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Description of an Initial State of Electron before Absorbing Infinite Number of Photons

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Abstract

In our investigation, we examine the initial state of an electron that is represented as a massless point-like charge before it absorbs an infinite number of photons. We consider this state as an eigen-function corresponding to the electron charge as an eigenvalue. As a result, we obtain a three-dimensional delta function as expected.

Keywords

Initial State of Electron, Three Dimensional Delta Function

1. Introduction

The theory that particle mass is generated by Higgs mechanism is widely accepted. Especially after the discovery of the Higgs particle [1], the Higgs mechanism has been convinced as the true origin of bosons. Encouraged by this discovery, many physicists have been trying to show that the Higgs mechanism is also true for fermions, specifically by showing a mass hierarchy for quarks and leptons [2]. However, some physicists are still not convinced that the Higgs mechanism is true for fermions, primarily because of lepton consideration. In this regard, it becomes very important whether an electron charge is a point or spreads out to finite volume because Weinberg shows that electron charge spreads out to finite volume simultaneously when it gains mass by absorbing infinite number of photons by estimating its charged radius in his famous book of Quantum Theory of Field I [3], for one instance. This point has become more important after ACME collaboration reports that they could not find a dipole moment inside electron [4]. Recently, this problem has attracted to scientists in other fields beyond particle physics, especially electron Laser physics [5]. We also investigated this problem considering an electron charge distribution as a corresponding eigen-function of electron mass that plays as an eigenvalue stimulating by Weinberg' description of electron in previous article [6]. The purpose of this paper is to show that our method used in the previous article is reasonable by showing that an initial state of electron is massless point particle as described in Weinberg's book [3] using same method in Ref. [6].

2. Formulation

We investigated an electron charge distribution function stimulated by Weinberg's consideration [3] in the previous article [6]. Weinberg considered that an electron is point-like charge before absorbing infinite number of photons and by absorbing infinite photons it obtains mass and simultaneously its charge is spread out finite volume. In our previous article, by introducing a charge matrix and by considering the equation of motion, we obtained an electron charge distribution function in its final state as a corresponding eigen-function to electron mass as an eigenvalue. In this paper, we investigate an initial state of electron corresponding to Weinberg's electron state before absorbing infinite photons. To approach this problem, we use the same method as the previous article because the obtained result may show whether our charge distribution function in Ref. [6] is actuary comparable to Weinberg's description of electron or not. To consider this problem, we assume that an electron charge before absorbing infinite photons is a bare charge e_b and that a vacuum permittivity is $\bar{\epsilon}_0$ that is different from normal vacuum permittivity ϵ_0. In fact normal vacuum permittivity is considered under finite volume. However, to follow Weinberg's consideration, we must consider the vacuum permittivity at $a_0 \to 0$ case that is defined under volume-less vacuum. To do this, we use the estimation method for vacuum permittivity by Mainland *et al.* [7]. According to Mainland *et al.*, the vacuum permittivity is estimated as

$$\epsilon_0 = \frac{1}{L_{0p}^3} \frac{\langle P^{VF} \rangle}{|E_0|}$$

where L_{0p}: zitterbewegung length (refer to Ref. [7])

$|E_0|$: electron field strength interacting with a single photon

$\langle P^{VF} \rangle$: expectation value of electric dipole moment

They use electron mass to estimate a zitterbewegung length L_{0p}. However, we are going to consider an initial state of electron case in which its mass is zero. Thus, we estimate L_{0p} from an eigen-function associated to an eigenvalue $P_0 a_0$ (a_0: electron charge radius) of an equation of motion describing in later part of this section.

As shown in later, the obtained eigen-function is $\mathrm{e}^{-\bar{r}^2} = \exp\left(-\frac{r^2}{a_0^2}\right)$. Thus, it is allowable for us to set as $L_{0p} \sim a_0$. The electron field strength interacting with a single photon $|E_0|$ can be expressed as

$$|E_0| = \frac{e_b}{a_0^2}$$

e_b : electron bare charge

By denoting an eigen-function associated with an eigenvalue ($P_0 a_0$) as $\phi(r)(r=|\vec{r}|)$, the expectation value of dipole moment $\langle P^{VF}\rangle$ is estimated as

$$\langle P^{VF}\rangle = \frac{4\pi\int_0^\infty \mathrm{d}r\, r^2 \phi^\dagger(r) e_b r \phi(r)}{4\pi\int_0^\infty \mathrm{d}r\, r^2 \phi^\dagger(r)\phi(r)}$$

As mentioned before, $\phi(r)$ is actuary $\exp\left(-\beta \frac{r^2}{a_0^2}\right)$ so that $\langle P^{VF}\rangle$ becomes

$$\langle P^{VF}\rangle = \sqrt{\frac{2}{\pi}}\frac{e_b}{\sqrt{\beta}}$$

As shown in later, β is expressed as $\beta^2 = 2\frac{e_b^2}{\bar{\epsilon}_0}\frac{P_0 a_0}{\hbar c}$. Thus, the vacuum permittivity at $a_0 \to 0$ case is described as

$$\bar{\epsilon}_0 \sim \frac{1}{a_0^3}\frac{e_b a_0^{\frac{3}{4}}}{\frac{e_b}{a_0^2}} \sim \frac{1}{a_0^{\frac{1}{4}}} \tag{1}$$

Recalling $\frac{e_b^2}{\bar{\epsilon}_0}$ is related to $\frac{e^2}{\epsilon_0}$ by renormalization, β is dimensionless so that $\bar{\epsilon}_0$ is dimensionless. Equation (1) shows that a_0 dependence of the vacuum permittivity $\bar{\epsilon}_0$ at $a_0 \to 0$ case. Equation (1) shows that $\bar{\epsilon}_0$ goes to infinity as $a_0 \to 0$. Therefore the quantity $\frac{e_b^2}{\bar{\epsilon}_0}$ is meaningful.

In previous article [6], we introduced a charge matrix state as

$$\rho_{\lambda\eta}(t,\vec{r}) = (-e)\langle 0| q_\lambda(t,\vec{r}) q_\eta^\dagger(t,\vec{r}) |\text{charge state}\rangle$$

where λ, η are Dirac indices. This kind of charge matrix (only operator part) is also used by Karnieli [5] as mentioned in Ref. [6]. For considering an initial state of electron case, we use the same charge matrix state except using charge e_b (*bare charge*) instead of *e*.

To obtain the equation of motion for an initial state of electron case, we use same argument as Ref. [6]. However, recalling that $\rho(t,\vec{r}) = \mathrm{e}^{-iP_0 t}\rho(r)(r=|\vec{r}|)$, we have to change the following two points.

First: Gauss' law becomes $\mathrm{div}\vec{E} = \frac{\rho_0}{\bar{\epsilon}_0}$ vacuum permittivity is changed from normal ϵ_0 to $\bar{\epsilon}_0$ for which we described before.

Second: necessary condition of $\rho_0(r)$ becomes

$4\pi\int_0^\infty \mathrm{d}r\, r^2 \rho_0(r) = -e_b$ and taking $\rho_0(r) = -e_b\bar{\rho}_0(r)$

$$4\pi\int_0^\infty \mathrm{d}r\, r^2 \bar{\rho}_0(r) = 1 \tag{2}$$

Decomposing $\rho(t,\vec{r})$ as

$$\rho(t,\vec{r}) = 1\rho_0(t,\vec{r}) + (-i\vec{\alpha}\cdot\hat{r})\rho_1(t,\vec{r}) + \beta\rho_2(t,\vec{r}) + \beta(i\vec{\alpha}\cdot\hat{r})\rho_3(t,\vec{r})$$

as shown in Ref. [6], we obtain the equation for eigen value P_0 as follows.

$$\begin{aligned} P_0\bar{\rho}_0 &= -\hbar c\frac{\partial\bar{\rho}_1}{\partial r} - \hbar c\frac{2}{r}\bar{\rho}_1 + \frac{e_b^2}{\epsilon_0}\bar{\rho}_0(r)\left[\frac{1}{r}\int_0^r \mathrm{d}r' r'^2\bar{\rho}_0(r')\right. \\ &\left. + \int_0^\infty \mathrm{d}r' r'\bar{\rho}_0(r') - \int_0^r \mathrm{d}r' r'\bar{\rho}_0(r')\right] \end{aligned} \tag{3}$$

$$P_0\bar{\rho}_1 = \hbar c\frac{\partial\bar{\rho}_0}{\partial r} \tag{4}$$

These form of equations are obtained in Ref. [6] except that charge part $\frac{e^2}{\epsilon_0}$ is changed to $\frac{e_b^2}{\epsilon_0}$. Recalling that we are considering an initial state of electron and that its quantity we know is only charge quantity-e after being renormalized. Thus we have to construct the equation of motion for eigen value of $[P_0 r]$ because dimension of e^2 is [*energy* × *length*]. To do this, we multiply r for both side of Equation (3). Then Equation (3) becomes as

$$\begin{aligned} P_0 r\bar{\rho}_0 &= -\hbar c r\frac{\partial\bar{\rho}_1}{\partial r} - \hbar c 2\bar{\rho}_1 + \frac{e_b^2}{\epsilon_0}\bar{\rho}_0(r)\left[\int_0^r \mathrm{d}r' r'^2\bar{\rho}_0(r')\right. \\ &\left. + r\int_0^\infty \mathrm{d}r' r'\bar{\rho}_0(r') - r\int_0^r \mathrm{d}r' r'\bar{\rho}_0(r')\right] \end{aligned} \tag{5}$$

From Equation (4) $\bar{\rho}_1$ is described as $\bar{\rho}_1 = \frac{\hbar c}{P_0}\frac{\partial\bar{\rho}_0}{\partial r}$, then we substitute this $\bar{\rho}_1$ into Equation (5). Then, taking new variable as $\bar{r} = \frac{r}{a_0}$ and after using some manipulation, we obtain the following equation for $\bar{\rho}_0(\bar{r})$.

$$\begin{aligned} \left(\frac{P_0 a_0}{\hbar c}\right)^2\bar{\rho}_0 &= -\frac{\partial^2\bar{\rho}_0}{\partial\bar{r}^2} - \frac{2}{\bar{r}}\frac{\partial\bar{\rho}_0}{\partial\bar{r}} + \frac{e_b^2}{\epsilon_0}\frac{(P_0 a_0)}{(\hbar c)^2}\bar{\rho}_0(\bar{r})a_0^3\left[\frac{1}{\bar{r}}\int_0^{\bar{r}} \mathrm{d}\bar{r}'\bar{r}'^2\bar{\rho}_0(\bar{r}')\right. \\ &\left. + \int_0^\infty \mathrm{d}r' r'\bar{\rho}_0(\bar{r}') - \int_0^{\bar{r}} \mathrm{d}r' r'\bar{\rho}_0(\bar{r}')\right] \end{aligned} \tag{6}$$

Note that a charge distribution function must satisfy Equation (2), and for $\bar{\rho}_0(\bar{r})$ case, Equation (2) is described as $\int_0^\infty \mathrm{d}\bar{r}\,\bar{r}^2\bar{\rho}_0(\bar{r}) = \frac{1}{4\pi a_0^3}$. Thus, $\bar{\rho}_0(\bar{r})$ can be expressed as $\bar{\rho}_0(\bar{r}) = \frac{1}{a_0^3}k(\bar{r})$ and $k(\bar{r})$ is dimensionless some function of $\bar{r}$. Thus, Equation (6) is consistent for dimensional argument.

Because $\bar{r}$ is dimensionless, we can use Tayler expansion for integral parts as shown in Ref. [6]. Then, Equation (6) becomes

$$\frac{\partial^2\bar{\rho}_0}{\partial\bar{r}^2} + \frac{2}{\bar{r}}\frac{\partial\bar{\rho}_0}{\partial\bar{r}} + \left(\frac{P_0 a_0}{\hbar c}\right)^2\bar{\rho}_0 - \frac{e_b^2}{\epsilon_0}\frac{(P_0 a_0)}{(\hbar c)^2}C_1\bar{\rho}_0 - \frac{1}{2}\frac{e_b^2}{\epsilon_0}\frac{(P_0 a_0)}{(\hbar c)^2}\bar{r}^2\bar{\rho}_0^2 = 0 \tag{7}$$

where $C_1 = a_0^3\int_0^\infty \mathrm{d}\bar{r}'\bar{r}'\bar{\rho}_0(\bar{r}')$

Note that because $\bar{\rho}_0(\bar{r})$ is described as $\bar{\rho}_0(\bar{r}) = \frac{1}{a_0^3}k(\bar{r})$ as mentioned before, C_1 is dimensionless.

We can deal with Equation (7) as an equation for new eigenvalue $V_{a_0} = P_0 a_0$.

Since we cannot find exact solution of Equation (7), we set following condition similar to that as shown in Ref. [6].

$$a_0^3 \bar{\rho}_0^2 = \bar{\rho}_0(\bar{r}) + a_0^3 f(\bar{r}) \tag{8}$$

Using the Condition (8), Equation (7) becomes

$$\begin{aligned} &\frac{\partial^2 \bar{\rho}_0}{\partial \bar{r}^2} + \frac{2}{\bar{r}}\frac{\partial \bar{\rho}_0}{\partial \bar{r}} + \left(\left(\frac{V_{a_0}}{\hbar c}\right)^2 - \frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}C_1\right)\bar{\rho}_0 - \frac{1}{2}\frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}\bar{r}^2\bar{\rho}_0 \\ &= \frac{1}{2}\frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}\bar{r}^2 a_0^3 f(\bar{r}) \end{aligned} \tag{9}$$

Equation (9) is an inhomogeneous second order differential equation. Thus, to find this solution, we first solve homogeneous equation, then we use Wronskian way to construct this solution as shown in Ref. [6]. As we mentioned in Ref. [6], this way of construction of solution is well known and we cite Ince's book [8] as an example of reference.

Homogeneous part of Equation (9) is expressed as

$$\frac{\partial^2 \bar{\rho}_0}{\partial \bar{r}^2} + \frac{2}{\bar{r}}\frac{\partial \bar{\rho}_0}{\partial \bar{r}} + \left(\left(\frac{V_{a_0}}{\hbar c}\right)^2 - \frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}C_1\right)\bar{\rho}_0 - \frac{1}{2}\frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}\bar{r}^2\bar{\rho}_0 = 0 \tag{10}$$

Taking $\bar{\rho}_0(\bar{r}) = \frac{1}{\bar{r}}W(\bar{r})$ and after some manipulation, we obtain the following equation for $W(\bar{r})$.

$$\frac{\partial^2 W}{\partial \bar{r}^2} + \left(\left(\frac{V_{a_0}}{\hbar c}\right)^2 - \frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}C_1\right)W - \frac{1}{2}\frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}\bar{r}^2 W = 0 \tag{11}$$

Changing variable as $\bar{r}^2 = z$ and recalling that $\frac{\partial^2}{\partial \bar{r}^2} = 4z\frac{\partial^2}{\partial z^2} + 2\frac{\partial}{\partial z}$, Equation (10) becomes as

$$\frac{\partial^2 W}{\partial z^2} + \frac{1}{2z}\frac{\partial W}{\partial z} + \frac{\left(\frac{V_{a_0}}{\hbar c}\right)^2 - \frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}C_1}{4z}W - \frac{1}{8}\frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}W = 0 \tag{12}$$

Taking $W(z) = z^{-\frac{1}{4}}\bar{W}(z)$ and after some manipulation, we obtain the following equation for $\bar{W}(z)$ as

$$\frac{\partial^2 \bar{W}}{\partial z^2} + \left[-\frac{1}{8}\frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2} + \frac{\frac{1}{4}\left(\left(\frac{V_{a_0}}{\hbar c}\right)^2 - \frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}C_1\right)}{z} + \frac{\frac{3}{16}}{z^2}\right]\bar{W} = 0 \tag{13}$$

Changing variable as $z = \beta\bar{z}$, Equation (13) becomes

$$\frac{\partial^2 \bar{W}}{\partial \bar{z}^2} + \left[-\frac{1}{8}\frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}\beta^2 + \frac{\frac{1}{4}\left(\left(\frac{V_{a_0}}{\hbar c}\right)^2 - \frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}C_1\right)\beta}{\bar{z}} + \frac{\frac{3}{16}}{\bar{z}^2} \right]\bar{W} = 0 \tag{14}$$

Taking $\frac{1}{8}\frac{\bar{e}_b^2}{\bar{\epsilon}_0}\frac{V_{a_0}}{(\hbar c)^2}\beta^2 = \frac{1}{4}$, this determines β as $\beta = \left(2\frac{\bar{e}_b^2}{\bar{\epsilon}_0}\frac{V_{a_0}}{(\hbar c)^2}\right)^{-\frac{1}{2}}$.

Inserting this β into Equation (14), we obtain the following equation.

$$\frac{\partial^2 \bar{W}}{\partial \bar{z}^2} + \left[-\frac{1}{4} + \frac{\frac{1}{4}\left(\left(\frac{V_{a_0}}{\hbar c}\right)^2 - \frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}C_1\right)\left(2\frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}\right)^{-\frac{1}{2}}}{\bar{z}} + \frac{\frac{3}{16}}{\bar{z}^2} \right]\bar{W} = 0 \tag{15}$$

Denoting as $\kappa = \frac{1}{4}\left(\left(\frac{V_{a_0}}{\hbar c}\right)^2 - \frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}C_1\right)\left(2\frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}\right)^{-\frac{1}{2}}$ and recalling Equation (15) is a standard form of Whittaker's differential equation [9]. We obtain basis solutions of Equation (15) as

$$M_{\kappa,\mu}(\bar{z}) \text{ and } M_{\kappa,-\mu}(\bar{z})$$

We do not use $W_{\kappa,\mu}(\bar{z})$ and $W_{\kappa,-\mu}(\bar{z})$ type solutions because we are seeking eigen values corresponded to Equation (15).

Standard form of $\frac{1}{\bar{z}^2}$ term is expressed as $\frac{-\mu^2 + \frac{1}{4}}{\bar{z}^2}$, thus μ value of the solution of Equation (15) is $\mu = \pm\frac{1}{4}$. Definition of $M_{\kappa,\mu}(\bar{z})$ is

$$M_{\kappa,\mu}(\bar{z}) = \bar{z}^{\mu+\frac{1}{2}} e^{-\frac{\bar{z}}{2}} F\left(\mu - \kappa + \frac{1}{2}, 2\mu + 1; \bar{z}\right) \text{ [9]} \tag{16}$$

where $F(\sigma,\gamma;z)$ is Kummer's hyper geometric series defined as [9]

$$F(\sigma,\gamma;z) = \sum_{n=0}^{\infty} \frac{\sigma(\sigma+1)\cdots(\sigma+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)}\frac{z^n}{n!} \tag{17}$$

Thus, when $\bar{z}$ becomes sufficiently large, $M_{\kappa,\mu}(\bar{z})$ behaves as $e^{\frac{\bar{z}}{2}}\bar{z}^{-\kappa}$. However, electron charge must be zero at r goes to ∞ so that its series have to be terminated. This gives the following condition equation.

$$\mu - \kappa + n + \frac{1}{2} = 0 \tag{18}$$

For electron, we take $n = 0$ as in Ref. [6]. Recalling that κ denotes as $\kappa = \frac{1}{4}\left(\left(\frac{V_{a_0}}{\hbar c}\right)^2 - \frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}C_1\right)\left(2\frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}\right)^{-\frac{1}{2}}$ and choosing $\mu = \frac{1}{4}$, that is $\kappa = \frac{3}{4}$, we obtain the determining equation of V_{a_0} as

$$\left(\left(\frac{V_{a_0}}{\hbar c}\right)^{\frac{3}{2}}-\frac{e_b^2}{\bar{\epsilon}_0}\frac{1}{\hbar c}\left(\frac{V_{a_0}}{\hbar c}\right)^{\frac{1}{2}}C_1\right)\left(2\frac{e_b^2}{\bar{\epsilon}_0}\frac{1}{\hbar c}\right)^{-\frac{1}{2}}=3 \tag{19}$$

Denoting $X=\left(\frac{V_{a_0}}{\hbar c}\right)^{\frac{1}{2}}$ and $\xi=\left(\frac{e_b^2}{\bar{\epsilon}_0\hbar c}\right)^{\frac{1}{2}}$, Equation (19) becomes

$$X^3-\xi^2C_1X-3\sqrt{2}\xi=0 \tag{20}$$

Equation (20) shows that X has a definite positive solution and we know that $\frac{e_b^2}{\bar{\epsilon}_0}$ is related to $\frac{e^2}{\epsilon_0}$ after being renormalized. Then we can set the following condition as

$$\frac{e_b^2}{\bar{\epsilon}_0}=\zeta\frac{e^2}{\epsilon_0} \quad \text{where } \zeta \text{ is some constant} \tag{21}$$

Note that this condition becomes meaningful only when a_0 approach 0.

Because electron charge is appeared as a quantity as $\frac{e^2}{4\pi\epsilon_0}$ in quantum electrodynamics (QED), we seek to find the corresponding eigenvalue as $V_{a_0}=\frac{e^2}{4\pi\epsilon_0}=\alpha\hbar c$.

Then, substituting $V_{a_0}=\alpha\hbar c$ into Equation (19) and using condition Equation (21), we obtain the equation for ζ as follows.

$$\zeta+\frac{3\sqrt{2}}{\sqrt{4\pi\alpha}}\sqrt{\zeta}-\frac{1}{4\pi C_1}=0 \tag{22}$$

Taking $\sqrt{\zeta}=\bar{\zeta}$, Equation (22) becomes

$$\bar{\zeta}^2+\frac{3\sqrt{2}}{\sqrt{4\pi\alpha}}\bar{\zeta}-\frac{1}{4\pi C_1}=0 \tag{23}$$

Because $\bar{\zeta}$ must be positive, we obtain $\bar{\zeta}=\frac{1}{2}\left(\sqrt{\frac{18}{4\pi\alpha}+\frac{1}{\pi C_1}}-\frac{3\sqrt{2}}{\sqrt{4\pi\alpha}}\right)$.

First, we construct a corresponding eigen-function using Wroskian method. However, we have to remind that this eigen-function is only formal one and becomes meaningful only after taking limitation of $a_0\to 0$ because of our definition of vacuum permittivity $\bar{\epsilon}_0$.

To find a function $f(\bar{r})$, we use Equation (8) and this gives

$$\bar{\rho}_0(\bar{r})=\frac{1}{2a_0^3}\left(1\pm\sqrt{1+4a_0^6f(\bar{r})}\right)$$

Because we are seeking a useful solution when a_0 goes to zero, we take $\bar{\rho}_0\sim -a_0^3f(\bar{r})$

Substituting this form of $\bar{\rho}_0$ into Equation (9), we obtain the following equation for $f(\bar{r})$ as

$$\frac{\partial^2 f}{\partial \bar{r}^2}+\frac{2}{\bar{r}}\frac{\partial f}{\partial \bar{r}}+\left(\left(\frac{V_{a_0}}{\hbar c}\right)^2-\frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}C_1\right)f=0 \tag{24}$$

Taking $f(\bar{r})=\frac{1}{\bar{r}}\bar{f}(\bar{r})$, equation for $\bar{f}(\bar{r})$ is obtained as

$$\frac{\partial^2 \bar{f}}{\partial \bar{r}^2}+\left(\left(\frac{V_{a_0}}{\hbar c}\right)^2-\frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}C_1\right)\bar{f}=0 \tag{25}$$

Usual solution of Equation (25) is $\sin(\delta\bar{r})$ where $\delta=\sqrt{\left(\frac{V_{a_0}}{\hbar c}\right)^2-\frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}C_1}>0$ under our required V_{a_0} value. However, we are considering a solution under $a_0\to 0$ condition so that we solve Equation (25) as follows. Changing variable as $u=\frac{1}{\bar{r}}$ and Equation (25) becomes

$$\frac{\partial^2 \bar{f}}{\partial u^2}+\frac{2}{u}\frac{\partial \bar{f}}{\partial u}+\frac{\delta}{u^4}\bar{f}=0 \tag{26}$$

Taking $\bar{f}(u)=\exp\left(-\frac{\alpha}{u}\right)F(u)$, we obtain the following equation of $F(u)$ from Equation (26).

$$\frac{\partial^2 F}{\partial u^2}+\left(\frac{2}{u^2}+\frac{2}{u}\right)\frac{\partial F}{\partial u}+\frac{\alpha^2+\delta}{u^4}F=0 \tag{27}$$

Taking $\alpha=i\sqrt{\delta}$ and $\bar{F}=\frac{\partial F}{\partial u}$, Equation (27) becomes

$$\frac{\partial \bar{F}}{\partial u}+\left(\frac{2}{u^2}+\frac{2}{u}\right)\bar{F}=0 \tag{28}$$

Equation (28) is first order differential equation so that a solution of Equation (28) is expressed as

$$\bar{F}=\int_0^u \mathrm{d}\bar{u}\exp\left(\frac{i2\sqrt{\delta}}{\bar{u}}\right)\frac{1}{\bar{u}^2} \tag{29}$$

Thus, a solution of Equation (26) is

$$\bar{f}(u)=\exp\left(-i\frac{\sqrt{\delta}}{u}\right)\int_0^u \mathrm{d}u'\frac{1}{u'^2}\exp\left(\frac{i2\sqrt{\delta}}{u'}\right) \tag{30}$$

Because we are interested in the behavior of a particular solution when a_0 approach 0, we check that the behavior of $\bar{f}(u)$ when u approach 0 recalling that $u=\frac{1}{\bar{r}}=\frac{a_0}{r}$.

To do this, changing variable as $u=-i(z+i\varepsilon)$, integral part of Equation (30) becomes

$$\lim_{\varepsilon\to 0}\left[\int_{-i\varepsilon}^{z}(-i)\mathrm{d}z'\frac{1}{-(z'+i\varepsilon)^2}\exp\left(-\frac{2\sqrt{\delta}}{z'+i\varepsilon}\right)\right]$$

When z goes to $-i\varepsilon$, this obviously goes to 0 because its integrand is well defined in the integral region. In addition, magnitude of $\exp\left(-i\frac{\sqrt{\delta}}{u}\right)$ is less and equal to 1. Thus, $\bar{f}(u)$ approaches 0 when a_0 goes to 0.

Recalling a solution of Equation (15) with $\mu=\frac{1}{4}$ is $M_{\frac{3}{4},\frac{1}{4}}(\bar{z})=\bar{z}^{\frac{3}{4}}\exp\left(-\frac{\bar{z}}{2}\right)$, the other solution is $M_{\frac{3}{4},-\frac{1}{4}}(\bar{z})$ and its series part is not terminated so that at sufficiently large $\bar{z}$, $M_{\frac{3}{4},-\frac{1}{4}}(\bar{z})$ becomes as $\bar{z}^{-\frac{1}{2}}\exp\left(\frac{\bar{z}}{2}\right)$. Then, basis solutions for $\bar{\rho}_0(\bar{r})$ are $\exp\left(-\frac{\bar{r}^2}{2\beta}\right)$ and $\bar{r}^{-\frac{5}{2}}\exp\left(\frac{\bar{r}^2}{2\beta}\right)$ (this form represents for sufficiently large $\bar{r}$ case)

Then Wronskian is expressed as

$$\text{Wroskian}=\begin{vmatrix} e^{-\frac{\bar{r}^2}{2\beta}} & -\frac{\bar{r}}{\beta}e^{-\frac{\bar{r}^2}{2\beta}} \\ \bar{r}^{-\frac{5}{2}}e^{-\frac{\bar{r}^2}{2\beta}} & -\left(\frac{5}{2}\bar{r}^{-\frac{7}{2}}+\frac{\bar{r}^{-\frac{3}{2}}}{2\beta}\right)e^{-\frac{\bar{r}^2}{2\beta}} \end{vmatrix}=\frac{2}{\beta}\bar{r}^{-\frac{3}{2}}-\frac{5}{2}\bar{r}^{-\frac{7}{2}} \tag{31}$$

A particular solution of Equation (9) without coefficient term $S(\bar{r})$ is written as

$$S(\bar{r})=\left[\int_\infty^{\bar{r}}\mathrm{d}\bar{r}'\frac{\bar{r}'^2 f(\bar{r}')e^{-\frac{\bar{r}'^2}{2\beta}}}{\frac{2}{\beta}\bar{r}'^{-\frac{3}{2}}-\frac{5}{2}\bar{r}'^{-\frac{7}{2}}}\right]\bar{r}^{-\frac{5}{2}}e^{\frac{\bar{r}^2}{2\beta}}-\left[\int_\infty^{\bar{r}}\mathrm{d}\bar{r}'\frac{\bar{r}'^2 f(\bar{r}')\bar{r}'^{-\frac{5}{2}}e^{\frac{\bar{r}'^2}{2\beta}}}{\frac{2}{\beta}\bar{r}'^{-\frac{3}{2}}-\frac{5}{2}\bar{r}'^{-\frac{7}{2}}}\right]e^{-\frac{\bar{r}^2}{2\beta}} \tag{32}$$

Considering that $\bar{r}'$ is large to evaluate integral parts, Equation (32) becomes

$$\left[\int_\infty^{\bar{r}}\mathrm{d}\bar{r}'\bar{r}'^{\frac{7}{2}}f(\bar{r}')e^{-\frac{\bar{r}'^2}{2\varsigma}}\right]\bar{r}^{-\frac{5}{2}}e^{\frac{\bar{r}^2}{2\beta}}-\left[\int_\infty^{\bar{r}}\mathrm{d}\bar{r}'\bar{r}'f(\bar{r}')e^{\frac{\bar{r}'^2}{2\beta}}\right]e^{-\frac{\bar{r}^2}{2\beta}} \tag{33}$$

Taking $u=\frac{1}{\bar{r}}$, Equation (33) is rewritten as

$$\left[-\int_0^u \mathrm{d}u'u'^{-\frac{9}{2}}\bar{f}(u')e^{-\frac{1}{2\beta}\frac{1}{u'^2}}\right]u^{\frac{5}{2}}e^{\frac{1}{2\beta}\frac{1}{u^2}}-\left[-\int_0^u \mathrm{d}u'u'^{-3}\bar{f}(u')e^{\frac{1}{2\beta}\frac{1}{u'^2}}\right]e^{-\frac{1}{2\beta}\frac{1}{u^2}} \tag{34}$$

Because that our concern is only how the particular solution behaves when u goes to 0 ($a_0\to 0$), we consider the following two quantities as

$$\frac{\int_0^u \mathrm{d}u'u'^{-\frac{9}{2}}\bar{f}(u')e^{-\frac{1}{2\beta}\frac{1}{u'^2}}}{u^{-\frac{5}{2}}e^{-\frac{1}{2\beta}\frac{1}{u^2}}}\xrightarrow[u\to 0]{}\frac{u\bar{f}(u)}{-\frac{5}{2}u^2-\frac{1}{\beta}}\xrightarrow[u\to 0]{}0 \tag{35}$$

To obtain the result of Equation (35), we use Roll's theory under the condition u approaches 0. For the second term of Equation (34), using the same method we obtain this term also becomes zero under $u \to 0$. This means that our particular solution goes to zero when a_0 goes to zero.

Denoting the particular solution $T(\bar{r})$ as

$$T(\bar{r}) = \frac{1}{2}\frac{e_b^2}{\epsilon_0}\frac{V_{a_0}}{(\hbar c)^2}a_0^3 S(\bar{r})$$

A general solution of Equation (9) is formally written as follows.

$$\bar{\rho}_0(\bar{r}) = \frac{1}{a_0^3}\left[T(\bar{r}) + A\exp\left(-\frac{\bar{r}^2}{2\beta}\right) + B\frac{1}{\bar{r}}\exp\left(-\frac{\bar{r}^2}{4\beta}\right)F\left(-\frac{1}{2},\frac{1}{2};\frac{\bar{r}^2}{2\beta}\right)\right] \tag{36}$$

where A and B are arbitrary constants

The term of $\frac{1}{a_0^3}$ comes from the condition Equation (2) as mentioned before.

Because A and B are arbitrary constants and recalling the fact that formal solution of Equation (36) becomes meaningful only when a_0 approaches 0 as mentioned before, we can choose $A = 1$ and $B = 0$. Then our solution of Equation (9) becomes

$$\bar{\rho}_0(\bar{r}) = \lim_{a_0\to 0}\frac{1}{a_0^3}\left[T(\bar{r}) + \exp\left(-\frac{\bar{r}^2}{2\beta}\right)\right] \tag{37}$$

Then, we obtain $\bar{\rho}_0(\bar{r})$ as follows.

$$\bar{\rho}_0(\bar{r}) = \lim_{a_0\to 0}\frac{1}{a_0^3}\exp\left(-\frac{\bar{r}^2}{2\beta}\right) \tag{38}$$

To obtain above form, we use the property that the first term of Equation (37) goes to zero when a_0 approaches 0 because $\frac{1}{a_0^3}T(\bar{r}) = constant\,S(\bar{r})$ and $S(\bar{r})$ goes to zero when a_0 goes to zero as shown before.

Recalling $\bar{r} = \frac{r}{a_0}$, Equation (38) becomes

$$\bar{\rho}_0(r) = \lim_{a_0\to 0}\frac{1}{a_0^3}\exp\left(-\frac{1}{2\beta}\frac{r^2}{a_0^2}\right) \tag{39}$$

Recalling $r^2 = x^2 + y^2 + z^2$ and the fact that usual definition of one dimensional delta function is

$$\delta(x) = \lim_{a_0\to 0}\frac{1}{a_0}\exp\left(-\frac{x^2}{a_0^2}\right)\frac{1}{\sqrt{\pi}} \tag{40}$$

we can express Equation (39) as follows.

$$\begin{aligned}\bar{\rho}_0(r) &= constant\,\delta\left(\frac{x}{\sqrt{2\beta}}\right)\delta\left(\frac{y}{\sqrt{2\beta}}\right)\delta\left(\frac{z}{\sqrt{2\beta}}\right)\\ &= constant\,\delta^3\left(\frac{1}{\sqrt{2\beta}}\vec{r}\right) = constant\,\delta^3(\vec{r})\end{aligned} \tag{41}$$

where $\delta^3(\vec{r})$ denotes three dimensional delta function

To obtain the final form of Equation (41), we use the fact that $\delta(\gamma x) = \frac{1}{\gamma}\delta(x)$

Note that this eigen-function is corresponded to eigenvalue $V_{a_0} = \frac{e^2}{4\pi\epsilon_0}$ under choosing certain $\bar{\zeta}$ value determined by Equation (23).

To determine C_1, recalling that meaningful eigen-function can be obtained when a_0 goes to 0, the characteristic eigen-function is only $\exp\left(-\frac{\bar{r}^2}{2\beta}\right)$ and we also use this consideration to satisfy charge condition Equation (2). Then from Equation (2), we can represent $\bar{\rho}_0(\bar{r})$ as $\bar{\rho}_0(\bar{r}) = \sqrt{\frac{8}{\beta\pi}}\frac{1}{4\pi a_0^3}\exp\left(-\frac{\bar{r}^2}{2\beta}\right)$. Then we can determine C_1 as $C_1 = \frac{1}{4\pi}\sqrt{\frac{8\beta}{\pi}}$. However, we have to insist that these value is only formal one. We show these values because we need to show that our argument is closed within our framework.

Thus, we can insist that an initial state of electron before absorbing infinite number of photons is described as three dimensional delta-function.

3. Results

Introducing vacuum permittivity at $a_0 \to 0$ and bare charge, we construct a second ordereigen value differential equation for $V_{a_0} = P_0 a_0$. Recalling this differential equation is meaningful only when a_0 approaches 0, we obtain an eigen-function corresponded to eigenvalue $\frac{e^2}{4\pi\epsilon_0}$ as three dimensional delta function. This is clearly corresponding to Weinberg's initial state of electron that is a massless point-like charge. This completes our claim that our methods to find both electron charge distribution functions at an initial state (massless) and at normal state (massive) by considering that these functions are corresponded eigen-functions to eigenvalues the former for charge and the latter for mass, respectively, are reasonable one so that the obtained results in Ref. [6] may reflect Weinberg's description of electron. As a final comment, we want to insist that charge distribution functions for an initial state (massless) and for normal state (massive) can be described as eigen-functions corresponding to each eigenvalue the former for charge and the latter for mass, respectively.

4. Discussion

To obtain the final three dimensional delta function, we choose Gaussian function ($\mu = \frac{1}{4}$) for Equation (15). This is consistent to basic solution of a charge distribution function in Ref. [6] because basic solution of ref. [6] is $\frac{1}{\bar{r}}\exp\left(-\sigma\bar{r}^2\right)$ type solution. This is corresponded to eigenvalue P_0. In this paper, we search a solution corresponded to eigenvalue $P_0 r = P_0 a_0 \bar{r}$. Thus,

$P_0 a_0 \bar{r} \frac{1}{\bar{r}} \exp\left(-\sigma \bar{r}^2\right) = \left(P_0 a_0\right) \exp\left(-\sigma \bar{r}^2\right) = V_{a_0} \exp\left(-\sigma \bar{r}^2\right)$. This shows that Gaussian type solution is consistent to that in Ref. [6]. This property is another reason why we consider that the obtained results in Ref. [6] may reflect Weinberg's description of electron.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Cho, A. (2012) *Science*, **338,** 1524-1525. https://doi.org/10.1126/science.338.6114.1524

[2] Gonzalez, E., Kane, G., Nguyen, K.D. and Perry, M. (2021) *Physical Review D*, **103**, Article ID: 126027. https://doi.org/10.1103/PhysRevD.103.126027

[3] Weinberg, S. (1995) The Quantum Theory of Field I. Cambridge University Press, Cambridge.

[4] Andreev, V., Ang, D.G., DeMille, D., *et al.* (2018) *Nature*, **562**, 355-360. https://doi.org/10.1038/s41586-018-0599-8

[5] Karnieli, A., Rivera, N., Arie, A. and Kaminer, I. (2021) *Science Advances*, **7**, No. 18. https://doi.org/10.1126/sciadv.abf8096

[6] Kurai, T. (2023) *Journal of Modern Physics*, **14**, 111-126. https://doi.org/10.4236/jmp.2023.142008

[7] Mainland, G.B. and Mulligan, B. (2020) *Foundations of Physics*, **50**, 457-480. https://doi.org/10.1007/s10701-020-00339-3

[8] Ince, E.L. (1956) Ordinary Differential Equations. Dover Publication, Mineola.

[9] Moriguchi, S., Udagawa, L. and Hitotsumatsu, S. (1975) Mathematics Formula III. Iwanami, Tokyo.