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Spherically Symmetric Problem of General Relativity for an Elastic Solid Sphere

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Abstract

The paper is devoted to a spherically symmetric problem of General Relativity (GR) for an elastic solid sphere. Originally developed to describe gravitation in continuum (vacuum, gas, fluid and solid) GR does not provide the complete set of equations for solids and, in contrast to the Newton gravitation theory, does not allow us to study the stresses induced by gravitation in solids, because the compatibility equations which are attracted in the Euclidean space for this purpose do not exist in the Riemannian space. To solve the problem within the framework of GR, a special geometry of the Riemannian space induced by gravitation is proposed. According to this geometry, the four-dimensional Riemannian space is assumed to be Euclidean with respect to the space coordinates and Riemannian with respect to the time coordinate. Such interpretation of the Riemannian space in GR allows us to supplement the conservation equations for the energy-momentum tensor with compatibility equations of the theory of elasticity and to arrive to the complete set of equations for stresses. The analytical solution of the Einstein equations for the empty space surrounding the sphere and the numerical solution for the internal space inside the sphere with the proposed geometry are presented and discussed.

Keywords

General Relativity, Spherically Symmetric Problem, Elastic Sphere

1. Introduction

The paper is concerned with the symmetric GR problem for an elastic solid sphere. To demonstrate it, consider the traditional formulation of the problem in General Relativity which is a phenomenological theory based on traditional models of space as a homogeneous isotropic continuum (vacuum, gas, fluid or solid) whose actual microstructure is ignored. The basic GR equations have the following form:

$$E_i^j = R_i^j - \frac{1}{2} g_i^j R = \chi T_i^j \tag{1}$$

in which R_i^j are the components of the Ricci curvature tensor depending on the metric tensor of the four-dimensional Riemannian space with the line element

$$\mathrm{d}s^2 = g_{ij} \mathrm{d}x^i \mathrm{d}x^j \quad (i, j = 1,2,3,4) \tag{2}$$

T_i^j is the energy-momentum tensor and

$$\chi = 8\pi G / c^4 \tag{3}$$

is the relativity gravitational constant expressed in terms of the classical gravitation constant G and the velocity of light c. The energy-momentum tensor must satisfy four conservation equations

$$\nabla_k T_i^k = 0 \quad (i = 1,2,3,4) \tag{4}$$

For $i, j = 1,2,3$, the energy-momentum tensor is composed of elastic stresses σ_i^j and kinetic terms depending on the velocities of the continuum points. For static problems, $T_i^j = \sigma_i^j$ and Equation (4) reduce to three equilibrium equations of the theory of elasticity. As known [1], these equations are not sufficient to determine the stresses. In the theory of elasticity, equilibrium equations are supplemented with compatibility equations written in terms of stresses. Classical compatibility equations have a simple geometric interpretation according to which the curvature tensor of the deformed space is zero. Such equations cannot exist in the Riemannian space in which Equations (1) and (4) are written. Thus, the General Relativity theory, in contrast to the Newton theory of gravitation, in principle, does not allow us to obtain the stresses induced by gravitation in solids. The solution is possible in two particular cases—for vacuum for which the stresses are zero and for a perfect fluid for which the stress tensor has only one component—the pressure that can be found from the equilibrium equation. It makes sense to mention that numerous existing monographs and textbooks describe the application of General Relativity to vacuum, and perfect gas or fluid, whereas the solid continuum is, as a rule, ignored.

In this paper, the problem of gravitation in solids is considered on the basis of a special model of Riemannian space according to which the space is Euclidean with respect to space coordinates x^1, x^2, x^3 and is Riemannian with respect to time coordinate x^4 only [2]. To support the proposed interpretation of the Riemannian space, first assume that we observe a massive spherical object. Suppose that the level of gravitation inside this object is high enough to induce in accordance with GR the internal Riemannian space. In this space, the ratio of the big circle length to the diameter is not equal to π. However, since we observe the sphere in a three-dimensional space, this ratio must be equal to π. Thus, we can conclude that the space inside the sphere is not Riemannian. Second, consider a static problem for a solid body which does not experience gravitation and is loaded with some surface forces. Since $T_i^j = \sigma_i^j$, Equation (1) allows us to conclude that the stresses induce inside the body the Riemannian space even in the absence of gravitation. However, we can observe the stressed solid only if the inside space is Euclidean (the three-dimensional Riemannian space can be imbedded into the Euclidean space with six dimensions). Thus, the space in GR must be Euclidean with respect to space coordinates. This allows us to supplement the equilibrium equations following for GR Equations (4) with compatibility equations of the theory of elasticity and to obtain the complete set of equations for stresses. Within the framework of the foregoing interpretation of the Riemannian space, the solution of GR equations is further obtained and discussed for the spherically symmetric problem which reduces to ordinary differential equations that can be solved analytically or numerically.

2. Gravitation Stresses in a Solid Elastic Sphere Following from the Newton Theory

In a solid sphere with radius *R*, gravitation induces radial and circumferential stresses σ_r and σ_θ which depend on the radial coordinate *r* and satisfy the following equilibrium equation [3]:

$$r\sigma_r' + 2\left(\sigma_r - \sigma_\theta\right) = \frac{\mu r_g c^2}{2R^3} r^2 \tag{5}$$

Here, $(\cdot)' = \mathrm{d}(\cdot)/\mathrm{d}r$, μ is the material density and

$$r_g = 2Gm/c^2 \tag{6}$$

is the so-called gravitation radius, depending on the sphere mass

$$m = \frac{4}{3}\pi\mu R^3 \tag{7}$$

In the Euclidean space, we can introduce the radial displacement $u_r(r)$ and the strains

$$\varepsilon_r = u_r', \ \varepsilon_\theta = u_r/r \tag{8}$$

These equations are valid for any displacement and strains, *i.e.*, Equation (8) can be applied for linear and nonlinear problems. Eliminating u_r from Equation (8), we arrive at the following compatibility equation:

$$\left(r\varepsilon_\theta\right)' = \varepsilon_r \tag{9}$$

Assume that the sphere consists of a linear elastic material with zero Poisson's ratio. Then, $\sigma_r = E\varepsilon_r$, $\sigma_\theta = E\varepsilon_\theta$, where E is the elastic modulus, and Equation (9) gives

$$\sigma_r = \left(r\sigma_\theta\right)' = \varphi' \tag{10}$$

Introducing dimensionless variables

$$\bar{\sigma}_r = \frac{\sigma_r}{\mu c^2}, \ \bar{\sigma}_\theta = \frac{\sigma_\theta}{\mu c^2}, \ \bar{r} = \frac{r}{R}, \ \bar{r}_g = \frac{r_g}{R} \tag{11}$$

we can reduce Equations (5) and (10) to the following equation for the function φ:

$$\varphi'' + \frac{2}{\bar{r}}\varphi' - \frac{2}{\bar{r}^2}\varphi = \frac{1}{2}\bar{r}_g\bar{r} \quad (12)$$

The solution of this equation which satisfies the boundary conditions $\bar{\sigma}_r(0) = \bar{\sigma}_\theta(0)$ and $\bar{\sigma}_r(1) = 0$ yields the following stresses:

$$\bar{\sigma}_r = -\frac{3\bar{r}_g}{20}\left(1-\bar{r}^2\right),\ \bar{\sigma}_\theta = -\frac{\bar{r}_g}{20}\left(3-\bar{r}^2\right) \quad (13)$$

In terms of GR, the metric tensor corresponding to the Newton gravitation theory has the following components [4]:

$$g_{11}^n = 1,\ g_{22}^n = r^2,\ g_{44}^n = 1 - \frac{r_g}{r} \quad (14)$$

3. Introduction of the Special Space Geometry

For the special geometry of the Riemannian space introduced above, Equation (2) for the line element becomes [2]

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 + 2g_{14}c\mathrm{d}r\mathrm{d}t - g_{44}c^2\mathrm{d}t^2,\ \mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2 \quad (15)$$

Assume that the metric tensor depends on r and t. Then, the field equations, Equation (1), take the following form:

$$E_1^1 = -\frac{1}{r^2g^2}\left[g\left(rg'_{44} - g_{14}^2\right) + 2rg_{44}\dot{g}_{14} - rg_{14}\dot{g}_{44}\right] = \chi T_1^1 \quad (16)$$

$$\begin{aligned} E_2^2 = &\frac{1}{4rg^2}\big(4g_{14}g_{44}g'_{14} - 4g_{14}^2g'_{44} - 2g_{44}g'_{44} - 2rg_{44}g''_{44} + rg'^2_{44} - 2rg_{14}^2g''_{44} \\ &+ 2rg_{14}g'_{14}g'_{44} - 4g_{44}\dot{g}_{14} + 2g_{14}\dot{g}_{44} + 4rg_{14}g'_{14}\dot{g}_{14} + 2rg'_{14}\dot{g}_{44} - 4rg\dot{g}'_{14}\big) \\ = &\chi T_2^2 \end{aligned} \quad (17)$$

$$E_4^4 = \frac{g_{14}}{r^2g^2}\left(2rg_{44}g'_{14} + gg_{14} - rg_{14}g'_{44}\right) = \chi T_4^4 \quad (18)$$

$$E_1^4 = -\frac{g_{14}g'}{rg^2} = \chi T_1^4 \quad (19)$$

$$E_4^1 = -\frac{g_{14}}{rg^2}\left(2g_{44}\dot{g}_{14} - g_{14}\dot{g}_{44}\right) = \chi T_4^1,\quad g = g_{44} + g_{14}^2 \quad (20)$$

Here, $(\cdot)' = \partial(\cdot)/\partial r$ and $(\dot{\cdot}) = \partial/c\partial t$. The energy-momentum tensor satisfies conservation Equation (4), *i.e.*,

$$\left(T_1^1\right)' + \frac{2}{r}\left(T_1^1 - T_2^2\right) + \frac{g'_{44}}{2g}\left(T_1^1 - T_4^4\right) + \frac{g_{14}g'_{44}}{2g}T_1^4 + \frac{g'_{14}}{g}T_4^1 + \dot{T}_1^4 + \frac{\dot{g}}{2g}T_1^4 = 0 \quad (21)$$

$$\begin{aligned} &rg'_{44}\left[g_{14}\left(T_1^1 - T_4^4\right) - g_{44}T_1^4\right] + 2g\left[r\dot{T}_4^4 + r(T_4^1)' + 2T_4^1\right] + 2rg_{14}g'_{14}T_4^1 \\ &- 2rg_{44}\dot{g}_{14}T_1^4 + rg_{14}\dot{g}_{44}T_1^4 = 0 \end{aligned} \quad (22)$$

As follows from Equations (1) and (4), the Einstein tensor E_i^j expressed in terms of the metric tensor by Equations (16)-(20) satisfies Equations (21) and

(22) which means that only three of five Equations (16)-(20) are mutually independent.

4. Solution of the External Problem

Consider the external space surrounding the sphere with radius *R*. For an empty space, we have $T_i^j = 0$. Hence, Equations (16)-(20) are homogeneous and Equations (21), (22) are satisfied identically. Obtain the solution of Equations (16)-(20). As follows from Equation (20) in which $T_4^1 = 0$, g_{14} and g_{44} do not depend on *t*. Thus, the solution of the external problem is static. This result is analogous to the Birkhoff theorem in the Schwarzschild solution. For a static problem, Equations (16)-(19) yield

$$rg'_{44} - g_{14}^2 = 0 \tag{23}$$

$$4g_{14}\left(g_{44}g'_{14} - g_{14}g'_{44}\right) - 2g_{44}\left(rg'_{44}\right)' + rg'^2_{44} - 2rg_{14}\left(g''_{44} - g'_{14}g'_{44}\right) = 0 \tag{24}$$

$$2rg_{44}g'_{14} + g_{14}\left(g_{44} + g_{14}^2\right) - rg_{14}g'_{44} = 0 \tag{25}$$

$$g'_{44} + 2g_{14}g'_{14} = 0 \tag{26}$$

Equation (23) gives $g'_{44} = g_{14}^2/r$. Then, Equation (26) reduces to $2rg'_{14} + g_{14} = 0$. The solution must satisfy the asymptotic conditions and reduce for $r \to \infty$ to Equation (14) corresponding to the Newton theory, *i.e.*, $g_{14}(r \to \infty) = 0$ and $g_{44}(r \to \infty) \to g_{44}^n$. Finally, we get [2]

$$g_{14} = \sqrt{\frac{r_g}{r}}, \quad g_{44} = 1 - \frac{r_g}{r} \tag{27}$$

Substituting Equation (27) in Equations (23) and (25), we can conclude that these equations are satisfied identically. Thus, Equations (27) specify the GR solution for the line element in Equation (15).

The metric coefficients in Equation (27) correspond to the so-called Gullstand-Painlever coordinates [5] [6] proposed as a result of the coordinate transformation of the geometry corresponding to the Schwarzschild metrics with the line element

$$ds^2 = g_{11}^s dr^2 + r^2 d\Omega^2 - g_{44}^s c^2 dt^2 \tag{28}$$

in which

$$g_{11}^s = \frac{r}{r - r_g}, \quad g_{44}^s = 1 - \frac{r_g}{r} \tag{29}$$

is the solution of the following field equations [3]:

$$E_1^1 = \frac{1}{r^2} - \frac{1}{g_{11}^s}\left(\frac{1}{r^2} + \frac{\left(g_{44}^s\right)'}{rg_{44}^s}\right) = 0 \tag{30}$$

$$E_4^4 = \frac{1}{r^2} - \frac{1}{g_{11}^s}\left(\frac{1}{r^2} - \frac{\left(g_{11}^s\right)'}{rg_{11}^s}\right) = 0 \tag{31}$$

Using identical transformations, we can reduce Equation (15) for the line element to the following form:

$$\mathrm{d}s^2 = \left(1+\frac{g_{14}^2}{g_{44}}\right)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 - g_{44}\left(c-\frac{g_{14}}{g_{44}}\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 \mathrm{d}t^2 \tag{32}$$

For the metric coefficients in Equation (27), we have [2]

$$\frac{\mathrm{d}r}{\mathrm{d}t} = c\frac{r-r_g}{r+r_g}\sqrt{\frac{r_g}{r}}$$

Using this result in conjunction with Equation (27), we can reduce Equation (32) to Equation (28) in which

$$g_{11}^s = \frac{r}{r-r_g}, \quad g_{44}^s = \frac{r\left(r-r_g\right)}{\left(r+r_g\right)^2} \tag{33}$$

As can be seen, coefficients g_{11}^s in Equations (28) and (32) are the same, whereas g_{44}^s are different. The metric coefficients in Equation (33), in contrast to the Schwarzschild solution in Equation (29), do not satisfy the field Equations (30) and (31). The metric form in Equation (15) can be formally reduced to Equation (28) if we change $\mathrm{d}t$ to the new time variable $\mathrm{d}\tau$ as

$$\mathrm{d}t = \left(1+\frac{r_g}{r}\right)\mathrm{d}\tau \tag{34}$$

In this paper, the proposed line element in Equation (15) is not associated with the transformation of the Schwarzschild metrics and follows from the special model of the Riemannian space introduced above. The obtained metric coefficients in Equation (27) are not singular. The sphere with radius $R = r_g$ is invisible as the black hole in the Schwarzschild solution [2].

5. Solution for the Internal Problem

Consider the internal space of a solid elastic sphere for which $T_i^j \neq 0$ in Equations (16)-(22). Full expressions which specify T_i^j for solid continuum are rather cumbersome, because they take into account the effects of Special Theory of Relativity [7] associated with high radial velocity of a sphere point v and the dependency of density on velocity. However, if $\left(v/c\right)^2 \ll 1$, they can be simplified and presented as

$$T_1^1 = \sigma_1^1 - \mu v_1 v^1, \; T_2^2 = \sigma_2^2, \; T_1^4 = -\mu v_1 v^4, \; T_4^1 = -\mu v_4 v^1, \; T_4^4 = \mu v_4 v^4 \tag{35}$$

where v is the 4-velosity vector which satisfies the condition $v_i v^i = c^2$. For the Schwarzschild solution based on the line element in Equation (28), $g_{14} = 0$ which means that space is "orthogonal" to time. For a static problem, we have $v_1 = v^1 = 0$ and $v_4 = v^4 = c$. However, for the line element in Equation (15) the space is not "orthogonal" to time. Using the definitions of covariant and contravariant vector components, we can conclude that in this case $v^1 = 0$, but $v_1 \neq 0$, because the velocity v_4 directed along the time axis gives the projection on the radial axis. To demonstrate this effect, consider a model space with the line element similar to Equation (15), *i.e.*,

$$\mathrm{d}s^2 = \left(\mathrm{d}x^1\right)^2 + 2g_{14}\mathrm{d}x^1\mathrm{d}x^4 - g_{44}\left(\mathrm{d}x^4\right)^2$$

Introduce orthogonal coordinates x_0^1, x_0^4 applying the following transformation:

$$x^1 = x_0^1,\ x^4 = x_0^4 + f\left(x^1\right)$$

Then, the line element becomes

$$\mathrm{d}s^2 = \left[1 + 2g_{14}f' - g_{44}\left(f'\right)^2\right]\left(\mathrm{d}x_0^1\right)^2 + 2\left(g_{14} - f'g_{44}\right)\mathrm{d}x_0^1\mathrm{d}x_0^4 - g_{44}\left(\mathrm{d}x_0^4\right)^2$$

in which $f' = \mathrm{d}f/\mathrm{d}x^1 = \mathrm{d}f/\mathrm{d}x_0^1$. Since coordinates x_0^1, x_0^4 are "orthogonal", we must take $f' = g_{14}/g_{44}$ and arrive at

$$\mathrm{d}s^2 = \left(1 + \frac{g_{14}^2}{g_{44}}\right)\left(\mathrm{d}x_0^1\right)^2 - g_{44}\left(\mathrm{d}x_0^4\right)$$

Assume that the velocity vector $\bar{v}_0$ with the following components: $v_0^1 = v_{01} = v_0$ and $v_0^4 = v_{04} = c$ exists in the space with this line element. Transforming these components to the velocities in the initial "oblique" coordinates x^1, x^4 in accordance with

$$v^i = v_0^k \frac{\partial x^i}{\partial x_0^k},\ v_i = v_{0k}\frac{\partial x_0^k}{\partial x^i}$$

we get

$$v^1 = v_0^1 = v_0,\ v_1 = v_{01} - v_{04}f' = v_0 - c\frac{g_{14}}{g_{44}},$$

$$v^4 = v_0^1 f' + v_0^4 = v_0 + c\frac{g_{14}}{g_{44}},\ v_4 = v_{04} = c$$

Now assume that $v_0 = 0$ in "orthogonal" coordinates. Then, in "oblique" coordinates we have $v^1 = 0$ and $v_1 = -cg_{14}/g_{44} \neq 0$.

To proceed, we should take into account that the mixed stress tensor components coincide in spherical coordinates with physical stresses. Then, we can reduce Equation (35) to

$$T_1^1 = \sigma_r,\ T_2^2 = \sigma_\theta,\ T_1^4 = -\mu c v_1,\ T_4^1 = 0,\ T_4^4 = \mu c^2 \tag{36}$$

Consider the field equations, Equation (16)-(20). Since $T_4^1 = 0$, Equation (20) allows us to conclude that g_{14} and g_{44} do not depend on time. But the problem is not static, because v_1 is not zero. Using Equation (36), we can present Equations (16)-(20) as

$$\frac{1}{r^2 g}\left(g_{14}^2 - rg'_{44}\right) = \chi\sigma_r \tag{37}$$

$$\frac{1}{4rg^2}\left[4g_{14}\left(g_{44}g'_{14} - g_{14}g'_{44}\right) - 2g_{44}\left(rg'_{44}\right)' + r\left(g'_{44}\right)^2 - 2rg_{14}\left(g''_{44} - g'_{14}g'_{44}\right)\right] = \chi\sigma_\theta \tag{38}$$

$$\frac{g_{14}}{r^2 g^2}\left[g_{14}g + r\left(2g_{44}g'_{14} - g_{14}g'_{44}\right)\right] = \chi\mu c^2 \tag{39}$$

$$-\frac{g_{14}g'}{rg^2} = T_1^4 = -\chi\mu c v_1, \; g = g_{44} + g_{14}^2 \tag{40}$$

Respectively, Equations (21) and (22) become

$$\sigma_r' + \frac{2}{r}(\sigma_r - \sigma_\theta) + \frac{g_{44}'}{2g}(\sigma_r - \mu c^2) + \frac{g_{14}g_{44}'}{2g}T_1^4 = 0 \tag{41}$$

$$T_1^4 = \frac{g_{14}}{g_{44}}(\sigma_r - \mu c^2) = -\mu c v_1 \tag{42}$$

Determine the metric coefficients. Consider Equation (39) and introduce a new function

$$f = \frac{g}{g_{44}} = 1 + \frac{g_{14}^2}{g_{44}}$$

Then,

$$g_{14} = \sqrt{(f-1)g_{44}}, \; g_{14}' = \frac{f'g_{44} + (f-1)g_{44}'}{2\sqrt{(f-1)g_{44}}} \tag{43}$$

Substituting Equation (43) in Equation (39), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r - \frac{r}{f}\right) = \chi\mu c^2 r^2 \tag{44}$$

Using Equations (3), (6) and (7) for χ, r_g and *m*, we get

$$\chi\mu c^2 = 3r_g / R^3 \tag{45}$$

Then, the solution of Equation (44) which is regular at the sphere center becomes

$$f = \frac{1}{1 - r_g r^2 / R^3} \tag{46}$$

Consider Equation (37). Substituting Equation (43), we get

$$\frac{g_{44}'}{g_{44}} = \frac{1}{r}(f-1) - \chi f r \sigma_r \tag{47}$$

Using Equations (43), (46) and (47) and transforming Equations (38) and (40) with the aid of Equations (41) and (42), we can prove that Equations (38) and (40) are satisfied identically. Thus, Equations (43), (46) and (47) specify the solution of the field equations.

Having the metric coefficients, we can determine the stresses induced in a solid sphere by gravitation. Substituting the first part of Equation (42) in Equation (41), we arrive at the following equilibrium equation:

$$\sigma_r' + \frac{2}{r}(\sigma_r - \sigma_\theta) + \frac{g_{44}'}{2g_{44}}(\sigma_r - \mu c^2) = 0 \tag{48}$$

Substitution of Equation (47) yields

$$\sigma_r' + \frac{2}{r}(\sigma_r - \sigma_\theta) + \frac{1}{2}\left[\frac{1}{r}(f-1) - \chi r f \sigma_r\right](\sigma_r - \mu c^2) = 0$$

Using notations (11) and Equation (45), we can reduce this equation to the following form:

$$\bar{\sigma}_r' + \frac{2}{\bar{r}}\left(\bar{\sigma}_r' - \bar{\sigma}_\theta\right) - \frac{\bar{r}_g \bar{r}}{2\left(1-\bar{r}_g\bar{r}^2\right)}\left(1-3\bar{\sigma}_r\right)\left(1-\bar{\sigma}_r\right) = 0 \tag{49}$$

in which $(\cdot)' = \mathrm{d}(\cdot)/\mathrm{d}\bar{r}$. Now recall that in accordance with the proposed model of the Riemannian space-time, the space is Euclidean with respect to spherical coordinates r,θ,φ. In this space, the stresses must satisfy the compatibility Equation (10), *i.e.*,

$$\bar{\sigma}_r = \left(\bar{r}\bar{\sigma}_\theta\right)'$$

Using this equation to eliminate $\bar{\sigma}_r$ from Equation (49), we arrive at

$$\varphi'' + \frac{2\varphi'}{\bar{r}\left(1-\bar{r}_g\bar{r}^2\right)} - \frac{3\bar{r}_g\bar{r}}{2\left(1-\bar{r}_g\bar{r}^2\right)}\left(\varphi'\right)^2 - \frac{2\varphi}{\bar{r}^2} = \frac{\bar{r}_g\bar{r}}{2\left(1-\bar{r}_g\bar{r}^2\right)} \tag{50}$$

where, as earlier, $\varphi = \bar{r}\bar{\sigma}_\theta$ and $\bar{\sigma}_r = \varphi'$. For small ratio $\bar{r}_g = r_g/R$, we can neglect the term $\bar{r}_g\bar{r}^2$ in comparison with unity. Then, omitting the nonlinear term, we can reduce Equation (50) to Equation (12) corresponding to the Newton gravitation theory.

Equation (50) is solved numerically under the conditions $\bar{\sigma}_r(0) = \bar{\sigma}_\theta(0)$ and $\bar{\sigma}_r(1) = 0$. The dependences of stresses $\bar{\sigma}_r(\bar{r})$ and $\bar{\sigma}_\theta(\bar{r})$ for $\bar{r}_g = 0.05; 0.1; 0.25; 0.5; 0.75; 0.9$ are shown in **Figures 1-3** with solid lines. **Figure 4** demonstrates the behavior of the stresses at the sphere center on the gravitation radius $\bar{r}_g$. As can be seen, these stresses tend to become infinitely high for

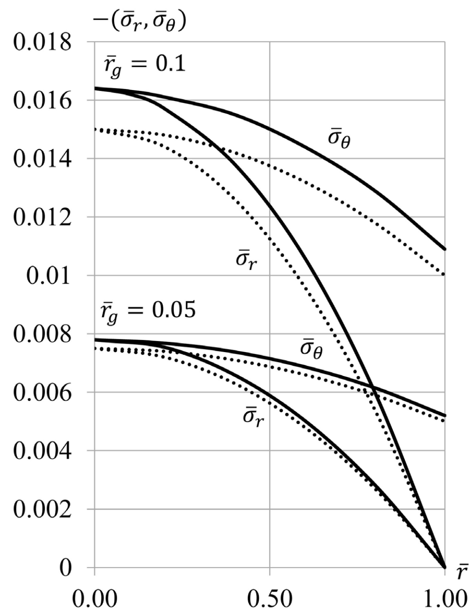

Figure 1. Dependences of the normalized stresses on the radial coordinate for $\bar{r}_g = 0.05$ and $\bar{r}_g = 0.1$. ——: Numerical GR solution. ······: Analytical solution for small $\bar{r}_g$.

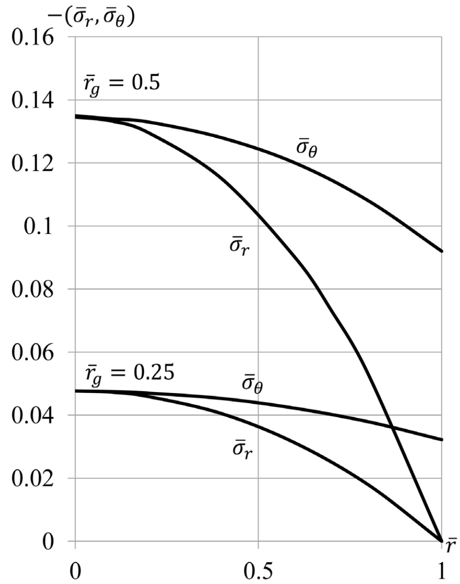

Figure 2. Dependences of the normalized stresses on the radial coordinate for $\bar{r}_g = 0.25$ and $\bar{r}_g = 0.5$.

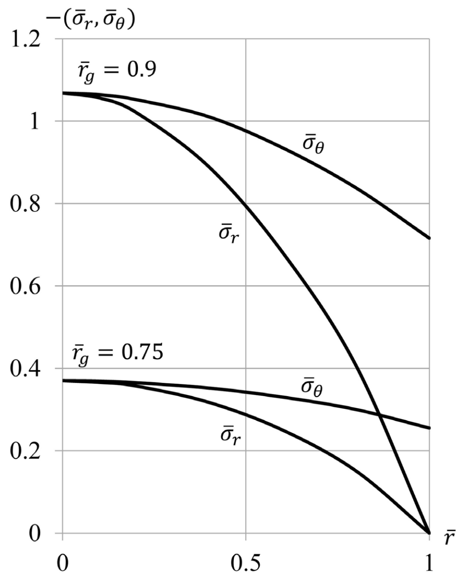

Figure 3. Dependences of the normalized stresses on the radial coordinate for $\bar{r}_g = 0.75$ and $\bar{r}_g = 0.9$.

$\bar{r}_g \to 1$. Integration of Equation (47) under the condition $g_{44}(1) = 1 - \bar{r}_g$ following from Equation (27) allows us to determine g_{44} as

$$g_{44} = \sqrt{\frac{\left(1-\bar{r}_g\right)^3}{1-\bar{r}_g\bar{r}^2}} \exp\left(-3\bar{r}\int_1^{\bar{r}} \frac{\bar{r}\bar{\sigma}_r \mathrm{d}\bar{r}}{1-\bar{r}_g\bar{r}^2}\right) \tag{51}$$

Then, g_{14} can be found from Equation (43), *i.e.*,

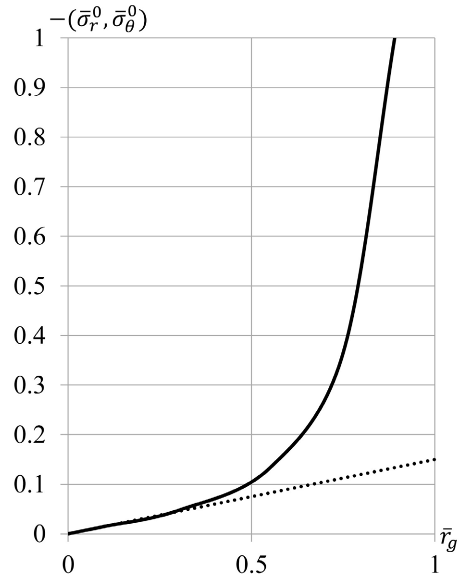

Figure 4. Dependences of the normalized stresses at the sphere center on the gravitation radius. ——: Numerical GR solution; ……: Analytical solution for small $\bar{r}_g$.

$$g_{14} = \bar{r}\sqrt{\frac{\bar{r}_g g_{44}}{1-\bar{r}_g \bar{r}^2}} \tag{52}$$

Dependences $g_{44}(\bar{r})$ and $g_{14}(\bar{r})$ are presented in **Figure 5** and **Figure 6** with solid lines.

Finally, using Equation (42), we can obtain the velocity $v_1(\bar{r})$ as

$$v_1 = c\frac{g_{14}}{g_{44}}(1-\bar{\sigma}_r) \tag{53}$$

Since $\bar{\sigma}_r \le 0$ (**Figures 1-3**) and $g_{14}(0) = 0$ (**Figure 6**), $v_1 = 0$ at the sphere center and is directed along the radial coordinate. At the sphere surface, we have $g_{14}(1) = \sqrt{\bar{r}_g}$, $g_{44}(1) = 1-\bar{r}_g$, $\bar{\sigma}_r = 0$ and Equation (53) yields

$$v_1(1) = \frac{c\sqrt{\bar{r}_g}}{1-\bar{r}_g}$$

As follows from this expression, $v_1(1) \ge c$ for $\bar{r}_g \ge 0.382$. This result does not look correct, because Equation (35) are valid if $(v_1/c)^2 \ll 1$. Whether v_1 corresponds to an actual movement or is a formal result of coordinate transformation is under question. An interesting interpretation of this velocity—the "river model" according to which space flows through a flat background with the velocity that can be higher than c is proposed by Hamilton and Lisle [8].

Most probably, v_1 does not have any physical meaning. This velocity is the projection of $v_4 = c$ directed along the time axis on the radial axis in coordinates for which $g_{14} \ne 0$. However, v_4 is not associated with any actual movement of the sphere points. Since the space in coordinates r, θ, φ is Euclidean

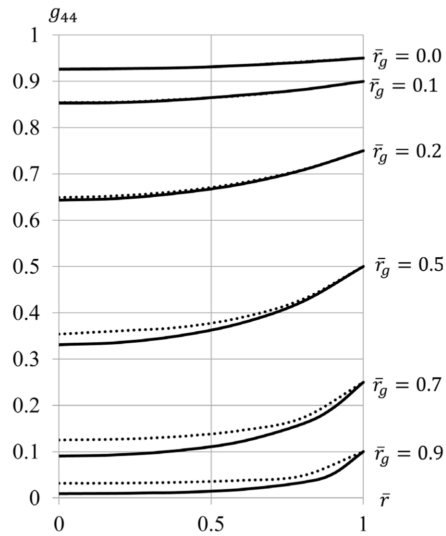

Figure 5. Dependences of g_{44} on the radial coordinate corresponding to various $\bar{r}_g$ values. ——: Numerical GR solution; ······: Analytical solution for small $\bar{r}_g$.

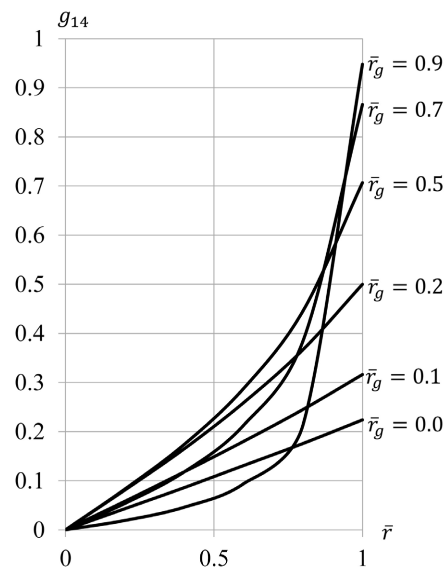

Figure 6. Dependences of g_{14} on the radial coordinate corresponding to various $\bar{r}_g$ values.

and the stresses do not depend on time, we can find the strains and the radial displacement of the sphere points which does not depend on time and corresponds to zero physical radial velocity.

For relatively small ratios $\bar{r}_g = r_g/R$, we can obtain an approximate analytical solution. Consider **Figure 1** in which the dotted lines correspond to the linear solutions in Equation (13) following from the Newton gravitation theory. As can be seen, for $\bar{r}_g \leq 0.5$ this solution is in fair agreement with the GR numerical solution (solid lines). Thus, substituting $\bar{\sigma}_r$ from Equation (13) in Equation (51), we get for small $\bar{r}_g$ values

$$g_{44} \approx \sqrt{\frac{\left(1-\bar{r}_g\right)^3}{1-\bar{r}_g\bar{r}^2}}\left[1+\frac{9\bar{r}_g^2}{20}\int_1^{\bar{r}}\frac{\bar{r}\left(1-\bar{r}^2\right)\mathrm{d}\bar{r}}{1-\bar{r}_g\bar{r}^2}\right] \approx \sqrt{\frac{\left(1-\bar{r}_g\right)^3}{1-\bar{r}_g\bar{r}^2}} \quad (54)$$

As follows from Equation (54), the radial stress does not affect g_{44} for small $\bar{r}_g$. The results of calculation are shown in **Figure 5** with dotted lines. Using Equations (52) and (54), we get

$$g_{14} = \bar{r}\sqrt{\bar{r}_g}\sqrt[4]{\left(\frac{1-\bar{r}_g}{1-\bar{r}_g\bar{r}^2}\right)^3} \quad (55)$$

Finally, neglecting $\bar{\sigma}_r$ in comparison with unity in Equation (53) and applying Equations (54) and (55), we arrive at

$$v_1 = \frac{c\bar{r}\sqrt{\bar{r}_g}}{\sqrt[4]{\left(1-\bar{r}_g\right)^3\left(1-\bar{r}_g\bar{r}^2\right)}}$$

In conclusion, return to velocity v_1 and consider the static problem of the theory of elasticity for which v_1 cannot exist in any coordinates. Taking $T_1^1 = \sigma_r(r)$, $T_2^2 = \sigma_\theta(r)$, $T_4^4 = \mu c^2$, $T_1^4 = -\mu c v_1(r)$, $T_4^1 = -\mu c v^1(r)$ and $g_{14} = g_{14}(r)$, $g_{44} = g_{44}(r)$, we can present Equations (21) and (22) in the following explicit form:

$$\sigma_r' + \frac{2}{r}\left(\sigma_r - \sigma_\theta\right) + \frac{g_{44}'}{2g}\left(\sigma_r - \mu c^2\right) - \frac{g_{14}g_{44}'}{2g}\mu c v_1 - \frac{g_{14}'}{g}\mu c v^1 = 0 \quad (56)$$

$$r g_{44}'\left[g_{14}\left(\sigma_r - \mu c^2\right) + g_{44}\mu c v_1\right] - 2g\mu c\left[r\left(v^1\right)' + 2v^1\right] - 2r g_{14} g_{14}' \mu c v^1 = 0 \quad (57)$$

These equations have a simple physical meaning [9]—Equation (56) is the motion equation, whereas Equation (57) provides the matter conservation.

Consider the static problem in "orthogonal" coordinates for which $g_{14} = 0$, $g_{44} = 1$ and $v_1 = v^1 = v_r$ is the radial coordinate velocity. Then, Equation (56) reduces to the equilibrium equation of the theory of elasticity

$$r\sigma_r' + 2\left(\sigma_r - \sigma_\theta\right) = 0 \quad (58)$$

whereas Equation (57) yields $r v_r' + 2v_r = 0$. This result can be presented in the following form: $4\pi\mu r^2 v_r = const$ which means that the matter flow through spherical surfaces is the same. For a static problem, $v_r = 0$ and in the absence of gravitation GR problem reduces to the problem of the theory of elasticity.

Now, assume that $g_{14} \neq 0$. As shown above, for a static problem $v^1 = 0$ and $v_1 = v_1(r)$. Then, Equations (56) and (57) become

$$\sigma_r' + \frac{2}{r}\left(\sigma_r - \sigma_\theta\right) + \frac{g_{44}'}{2g}\left(\sigma_r - \mu c^2\right) - \frac{g_{14}g_{44}'}{2g}\mu c v_1 = 0 \quad (59)$$

$$r g_{44}'\left[g_{14}\left(\sigma_r - \mu c^2\right) + g_{44}\mu c v_1\right] = 0 \quad (60)$$

In the absence of gravitation, $g_{44}=1$, Equation (59) reduces to Equation (58) whereas Equation (60) is satisfied identically for any function $v_1(r)$. If in addition g_{14} does not depend on *r*, Equation (19) gives $T_1^4=0$ and hence, $v_1=0$. Thus, the velocity v_1 appears only if the time metric coefficients g_{14} and g_{44} depend on the radial coordinate. Traditional velocity occurs if the space coordinate depends on time, whereas the velocity v_1 is associated with the dependency of the time metric coefficient on the space coordinate and appears only in gravitation problems described by the General Relativity Theory.

6. Conclusion

A spherically symmetric problem of General Relativity is considered for a solid elastic sphere within the framework of the special model of the Riemannian space-time which is Euclidean with respect to space coordinates and Riemannian with respect to time. In this version of the Riemannian space, the equilibrium equation is supplemented with the compatibility equation and the obtained set of two equations allows us to determine the stresses induced by gravitation in a solid elastic sphere. The numerical solution that specifies the dependences of stresses and metric coefficients on the radial coordinate for various values of the gravitation radius are presented and discussed. Extrapolation of the obtained solution allows us to conclude that the stresses at the sphere center become infinitely high if the sphere radius becomes close to the gravitation radius. The approximate analytical solution for stresses and metric coefficients is obtained is the sphere radius is much less than the gravitation radius. The additional velocity directed along the time axis which appears in GR with a metric form with a mixed space-time coefficient is discussed.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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