of $s(\lambda)$ that has given rise to the mistaken notion that the probabilities of quantum behaviour specify the expectation value $E(s) = 2\sqrt{2}$, defying the Bell inequality which recognizes that $-2 \le E(s) \le +2$.

Of course it is true that the expectation of $s(\lambda)$ must equal a linear combination of the expectations of its components. It is also true that quantum theory identifies the expectations of individual polarization products in Aspect's experiments on four distinct pairs of photons as $E\left[A(\mathbf{a})B(\mathbf{b})\right] = E\left[A(\mathbf{a}')B(\mathbf{b})\right] = E\left[A(\mathbf{a}')B(\mathbf{b}')\right] = \cos(\pi/4) = 1/\sqrt{2}$, and $E\left[A(\mathbf{a})B(\mathbf{b}')\right] = \cos(-3\pi/4) = -1/\sqrt{2}$. However in the gedankenexperiment that gives rise to Bell's inequality, only any three of these free expectations would be relevant to the evaluation of $E(s)$. The expectation of the fourth would involve assessing a function of these three variables, and must be evaluated by linear programming problems. In brief, this is how they are formulated. The expectation for the vector of quantities detailed in $\mathbf{R}$ must equal a convex combination of its columns, $\mathbf{Rq}_8$, where $\mathbf{q}_8$ is a vector of non-negative coefficients that sum to 1. This combination would be constrained by the quantum theoretic expectations specified for each three of the four polarization products, observable on its own without involvement in any gedankenexperiment.

Quantum theoretic distributions do identify precise expectations for the polarization products that might result from emitting a pair of photons at any one of the four angle pairings considered in the Aspect/Bell problem. Any three of these expectation values would constitute linear restrictions on the components of $\mathbf{q}_8$ in a gedankenexperiment which assesses results from a single pair of photons approaching all four angle pairings. Conforming to these restrictions is a 4-D polytope of vectors $\mathbf{q}_8$. It is spanned by eight solution vectors $\mathbf{q}_8^*$ to four pairs of min and max LP-problems for an objective function specified by the expectation of Bell's quantity, $E\left[s(\lambda)\right]$. Each of these problem pairs arises from a distinct choice of three polarization angles from the four that are involved in Bell's gedankenexperiment to specify the constraints. The eight 8-D vectors that solve these extreme value problems are those displayed as columns of the matrix $\mathbf{q}_8^*\mathbf{Mat}$, shown below. It was published in Lad [2] Section 7.2, atop page 1132, an article which presents more expansive details.

$$\mathbf{q}_8^*\mathbf{Mat} = \begin{pmatrix} 0 & 0.1464 & 0 & 0.1464 & 0.7803 & 0.5607 & 0 & 0.1464 \\ 0.7803 & 0.5607 & 0 & 0.1464 & 0 & 0.1464 & 0 & 0.1464 \\ 0.0732 & 0 & 0.0732 & 0 & 0.0732 & 0 & 0 & 0 \\ 0 & 0.1464 & 0.7803 & 0.5607 & 0 & 0.1464 & 0 & 0.1464 \\ 0 & 0.1464 & 0 & 0.1464 & 0 & 0.1464 & 0.7803 & 0.5607 \\ 0.0732 & 0 & 0 & 0 & 0.0732 & 0 & 0.0732 & 0 \\ 0.0732 & 0 & 0.0732 & 0 & 0 & 0 & 0.0732 & 0 \\ 0 & 0 & 0.0732 & 0 & 0.0732 & 0 & 0.0732 & 0 \end{pmatrix} \tag{2}$$

The columns of $\mathbf{q}_8^*\mathbf{Mat}$ are the minimum and maximum solution vectors to linear programming problems that yield the extreme value of $E(s)$ when the "fourth" dependent variable of the function $G(\cdot,\cdot,\cdot)$ is, sequentially, $A(\mathbf{a}')B(\mathbf{b}')$, $A(\mathbf{a}')B(\mathbf{b})$, $A(\mathbf{a})B(\mathbf{b}')$, and $A(\mathbf{a})B(\mathbf{b})$. As befits the symmetry of the problem structures, the columns 1, 3, 5, and 7 of minimum LP solutions are permutations of one another, as are the columns 2, 4, 6, and 8 of maximum solutions. The minimum and maximum values of $E(s(\lambda))$ that these column solutions support are alternately 1.1213 and 2. The associated (entropy, extropy) pairs the alternating columns inhere, respectively, are (0.7678, 0.5444) and (1.1684, 0.7668).

The *extropy* of a probability mass function is a measure that is complementary and dual to its entropy. Introduced and analyzed in [9], the extropy is a measure assessing the probabilities of non-occurrence of possible values of an observation vector. Computationally it equals $-\sum_{i=1}^{N}(1-p_i)\log(1-p_i)$ as opposed to the entropy $-\sum_{i=1}^{N} p_i \log(p_i)$. Algebraically it is equivalent to the (rescaled) entropy in the mass function complementary to $\mathbf{p}_N$, viz., the entropy in the mass function $(\mathbf{1}_N - \mathbf{p}_N)/(N-1)$.

The "maxent distribution" within this polytope is specified by that convex combination of these eight solution vectors whose entropy is a maximum. It has been identified using Monte Carlo methods that scan convex coefficient vectors over the unit simplex $\mathcal{S}^7$. It turns out that in this problem the maximum entropy distribution is also the maximum extropy distribution among those that share the entropy value, a feature that does not necessarily hold.

Two other distributions have also been studied, for comparison. One of them is the uniform combination of the $\mathbf{q}_8^*\mathbf{Mat}$ solution vectors. The other is a mass function vector underlying one component of the Monte Carlo simulation of Aspect's experiments that was analysed in Section 7 of the "Quantum violations" article [2]. The three NAMED vectors appear as the first three columns of the matrix shown in Table 2. They are followed in the Table by the 8-D vectors of linear coefficients for the columns of $\mathbf{q}_8^*\mathbf{Mat}$ that define them. We shall refer to such coefficient vectors as $\mathbf{c}_8 NAME$, displayed here as column vectors. Computational procedures and a discussion will follow the display.

2.1. Computational Methods

The *maxent* pmf vector was identified through a two-stage procedure. To begin, twenty runs of one hundred million Monte Carlo selections of 8-D convex combination vectors were engaged using a uniform distribution over the unit-simplex

Table 2. Three probability vectors constructed as mixtures of the extreme solutions to the Aspect/Bell expectation $E(s(\lambda))$, followed by the linear coefficients $\mathbf{c}_8$ on the extreme solution vectors that support them. Each probability vector equals the matrix $\mathbf{q}_8^*\mathbf{Mat}$ multiplied by the correspondingly named $\mathbf{c}_8$ vector. Below each mixed probability vector is its entropy value, *H*, its extropy value, *J*, and the expectation value $E(s)$ it specifies.

$\mathbf{q}_8$	*MAXNT*	*UNFORM*	*SIMUL*	$\mathbf{c}_8$*MXNT*	$\mathbf{c}_8$*UNFRM*	$\mathbf{c}_8$*SIMUL*
q_1	0.1956	0.2225	0.1067	0.2476	0.125	0.2214
q_2	0.1956	0.2225	0.6219	0.0024	0.125	0.8241
q_3	0.0544	0.0275	0.0183	0.2476	0.125	0.0143
q_4	0.1956	0.2225	0.1067	0.0024	0.125	−0.0295
q_5	0.1956	0.2225	0.1067	0.2476	0.125	0.0143
q_6	0.0544	0.0275	0.0183	0.0024	0.125	−0.0295
q_7	0.0544	0.0275	0.0183	0.2476	0.125	0.0143
q_8	0.0544	0.0275	0.0031	0.0024	0.125	−0.0295
$H(\mathbf{q}_8)$	1.9101	1.7325	1.2495			
$J(\mathbf{q}_8)$	0.9119	0.8911	0.7277			
$E(s(\lambda))$	1.1296	1.5607	1.7678			

$\mathcal{S}^7$. Each selection resulted in a non-negative vector summing to 1, a probability mass function vector (pmf). Each of these selections specifies a distinct convex combination of the LP solution vectors composing $\mathbf{q}_8^*\mathbf{Mat}$ whose *entropy and extropy* were then computed. The maximum entropy value among those of the pmf vectors so determined was $maxH = 1.9089$. Interestingly enough, that pmf entailed the maximum extropy value among them as well, this being $maxJ = .9117$. However, the maxent and maxext pmf displayed above is *not* exactly this one derived simply from the outcome of the Monte Carlo search. The one displayed derived from an adjustment and a perturbation, the motivation for which is now described.

To begin computations, the *uniform* mixture of the columns of $\mathbf{q}_8^*\mathbf{Mat}$ was computed as a simple average of its columns. This was found to entail an entropy value of $HU = 1.7325$ and extropy $JU = 0.8911$. These values were lower than the maximum values reported in the paragraph above on both counts, as expected. On examining this *uniform* mixture of the columns of $\mathbf{q}_8^*\mathbf{Mat}$, it was noticed that two of its component values (0.2225 and 0.0275) were each repeated four times in the vector. Examining then the values of the maxent vector discovered via the Monte Carlo runs, it was noticed that the values corresponding to each of the repeated components in the uniform mixture were also very nearly equal among themselves in both cases. Members of these two groups were then averaged separately to replace them, yielding a mass function with slightly higher entropy. Finally, this function was then perturbed a bit to identify the *MAXNT* mass function displayed in Table 2. All computations displayed have been rounded to four decimal places from the standard sixteen place computation, except for *MAXNT* which I specified exactly to four places. I have not yet discovered another $\mathbf{q}_8$ vector with higher entropy, but the one that is displayed may be inaccurate as the true maxent distribution beyond the fourth decimal place. It is interesting that the value of $E\left[s(\lambda)\right]$ it supports is quite close to the minimum value allowed by quantum theoretic prescriptions, which is 1.1213. Nonetheless, the entropy of any odd numbered column of the $\mathbf{q}_8^*\mathbf{Mat}$ matrix which support this minimum value of $E\left[s(\lambda)\right]$ is only 0.7678, and the extropy is 0.5444.

Finally, the $\mathbf{q}_8$ vector displayed as the *SIMUL* vector in Table 1 was computed from a simulation design that was used to construct the results of the Quantum violations article [2] in its Section 8.2. This specified choosing any three of the simulated polarization product values $A(\mathbf{a}^*)B(\mathbf{b}^*)$ independently using probabilities appropriate to specifications of quantum theory. These derive from the well-known values of probabilities for polarization pairs computed as $\frac{1}{2}\cos^2(\mathbf{a}^*,\mathbf{b}^*)$ and $\frac{1}{2}\sin^2(\mathbf{a}^*,\mathbf{b}^*)$, reported as Equation (1) in [2] and of course elsewhere. The computed example on display in Table 2 is specific to following this procedure using the three relative polarizer angles $(\mathbf{a},\mathbf{b}),(\mathbf{a},\mathbf{b}')$, and $(\mathbf{a}',\mathbf{b})$, and then determining the polarization product at $(\mathbf{a}',\mathbf{b}')$ from them, so to ensure the value of the Aspect/Bell quantity s equals either −2 or +2 as required. For example, the values of q_1 accorded to the first column of the Realm matrix was computed via the prescription

$$P\left[A(\mathbf{a})B(\mathbf{b})=1\right]P\left[A(\mathbf{a})B(\mathbf{b}')=1\right]P\left[A(\mathbf{a}')B(\mathbf{b})=1\right]$$
$$=\cos^2(-\pi/8)\cos^2(-3\pi/8)\cos^2(\pi/8)=0.8536\times 0.1464\times 0.8536=0.1067$$

The remaining values of $\mathbf{q}_8$ for the *SIMUL* column were computed similarly, with adjustments of the quantum probability factors when the observation values of the polarization products are designated differently, yielding one, two, or three −1's among the three polarization products rather than having all equal to 1 as shown in the line above. This procedure simulates the experimental results of Aspect exactly, in the sense that he conducted his experimental runs relevant to any string of four relative angles independently. However, he ignored the functional relations embedded in the components of Bell's quantity pertaining to thought experiments on a single pair of photons; whereas I had used each choice of three independent simulations to generate the fourth polarization product via the required functional relationships denoted by $G(\cdot,\cdot,\cdot)$ among the four. Our simulation data would only *mimic* his experimental results, because they arise merely from the probabilistic specifications of quantum theory rather than from his experimentation. However, structurally they are similar.

2.2. Completing the Simulation Procedure

Each of the three $\mathbf{q}_8$ vectors MAXNT, UNFORM, and SIMUL can be expressed as a linear combination of the columns of our vertex matrix via $\mathbf{q}_8 NAME=\left(\mathbf{q}_8^*\mathbf{Mat}\right)\mathbf{c}_8 NAME$. Using the pseudo-inverse of the vertex matrix $\mathbf{q}_8^*\mathbf{Mat}$, the vectors $\mathbf{c}_8 NAME$ corresponding to each of them has been computed as $\mathbf{PseudoInv}\left(\mathbf{q}_8^*\mathbf{Mat}\right)\mathbf{q}_8 NAME$. Each of the three $\mathbf{c}_8 NAME$ vectors so computed sums to 1.

The fact that $\mathbf{c}_8 MXNT$ and $\mathbf{c}_8 UNFRM$ vectors are strictly positive identifies their $\mathbf{q}_8$ vectors as lying within the convex hull of solution vectors to the Bell LP problems. The disturbing fact that $\mathbf{c}_8 SIMUL$ has three negative components means that this *SIMUL* vector is not in the hull! It does not cohere with the prescriptions of quantum theory, the basis of the constraints that gave rise to those extreme LP solutions. Something is amiss, and we need to address it forthwith. It turns out that there is nothing wrong with what has been done, but merely that the procedure has been incomplete.

The simulation procedure we have formalised to mimic Aspect's empirical work requires completion. When using quantum probabilities for polarization products at three angles independently, the functional relation $G(\cdot,\cdot,\cdot)$ would specify a degenerate conditional distribution for the product at the fourth angle conditional on these. For the computation of the *SIMUL* vector in **Table 1** we chose freely the simulated polarization products at the three chosen angles $(\mathbf{a},\mathbf{b}),(\mathbf{a},\mathbf{b}')$, and $(\mathbf{a}',\mathbf{b})$, and then computed the fourth value of $A(\mathbf{a}')B(\mathbf{b}')$ according to the function $G\left[A(\mathbf{a})B(\mathbf{b}),A(\mathbf{a})B(\mathbf{b}'),A(\mathbf{a}')B(\mathbf{b})\right]$. But we could have chosen any one of the four to be determined by a choice of three other free angles using the same functional relationship among them. There would be a distinct procedure of simulating the four polarization measurements for each choice of the function value variable. The probabilities reported for SIMUL in Table 2 derive from only one of the ways to generate exemplars of the possible outcome vectors, specifying the choice of the polarization product at the relative angle $(\mathbf{a}', \mathbf{b}')$ as the product determined by the $G(\cdot,\cdot,\cdot)$ function.

The columns of Table 3 display quantum probability vectors $\mathbf{q}_8$ that arise from the same procedure that generated the SIMUL vector in Table 2, but each of them involves a different denomination of relative angle as "the fourth" whose polarization product is determined via the G function of the others. The first column replicates the column SIMUL from Table 1, but the others arise from different selections of a polarization product as the dependent variable. *The* $\mathbf{q}_8$ *components of the fifth column are the averages of corresponding row components in the first four columns.* It too yields a pmf vector. The entropy, extropy, and $E(s)$ values inhering in each $\mathbf{q}_8$ column appear below it.

As is expected from the symmetry of their construction, the first four columns of these $\mathbf{q}_8$ vectors are permutations of one another.

Now premultiplying the five pmf columns of Table 3 by the $\mathbf{PseudoInv}\left(\mathbf{q}_8^*\mathbf{Mat}\right)$ yields a column of coefficient vectors $\mathbf{c}_8$ that will orient them in the interior of the convex hull of solution vectors for the Bell LP problems. These are displayed as columns of Table 4.

What a pleasant surprise! While the $\mathbf{c}_8$ coefficients for any single component of the simulation procedure supports a $\mathbf{q}_8$ vector that is outside the convex hull

Table 3. Columns of simulation probabilities for Bell's gedanken possibility vectors, derived from alternative choices of Aspect's fourth polarization product to be determined by the $G(\cdot,\cdot,\cdot)$ function. These appear as the first four columns. The fifth column is their average. The final three rows display the entropy, extropy, and $E(s(\lambda))$ inhering in each.

G *map* →	$A(\mathbf{a}')B(\mathbf{b}')$	$A(\mathbf{a}')B(\mathbf{b})$	$A(\mathbf{a})B(\mathbf{b}')$	$A(\mathbf{a})B(\mathbf{b})$	Average $\mathbf{q}_8$
q_1	0.1067	0.1067	0.6219	0.1067	0.2355
q_2	0.6219	0.1067	0.1067	0.1067	0.2355
q_3	0.0183	0.0183	0.0183	0.0031	0.0145
q_4	0.1067	0.6219	0.1067	0.1067	0.2355
q_5	0.1067	0.1067	0.1067	0.6219	0.2355
q_6	0.0183	0.0031	0.0183	0.0183	0.0145
q_7	0.0183	0.0183	0.0031	0.0183	0.0145
q_8	0.0031	0.0183	0.0183	0.0183	0.0145
$H(\mathbf{q}_8)$	1.2495	1.2495	1.2495	1.2495	1.6079
$J(\mathbf{q}_8)$	0.7277	0.7277	0.7277	0.7277	0.8788
$E(s(\lambda))$	1.7678	1.7678	1.7678	1.7678	1.7678

Table 4. Coefficient vectors $\mathbf{c}_8$ that orient the simulation vectors $\mathbf{q}_8$ of Table 3 with respect to the convex Hull of Bell LP solutions.

G *map* →	$A(\mathbf{a}')B(b')$	$A(\mathbf{a}')B(\mathbf{b})$	$A(\mathbf{a})B(\mathbf{b}')$	$A(\mathbf{a})B(\mathbf{b})$	$\mathbf{c}_8$ For Average
c_1	0.2214	0.0143	0.0143	0.0143	0.0661
c_2	0.8241	−0.0295	−0.0295	−0.0295	0.1839
c_3	0.0143	0.2214	0.0143	0.0143	0.0661
c_4	−0.0295	0.8241	−0.0295	−0.0295	0.1839
c_5	0.0143	0.0143	0.2214	0.0143	0.0661
c_6	−0.0295	−0.0295	0.8241	−0.0295	0.1839
c_7	0.0143	0.0143	0.0143	0.2214	0.0661
c_8	−0.0295	−0.0295	−0.0295	0.8241	0.1839

of quantum solution vectors (negative components of the coefficient vector $\mathbf{c}_8$), a procedure that mixes the use of all four produces a $\mathbf{q}_8$ vector *inside* the hull. All the components of "$\mathbf{c}_8$ *For Average*" are positive, and they sum to 1.

3. Maxent Probability Assertions Eschewed by QM

It is well known that quantum theory abstains from providing a joint probability distribution for quantity vectors that cannot possibly be observed, denying assertions such as

$$P\left[\left(A(\mathbf{a})=a\right)\left(B(\mathbf{b})=b\right)\left(A(\mathbf{a}')=a'\right)\left(B(\mathbf{b}')=b'\right)\right] \tag{3}$$

applying to the 16 possibility vectors $[a,b,a',b']$ for which each component equals either +1 or to −1. Such a possibility vector cannot be observed because it is physically impossible to set up the gedakenexperiment. Rather, the theory promotes only the four paired probability assertions $P\left[\left(A(\mathbf{a}^*)=a\right)\left(B(\mathbf{b}^*)=b\right)\right]$ pertinent to a pair of photons for each of the polarizer angle pairings that can be observed. The choice of the maxent distribution among those that agree with accepted quantum theoretic prescriptions does not shy away from such a proposition. It is interesting to study the implications of such boldness.

To begin, the maxent joint distribution pertinent to the four polarizer products that specify Bell's quantity s also specifies a complete mass function for the vector of polarization observations themselves. We repeat these here from Table 1.

Maxent probabilities for vectors of possible gedanken observations

$A(\mathbf{a})$	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1
$B(\mathbf{b})$	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1
$A(\mathbf{a}')$	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1
$B(\mathbf{b}')$	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1
***** *mxntP*	0.0978	0.0978	0.0272	0.978	0.0978	0.0272	0.0272	0.0272	0.0272	0.0272	0.0272	0.978	0.978	0.0272	0.978	0.978

Without further explication, these probabilities for individual results of the four polarization observations derive from the maxent probabilities for polarization products reported in **Table 2**, supplemented by the fact that quantum probabilities specify further symmetry conditions for each angle pairing $(\mathbf{a}^*, \mathbf{b}^*)$ that

$$P\left[\left(A(\mathbf{a}^*), B(\mathbf{b}^*)\right) = (+1,+1)\right] = P\left[\left(A(\mathbf{a}^*), B(\mathbf{b}^*)\right) = (-1,-1)\right], \text{ and}$$

$$P\left[\left(A(\mathbf{a}^*), B(\mathbf{b}^*)\right) = (+1,-1)\right] = P\left[\left(A(\mathbf{a}^*), B(\mathbf{b}^*)\right) = (-1,+1)\right].$$

These equalities hold no matter what are the polarizer directions $\mathbf{a}^*$ and $\mathbf{b}^*$, identical on the two sides of either equation. The probabilities massed at the columns of joint polarization observations are merely the probabilities for the products they imply but divided by 2. We will need access to these probabilities to address the questions we pose now.

It is standard fare that any joint probability assertion can be factored into the product of a sequence of conditional probabilities, such as

$$\begin{aligned} &P\left[\left(A(\mathbf{a})=1\right)\left(B(\mathbf{b})=1\right)\left(A(\mathbf{a}')=1\right)\left(B(\mathbf{b}')=1\right)\right] \\ &= P\left[\left(A(\mathbf{a})=1\right)\right] P\left[\left(B(\mathbf{b})=1\right) \mid \left(A(\mathbf{a})=1\right)\right] \\ &\quad \times P\left[\left(A(\mathbf{a}')=1\right) \mid \left(A(\mathbf{a})=1\right)\left(B(\mathbf{b})=1\right)\right] \\ &\quad \times P\left[\left(B(\mathbf{b}')=1\right) \mid \left(A(\mathbf{a})=1\right)\left(B(\mathbf{b})=1\right)\left(A(\mathbf{a}')=1\right)\right] \end{aligned} \tag{4}$$

Now quantum theory would surely specify values for the first two of these four factored multiplicands if we were to conduct an experiment exclusively at the polarizing angle $(\mathbf{a}, \mathbf{b})$, say, as $P\left[\left(A(\mathbf{a})=1\right)\right] = 1/2$ and $P\left[\left(B(\mathbf{b})=1\right) \mid \left(A(\mathbf{a})=1\right)\right] = \cos^2(\mathbf{a}, \mathbf{b})$. Perhaps surprisingly, however, while this first specification for individual polarization values continues to hold for the thought experiment on the photon pair at all four angle pairings, the conditional probability $P\left[\left(B(\mathbf{b})=1\right) \mid \left(A(\mathbf{a})=1\right)\right]$ is no longer determined precisely in the gedanken context. Rather, coherency with the quantum theoretic probabilities specified for the four separate polarization angles merely bounds the conditional probability $P\left[\left(B(\mathbf{b})=1\right) \mid \left(A(\mathbf{a})=1\right)\right]$ within the interval $[0, 0.8536)$. This can be determined by an array of fractional programming computations implementing de Finetti's fundamental theorem of probability as it applies to conditional probabilities, see [10] Section 3.3. It can also be derived from the assessment of conditional probabilities associated with the expansions of the solution vectors to the linear programming problems we have identified as $\mathbf{q}_8^*\mathbf{Mat}$.

Though somewhat of an aside at the moment, it is worth noticing that the coherent bounds on the conditional probability $P\left[\left(B(\mathbf{b})=1\right) \mid \left(A(\mathbf{a})=1\right)\right]$ include the value of 0.5, which is the value of the quantum theoretic specification of $P\left(B(\mathbf{b})=1\right)$ on its own. This peculiarity is worth noticing because the difference of this conditional probability $P\left[\left(B(\mathbf{b})=1\right) \mid \left(A(\mathbf{a})=1\right)\right] = \cos^2(\mathbf{a}, \mathbf{b})$ from $P\left(B(\mathbf{b})=1\right) = 1/2$ in an actual experiment at a single paired polarization angle is the feature that characterizes the fabled entanglement of the photons' behaviour at the observation stations *A* and *B*. Just saying.

For now I should also mention that maxent specifies a precise value for the conditional probability $P\left[\left(B(\mathbf{b})=1\right)|\left(A(\mathbf{a})=1\right)\right]$ as 0.3206/5 = 0.6412, while the average simulation specifies it as a value rounding to 0.7288. Both of these values lie within the interval that coheres with the prescriptions of quantum theory, and both of these differ from the marginal probability of 0.5, as is specified by quantum theory in an actual experiment at a single relative polarizer angle.

To complete our analysis of the factorization of joint probability we have identified in Equation (4) we need to consider the third and fourth factors that appear therein. Once again, quantum theory itself specifies nothing about them, as their assessment involves consideration of unobservables. Nonetheless, the probabilities deriving from quantum theory do place bounds on the third and fourth factors if they are to cohere with the positive prescriptions it does provide. Specifically, $P\left[\left(A(\mathbf{a}')=1\right)|\left(A(\mathbf{a})=1\right)\left(B(\mathbf{b})=1\right)\right]$ is bounded to lie within the interval $[0, 0.9142]$, while $P\left[\left(B(\mathbf{b}')=1\right)|\left(A(\mathbf{a})=1\right)\left(B(\mathbf{b})=1\right)\left(A(\mathbf{a}')=1\right)\right]$ may lie anywhere in the unit interval $[0,1]$. These bounds also derive either from fractional programming computations or from the assessment of the vertices of the convex hull appearing in (2). In contrast to the quantum specification of mere intervals, these liberties are taken specifically by the MAXNT assessment as the values 0.6101 and 0.5, and by the average SIMUL assessment as 0.6532 and 0.5.

We have already noted that the specification of the second factor in (4) may portray the so-called entanglement of the two photons' behaviours via a conditional probability that differs from its marginal value $P\left(B(\mathbf{b})=1\right)=1/2$. While this is surely the case in a real experiment with a pair of photons engaging a specific polarization angle, it seems peculiar that this is no longer required (though it would be permitted) in the gedanken scenario. However quantum theory itself says nothing at all pertinent to the third and fourth factors which describe features of unobservable behaviour. Yet contemporary physical theory does entertain such considerations in the form of the so-called "many worlds" hypothesis. In the spirit of the tendered principle of local realism, one might consider proposing a condition such as

$$P\left[\left(A(\mathbf{a}')=1\right)|\left(A(\mathbf{a})=1\right)\left(B(\mathbf{b})=1\right)\right]=P\left[\left(A(\mathbf{a}')=1\right)|\left(B(\mathbf{b})=1\right)\right].$$

While the entanglement of distant photon behaviour is a recognized feature of experimentation [allowing that the value of $B(\mathbf{b})$ may be informative about $A(\mathbf{a}')$], it would be hard for many to accept that physical occurrences in another world in which the polarizer at A is directed at $\mathbf{a}$ could be informative in any way about the observation of $A(\mathbf{a}')$.

Similar, but more intricate quandary would be involved in the specification of the fourth factoring conditional mass function

$$P\left[\left(B(\mathbf{b}')=1\right)|\left(A(\mathbf{a})=1\right)\left(B(\mathbf{b})=1\right)\left(A(\mathbf{a}')=1\right)\right].$$

Which of the impossibly jointly conditioning observations should be considered relevant to $B(\mathbf{b}')$, that of $A(\mathbf{a})$ or that of $A(\mathbf{a}')$? …not to speak of the relevance of $B(\mathbf{b})$. Comparatively in the scenario we have been considering, the quantum theoretic specifications of $P\left[\left(B(\mathbf{b}')=1\right)|\left(A(\mathbf{a})=1\right)\right]$ and $P\left[\left(B(\mathbf{b}')=1\right)|\left(A(\mathbf{a}')=1\right)\right]$ are both different and complementary in distinct real experiments. They sum to 1 as the square of a cosine and the square of a sine.

Nonetheless, the maxent resolution of the incomplete quantum theoretic assessment of such issues is decisive. The maxent resolution specifies the values of $P\left[\left(A(\mathbf{a}')=1\right)|\left(A(\mathbf{a})=1\right)\left(B(\mathbf{b})=1\right)\right]=0.6101$, which differs from its specification of $P\left[\left(A(\mathbf{a}')=1\right)|\left(B(\mathbf{b})=1\right)\right]=0.6412$. Meanwhile it also asserts $P\left[\left(B(\mathbf{b}')=1\right)|\left(A(\mathbf{a})=1\right)\left(B(\mathbf{b})=1\right)\left(A(\mathbf{a}')=1\right)\right]=1/2$, while concomitantly asserting $P\left[\left(B(\mathbf{b}')=1\right)|\left(A(\mathbf{a})=1\right)\right]=0.3588$ along with $P\left[\left(B(\mathbf{b}')=1\right)|\left(A(\mathbf{a}')=1\right)\right]=0.6412$.

One wonders on what foundation do these strange assertions of maxent distributions rest, proclaiming information conveyed across worlds. It appears that supporters of a maxent resolution to incomplete quantum theory have some questions to answer. Rather than speculating, we shall conclude now, leaving such issues as a proposal for discussion. On my own account, I would resign myself to consider the quantum theoretic assertion of $E(s)$ merely as the interval $(1.1213, 2]$. I believe this would be the position held by de Finetti, and surely by the many proponents of interval probabilities as representative of scientific uncertainty.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Fine, A. (1982) *Physics Review Letters*, 48, 291-295. https://doi.org/10.1103/PhysRevLett.48.291

[2] Lad, F. (2021) *Journal of Modern Physics*, **12**, 1109-1144. https://doi.org/10.4236/jmp.2021.128067

[3] Aspect, A. (2002) Bell's Theorem: The Naive View of an Experimentalist. In: Bertlmann, R.A. and Zeilinger, A., Eds., *Quantum* [*Un*]*speakables*, Springer, Berlin, 119-153. https://doi.org/10.1007/978-3-662-05032-3_9

[4] Caticha, A. (2021) *Entropy*, **23**, Article 895. https://doi.org/10.3390/e23070895

[5] Redhead, M. (1987) Incompleteness, Nonlocality, and Realism A Prolegomenon to the Philosophy of Quantum Mechanics. Clarendon Press, Oxford. https://doi.org/10.1063/1.2811563

[6] Mermin, N.D. (2005) *Foundations of Physics*, **35**, 2073-2077. https://doi.org/10.1007/s10701-005-8663-2

[7] Gill, R.D. (2014) *Statistical Science*, **29**, 512-528. https://doi.org/10.1214/14-STS490

[8] Clauser, J.F., Horne, M.A., Shimony, A. and Holt, R.A. (1969) *Physical Review Letters*, **23**, 880-884. https://doi.org/10.1103/PhysRevLett.23.880

[9] Lad, F., Sanfilippo, G. and Agró, G. (2015) *Statistical Science*, **30**, 40-58. https://doi.org/10.1214/14-STS430

[10] Lad, F. (1996) Operational Subjective Statistical Methods: A Mathematical, Philosophical, and Historical Introduction. John Wiley, New York.