Journal of Modern Physics, 2023, 14, 147-159
https://www.scirp.org/journal/jmp
ISSN Online: 2153-120X
ISSN Print: 2153-1196

To the Solution of a Spherically Symmetric Problem of General Relativity

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How to cite this paper: Vasiliev, V.V. and Fedorov, L.V. (2023) To the Solution of a Spherically Symmetric Problem of General Relativity. *Journal of Modern Physics*, **14**, 147-159.
https://doi.org/10.4236/jmp.2023.142010

Received: November 27, 2022
Accepted: January 28, 2023
Published: January 31, 2023

Abstract

The paper is devoted to the spherically symmetric problem of General Relativity. Existing solutions obtained by K. Schwarzschild and V. Fock are presented and discussed. A special geometry of the Riemannian space induced by gravitation is proposed. According to this geometry the four-dimensional Riemannian space is assumed to be Euclidean with respect to the space coordinates and Riemannian with respect to the time coordinate. The solution of the Einstein equations for the empty space with this geometry coincides with the solution in Gullstand-Painlever coordinates. In application to the found solution, the problem of the light trajectory deviation in the vicinity of Sun and the problem of escape velocity are discussed.

Keywords

General Relativity, Spherically Symmetric Problem

1. Introduction

To demonstrate the problems discussed in the paper, consider the traditional formulation of the problem in General Relativity (GR). The basic equations of the general relativity specify the Einstein tensor which has the following form:

$$E^{ij} = R^{ij} - \frac{1}{2} g^{ij} R \tag{1}$$

in which R^{ij} are the components of the Ricci curvature tensor, $R = g_{ij} R^{ij}$ and g_{ij} is the metric tensor specifying the line element

$$\mathrm{d}s^2 = g_{ij} \mathrm{d}x^i \mathrm{d}x^j \quad (i, j = 1,2,3,4) \tag{2}$$

in a four-dimensional Riemannian space. The Einstein tensor is associated with the energy-momentum tensor as

$$E^{ij} = \chi T^{ij} \tag{3}$$

where

$$\chi = 8\pi G/c^4 \tag{4}$$

is the relativity gravitational constant expressed in terms of the classical gravitation constant G and the velocity of light c. We consider the original GR which is a phenomenological theory based on the traditional model of space as a homogeneous isotropic continuum whose microstructure is ignored. Within the framework of this model, the energy-momentum tensor has the following form:

$$T^{ij} = \sigma^{ij} - \mu v^i v^j \ (i, j = 1,2,3), \ T^{i4} = \mu c v^i, \ T^{44} = \mu c^2 \tag{5}$$

Here, σ^{ij} is the stress tensor induced in the continuum by gravitation, v^i is the 4-velocity vector and μ is the continuum density. The energy-momentum tensor must satisfy four conservation equations, *i.e.*,

$$\nabla_k T_i^k = 0 \ (i = 1,2,3,4) \tag{6}$$

The Einstein tensor in Equation (1) has a special structure. As follows from Equation (3), it satisfies Equations (6), *i.e.*,

$$\nabla_k E_i^k = 0 \ (i = 1,2,3,4) \tag{7}$$

For the continuum which simulates an empty space (vacuum), we have $\sigma^{ij} = 0$, $\mu = 0$. Thus, $T^{ij} = 0$ and, in accordance with Equation (3),

$$E^{ij} = 0 \tag{8}$$

In four-dimensional empty Riemannian space, ten Equations (8) contain ten unknown components of the metric tensor g_{ij}. As known [1], the set of Equations (8) is not complete—only six of these equations are mutually independent because the left-side parts of them identically satisfy Equations (7). Traditional approach is associated with four coordinate conditions [1] with which Equations (8) should be supplemented to arrive at the solution. However, the general form of these conditions is not known.

In the paper, this problem is demonstrated and discussed for the spherically symmetric problem which reduces to ordinary differential equations that can be solved analytically. The general form of the line element in Equation (2) in spherical coordinates r, θ, φ is

$$\mathrm{d}s^2 = g_{11}\mathrm{d}r^2 + g_{22}\mathrm{d}\Omega^2 - g_{44}c^2\mathrm{d}t^2, \ \mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2 \tag{9}$$

For a static problem, the components of the metric tensor depend on the radial coordinate only. Traditionally [1] [2] [3], Equation (9) is reduced to the following form:

$$\mathrm{d}s^2 = g_{11}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 - g_{44}c^2\mathrm{d}t^2 \tag{10}$$

in which $g_{22} = r^2$ corresponds to the Euclidean space. According to the traditional transformation, put $g_{22} = f^2(r)$ and introduce a new coordinate $r' = f(r)$. Omitting the prime, we get $g_{22} = r^2$ and arrive at Equation (10). However, the undertaken transformation is valid for the infinite space for which

$0 \le (r, r') < \infty$ and the difference between r and r' is not significant. If we have in space a spherical object with radius $r = R$, we cannot determine the corresponding value of r' and study the problem. We further consider the gravitation field for the sphere with given radius and cannot use the transformation resulting in Equation (10).

Consider the gravitation field in the outer empty space of the sphere with radius R ($R \le r < \infty$). For the line element in Equation (9), the field equations, Equations (1), become [4]

$$E_1^1 = \frac{1}{g_{22}} - \frac{1}{g_{11}}\left[\frac{1}{4}\left(\frac{g'_{22}}{g_{22}}\right)^2 + \frac{g'_{22}g'_{44}}{2g_{22}g_{44}}\right] = 0 \tag{11}$$

$$E_2^2 = -\frac{1}{2g_{11}}\left[\frac{g''_{44}}{g_{44}} - \frac{1}{2}\left(\frac{g'_{44}}{g_{44}}\right)^2 + \frac{g''_{22}}{g_{22}} - \frac{1}{2}\left(\frac{g'_{22}}{g_{22}}\right)^2 + \frac{g'_{22}}{2g_{22}}\left(\frac{g'_{44}}{g_{44}} - \frac{g'_{11}}{g_{11}}\right) - \frac{g'_{11}g'_{44}}{2g_{11}g_{44}}\right] = 0 \tag{12}$$

$$E_4^4 = \frac{1}{g_{22}} - \frac{1}{g_{11}}\left[\frac{g''_{22}}{g_{22}} - \frac{1}{4}\left(\frac{g'_{22}}{g_{22}}\right)^2 - \frac{g'_{11}g'_{22}}{2g_{11}g_{22}}\right] = 0 \tag{13}$$

where $(\cdot)' = \mathrm{d}(\cdot)/\mathrm{d}r$. As can be seen, Equations (13) and (11) allow us to express g_{11} and g_{44} in terms of g_{22}. The general solution of these equations is [5]

$$g_{11} = \frac{(g'_{22})^2}{4\left(g_{22} + C_1\sqrt{g_{22}}\right)}, \quad g_{44} = C_2\left(1 + \frac{C_1}{\sqrt{g_{22}}}\right) \tag{14}$$

in which C_1 and C_2 are the integration constants. Substituting Equations (14) in Equation (12), we can conclude that this equation is satisfied identically with any function $g_{22}(r)$. It looks like this function can be arbitrary chosen and the GR solution corresponding to the line element in Equation (9) is not unique. As mentioned above, Equations (11)-(13) should be supplemented with an additional coordinate condition. Thus, the first problem discussed further is the problem of the coordinate condition.

Consider the internal space ($0 \le r \le R$) of a solid sphere. The second problem is associated with conservation Equation (6). For the line element in Equation (9), this equation reduces to

$$\frac{\mathrm{d}T_1^1}{\mathrm{d}r} + \frac{g'_{22}}{g_{22}}\left(T_1^1 - T_2^2\right) + \frac{g'_{44}}{2g_{44}}\left(T_1^1 - T_4^4\right) = 0 \tag{15}$$

For a static problem, we have $T_1^1 = \sigma_r, T_2^2 = \sigma_\theta$, *i.e.*, the mixed components of the energy-momentum tensor coincide with the physical components of the stress tensor in spherical coordinates. Then, Equation (15) is the equilibrium equation analogous to the corresponding equation in the theory of elasticity [6]. In this theory, the equilibrium equation is supplemented with the compatibility equation [6] and the set of two equations allows us to determine two stresses σ_r and σ_θ. However, the compatibility equation which states that the space inside the stressed sphere is Euclidean does not exist in the Riemannian space. Thus, the second problem discussed further occurs—GR does not provide the complete set of equations to determine the stresses induced in a solid continuum by gravitation.

To solve the foregoing problems, a special geometry of the Riemannian space is proposed in the paper. According to this geometry, the four-dimensional Riemannian space is assumed to be Euclidean with respect to space coordinates and Riemannian with respect to the time coordinate.

2. Spherically Symmetric Problem—Existing Solutions

Consider the solution corresponding to the Newton gravitation theory. For the Euclidean space, we have $g_{11}=1, g_{22}=r^2, g_{44}=1$ and Equations (11)-(13) are satisfied identically. In terms of GR, the metric tensor corresponding to the Newton gravitation theory has the following components [1]:

$$g_{11}=1,\ \ g_{22}=r^2,\ \ g_{44}=1-\frac{r_g}{r} \tag{16}$$

Here,

$$r_g=\frac{2mG}{c^2} \tag{17}$$

is the so-called gravitation radius depending on the mass m of the spherical object which induces gravitation. The equilibrium Equation (15) reduces to [6]

$$\bar{r}\bar{\sigma}_r'-2\left(\bar{\sigma}_\theta-\bar{\sigma}_r\right)-\frac{1}{2}\bar{r}_g\bar{r}^2=0,\ \bar{r}=\frac{r}{R},\ \bar{\sigma}=\frac{\sigma}{\mu c^2},\ \left(\cdot\right)'=\frac{\mathrm{d}\left(\cdot\right)}{\mathrm{d}\bar{r}} \tag{18}$$

Since the space is Euclidean with respect to coordinates r,θ,φ, we can introduce the radial displacement $u_r\left(r\right)$ and the strains

$$\varepsilon_r=\bar{u}_r',\ \ \varepsilon_\theta=\bar{u}_r/\bar{r},\ \ \bar{u}_r=u_r/R \tag{19}$$

Eliminating $\bar{u}_r$, we arrive at the compatibility equation

$$\left(\bar{r}\varepsilon_\theta\right)'=\varepsilon_r \tag{20}$$

Note that this equation is valid for any displacement and strains which means that it can be used to solve linear and nonlinear problems. Assume that the sphere is linear elastic. Then the strains can be expressed in terms of stresses with the aid of Hooke's law

$$\varepsilon_r=\frac{1}{E}\left(\sigma_r-2\nu\sigma_\theta\right),\ \ \varepsilon_\theta=\frac{1}{E}\left[\left(1-\nu\right)\sigma_\theta-\nu\sigma_r\right]$$

Substituting the strains in Equation (20), we get the compatibility equation in terms of stresses. Adding this equation to Equation (18), we arrive at two equations for two stresses induced by gravitation.

To analyze the Newton solution, determine the escape velocity for the sphere with radius R. Using Equation (17), we can present the radial motion equation of the Newton theory as

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2}=-\frac{Gm}{r^2}=-\frac{c^2r_g}{2r^2} \tag{21}$$

Introduce a new variable u

$$u = \frac{\mathrm{d}r}{\mathrm{d}t}, \quad \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}r}u \tag{22}$$

Then, Equation (21) becomes

$$\frac{\mathrm{d}u^2}{\mathrm{d}r} = -\frac{c^2 r_g}{r^2}$$

Integration yields

$$u^2 = c^2 \frac{r_g}{r} + C \tag{23}$$

For the Euclidean space, the coordinate velocity u coincides with the physical velocity v. The minimum value of the escape velocity can be found from Equation (17) if we take $v(r \to \infty) = 0$. Then,

$$v_e = c\sqrt{\frac{r_g}{R}} \tag{24}$$

As can be seen, $v_e = c$ for a spherical object with radius $R = r_g$ and the sphere with such radius becomes invisible. This result was obtained in the end of the 18$^{\text{th}}$ century by J. Michel and P. Laplace.

As known [3], there exist experiments which allow us to check the solution. We use here one of these experiments, particularly, the experiment that determines the shift angle for the light ray deviation from the straight trajectory in the vicinity of Sun. The experimental value is 1.75" [7], whereas the Newton theory yields only one half of this result.

Consider the Schwarzschild solution of GR equations. Return to Equations (14) and determine the integration constants C_1 and C_2 from the asymptotic condition according to which the solution must reduce to the classical solution in Equations (16) for $r \to \infty$. Assuming that $g_{22}(r \to \infty) \to r^2$, we get $C_1 = -r_g$, $C_2 = 1$ and the solution in Equations (14) becomes

$$g_{11} = \frac{(g'_{22})^2}{4\left(g_{22} - r_g\sqrt{g_{22}}\right)}, \quad g_{44} = \left(1 - \frac{r_g}{\sqrt{g_{22}}}\right) \tag{25}$$

As can be seen, to obtain the final solution we need to assign the function $g_{22}(r)$. The problem was solved in 1916 by K. Schwarzschild under the condition $g_{22} = r^2$. We discuss the modern form of this solution [4]. Originally, K. Schwarzschild used a different condition which can be reduced to $g_{22} = r^2$ [8]. Putting $g_{22} = r^2$ in Equations (25), we arrive at the following metric coefficients for an empty space:

$$g_{11} = \frac{r}{r - r_g}, \quad g_{44} = 1 - \frac{r_g}{r} \tag{26}$$

This solution is singular at $r = r_g$. The sphere with radius r_g is referred to as the horizon of events of the Black Hole. This term is associated with the expression for the escape velocity corresponding to the Schwarzschild solution. The motion equation which generalizes Equation (21) has the following form [9]:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}-\frac{3r_g}{2r^2\left(1-r_g/r\right)}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2=-\frac{c^2 r_g}{2r^2}\left(1-\frac{r_g}{r}\right) \tag{27}$$

The solution of this equation that specifies the coordinate escape velocity is [9]

$$u_e=\frac{\mathrm{d}r}{\mathrm{d}t}=c\left(1-\frac{r_g}{r}\right)\sqrt{\frac{r_g}{r}}$$

Physical escape velocity can be found with the aid of Equations (26) as

$$v_e=u_e\sqrt{\frac{g_{11}}{g_{44}}}=c\sqrt{\frac{r_g}{r}} \tag{28}$$

This result coincides with Equation (24) for the Newton theory and means that the Black Hole becomes invisible if $R=r_g$. Analysis of the shift angle for the light beam in the vicinity of Sun yields the result that is in agreement with the experimental value [3]. The Schwarzschild solution is characterized with an interesting property. To demonstrate it, consider the dynamics problem for which metric coefficients g_{11} and g_{44} depend on the radial coordinate *r* and time *t*. Then, the static field equations, Equations (11)-(13), are generalized as

$$\left(E_1^1\right)_D=E_1^1+\frac{1}{g_{44}}\left[\frac{\ddot{g}_{22}}{g_{22}}-\left(\frac{\dot{g}_{22}}{2g_{22}}\right)^2-\frac{\dot{g}_{22}\dot{g}_{44}}{2g_{22}g_{44}}\right]$$

$$\left(E_2^2\right)_D=E_2^2+\frac{1}{4g_{44}}\left[\frac{2\ddot{g}_{22}}{g_{22}}+\frac{2\ddot{g}_{11}}{g_{11}}-\left(\frac{\dot{g}_{22}}{g_{22}}\right)^2-\left(\frac{\dot{g}_{11}}{g_{11}}\right)^2+\frac{\dot{g}_{11}\dot{g}_{22}}{g_{11}g_{22}}-\frac{\dot{g}_{11}\dot{g}_{44}}{g_{11}g_{44}}-\frac{\dot{g}_{22}\dot{g}_{44}}{g_{22}g_{44}}\right]$$

$$\left(E_4^4\right)_D=E_4^4+\frac{1}{4g_{44}}\left[\left(\frac{\dot{g}_{22}}{g_{22}}\right)^2+\frac{2\dot{g}_{11}\dot{g}_{22}}{g_{11}g_{22}}\right]$$

$$\left(E_1^4\right)_D=\frac{1}{2g_{11}}\left(\frac{2\dot{g}_{22}}{g_{22}}-\frac{\dot{g}_{11}g'_{22}}{g_{11}g_{22}}-\frac{\dot{g}_{22}g'_{22}}{g_{22}^2}-\frac{\dot{g}_{22}g'_{44}}{g_{22}g_{44}}\right),\ \left(E_1^4\right)_D=-\left(E_4^1\right)_D\frac{g_{11}}{g_{44}}$$

Here, E_i^i are specified by Equations (11)-(13) and $\left(\dot{\cdot}\right)=\partial(\cdot)/\partial t$. For the Schwarzschild solution, taking $g_{22}=r^2$, we can conclude that $\dot{g}_{22}=\ddot{g}_{22}=0$ and $\left(E_1^1\right)_D=E_1^1$, $\left(E_4^4\right)_D=E_4^4$. Thus, the equations for g_{11} and g_{44} are the same that for the static problem. Consequently, the static solution in Equations (26) is valid for the dynamic problem as well. This result is known as the Birkhoff theorem [4] according to which the solution of a spherically symmetric problem is static.

Consider the internal space of a solid sphere. The equilibrium Equation (15) becomes

$$\sigma'_r-\frac{2}{r}\left(\sigma_\theta-\sigma_r\right)+\frac{g'_{44}}{2g_{44}}\left(\sigma_r-\mu c^2\right)=0 \tag{29}$$

This equation includes two stresses. Compatibility Equation (20) does not exist in the Riemannian space. Thus, the stresses cannot be found for the space with the metric coefficient $g_{22} = r^2$. If the sphere consists of a perfect incompressible fluid, $\sigma_r = \sigma_\theta = -p$, where p is the pressure, Equation (29) reduces to

$$p' + \frac{g'_{44}}{2g_{44}}\left(p + \mu c^2\right) = 0$$

This equation, in principle, allows us to determine the pressure. We do not proceed, because a perfect incompressible fluid is not a proper continuum model in GR. In this model, perturbations propagate with infinitely high velocity, whereas in GR the velocity cannot be higher than *c*.

There exist some other solutions of a spherically symmetric GR problem based on various coordinate conditions [10] [11] [12] [13]. Particularly, consider the application of the so-called harmonic coordinate condition [3] which has the following form:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(g_{22}\sqrt{\frac{g_{44}}{g_{11}}}\right) = 2r\sqrt{g_{11}g_{44}}$$

The solution of this equation in conjunction with Equations (25) is [10]

$$g_{11} = \frac{r + r_g/2}{r - r_g/2},\quad g_{22} = \left(r + \frac{r_g}{2}\right)^2,\quad g_{44} = \frac{r - r_g/2}{r + r_g/2}$$

As can be seen, this solution is singular at $r = r_g/2$.

3. Spherically Symmetric Problem—Solution Based on a Special Geometry of Space-Time

To introduce the special geometry, undertake the following virtual experiment. Assume that we observe a solid sphere in space. According to GR, the gravitation inside the sphere induces internal Riemannian geometry for which the ratio of the sphere circumference to its diameter in not equal to *π*. As known [14], a three-dimensional object with Riemannian geometry can be embedded in a six-dimensional Euclidean space. However, observing the sphere in a three-dimensional Euclidean space, we can measure the length of the sphere circumference and the sphere diameter and conclude that the ratio of the measured parameters is equal to *π* just because we observe the sphere. The foregoing discussion allows us to suppose that gravitation induces the special Riemannian space such that the space is Euclidean with respect to space coordinates and is Riemannian with respect to time.

In spherical coordinates, the line element can be presented in this case as

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 + 2g_{14}c\mathrm{d}r\mathrm{d}t + 2g_{24}c\mathrm{d}\theta\mathrm{d}t - g_{44}c^2\mathrm{d}t^2 \tag{30}$$

The Einstein equations, Equations (1), have a cumbersome form and are not presented here (for example, the equation for E_{44} which is one of six equations contains 35 terms). The coefficients of these equations include coordinate θ which means that the metric form (30) does not correspond to a spherically symmetric problem. So, we have two possible versions—to take $g_{14} = 0$ or $g_{24} = 0$. The first three field equations in case $g_{14} = 0$ are

$$E_{11} = -\frac{1}{4re}\left(4r^2 g'_{44} + rg'^2_{24} + 4g_{24}g'_{24}\right) = 0,\ E_{12} = \frac{g_{24}\cos\theta}{2re\sin\theta}\left(2g_{24} - rg'_{24}\right)$$
$$E_{14} = -\frac{\cos\theta}{2e\sin\theta}\left(g_{44}g'_{24} - g_{24}g'_{44}\right),\ e = r^2 g_{44} + g^2_{24} \tag{31}$$

The rest equations are not presented here for the sake of brevity. The solution of the second Equation (31) is $g_{24} = Cr^2$. It does not satisfy the asymptotic condition $g_{24}(r \to \infty) = 0$. Thus, the only one possible form of the line element corresponding to $g_{24} = 0$ is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 + 2g_{14}c\mathrm{d}r\mathrm{d}t - g_{44}c^2\mathrm{d}t^2 \tag{32}$$

In this case, Equations (1) become (we use contravariant tensor components to simplify the equations)

$$E^{11} = \frac{1}{r^2 g^2}\left(g^2_{14}g_{44} - rg_{44}g'_{44}\right) = 0 \tag{33}$$

$$E^{22} = -\frac{1}{4r^3 g^2}\left(4g_{14}g'_{44} - 4g'_{44}g^2_{14} - 2g_{44}g'_{44} - 2rg^2 g''_{44} + 2rg_{14}g'_{44}g'_{14} - 2rg_{44}g''_{44} + rg'^2_{44}\right) = 0 \tag{34}$$

$$E^{14} = E^{41} = \frac{g_{14}}{r^2 g^2}\left(g^2_{14} - rg'_{44}\right) = 0 \tag{35}$$

$$E^{44} = -\frac{g_{14}}{r^2 g^2}\left(2rg'_{14} + g_{14}\right) = 0,\ g = g_{44} + g^2_{14} \tag{36}$$

For the space with the line element in Equation (32), tensor E satisfies two conservation equations that follow from Equation (7). These equations take the simplest form if we use mixed tensor components, *i.e.*,

$$\frac{\mathrm{d}E^1_1}{\mathrm{d}r} + \frac{2}{r}\left(E^1_1 - E^2_2\right) + \frac{g'_{44}}{2g}\left(E^1_1 - E^4_4\right) + \frac{g_{14}g'_{44}}{2g}E^4_1 = 0 \tag{37}$$

$$g_{14}\left(E^4_4 - E^1_1\right) + g_{44}E^4_1 = 0 \tag{38}$$

Thus, only two of Equations (33)-(36) are mutually independent. Consider Equations (35) and (36) which yield

$$g^2_{14} - rg'_{44} = 0,\ 2rg'_{14} + g_{14} = 0 \tag{39}$$

These equations include two unknown functions and the problem of additional coordinate conditions does not occur. The general solution of Equations (39) is

$$g_{14} = \frac{C_1}{\sqrt{r}},\ g_{44} = -\frac{C_1^2}{r} + C_2$$

Integration constants C_1 and C_2 can be found from the asymptotic condition according to which the obtained solution must reduce to equations (16) for $r \to \infty$. The result is $C_1 = \sqrt{r_g}, C_2 = 1$ and the final solution becomes

$$g_{14} = \sqrt{\frac{r_g}{r}},\ g_{44} = 1 - \frac{r_g}{r} \tag{40}$$

Substituting this solution in Equations (33) and (34), we can prove that they are satisfied identically. Thus, the Einstein equations and the asymptotic conditions are satisfied. The constructed space has the following line element:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 + 2c\sqrt{r_g/r}\,\mathrm{d}r\mathrm{d}t - c^2\left(1-\frac{r_g}{r}\right)\mathrm{d}t^2 \tag{41}$$

The line element in Equation (41) corresponds to the so-called Gullstand-Painlever coordinates and was found in 1921-22 [15] [16] [17] [18] as a result of the coordinate transformation of the line element in Equation (10) corresponding to the Schwarzschild solution (26). Here, Equation (41) is not associated with the Schwarzschild Riemannian space. It corresponds to the special space introduced above which consists in the general case of a three-dimensional Euclidean space and the non-Newtonian coordinate which appears only in time. The total space is Riemannian—the non-zero components of the curvature tensor are

$$R_{1212} = -\frac{r_g}{2r},\ R_{1313} = -\frac{r_g}{2r}\sin^2\theta,\ R_{1224} = \frac{r_g}{2r}\sqrt{\frac{r_g}{r}},\ R_{1334} = \frac{r_g}{2r}\sqrt{\frac{r_g}{r}}\sin^2\theta$$

$$R_{1414} = -\frac{r_g}{r^3},\ R_{2323} = rr_g\sin^2\theta,\ R_{2424} = \frac{r_g}{2r^2}\left(r-r_g\right),\ R_{3434} = \frac{r_g}{2r^2}\left(r-r_g\right)\sin^2\theta$$

To analyze the obtained solution, determine the escape velocity. The motion equation analogous to Equation (27) is

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{c^2 r_g}{2r^2}\left[1-\frac{r_g}{r}-\frac{3}{c^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\right] + \frac{1}{c}\sqrt{\frac{r_g}{r}}\left[\frac{3c^2 r_g}{2r^2}+\frac{1}{2r}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\right]\frac{\mathrm{d}r}{\mathrm{d}t}$$

Introducing a new variable $u(r)$ in accordance with Equations (22), we get

$$u\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{3r_g}{2r^2}u^2 - \frac{c}{2r}\sqrt{\frac{r_g}{r}}\left(\frac{3r_g}{r}+\frac{u^2}{c^2}\right)u = -\frac{c^2 r_g}{2r^2}\left(1-\frac{r_g}{r}\right)$$

Using the following transformation

$$u(r) = \frac{1}{y(r)},\quad \frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{y'}{y^2}$$

We can reduce this equation to

$$y' - \frac{c^2 r_g}{2r^2}\left(1-\frac{r_g}{r}\right)y^3 + \frac{3cr_g}{2r^2}\sqrt{\frac{r_g}{r}}y^2 + \frac{3r_g}{2r^2}y = -\frac{1}{2cr}\sqrt{\frac{r_g}{r}}$$

This is the Abel equation whose solution which satisfies the condition $u(r\to\infty)=0$ is [19]

$$u = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{y(r)} = c\sqrt{\frac{r_g}{r}}\frac{1-r_g/r}{1+r_g/r} \tag{42}$$

To determine the physical escape velocity, we need to transform the matrix of the metric coefficients to the diagonal form. To specify the line element of the space-time radial trajectory, take $\mathrm{d}\theta = \mathrm{d}\varphi = 0$ in Equation (32) to get

$$\mathrm{d}s_e^2 = \mathrm{d}r^2 + 2g_{14}c\mathrm{d}r\mathrm{d}t - g_{44}c^2\mathrm{d}t^2$$

Using identical transformations, we have

$$\mathrm{d}s_e^2=\left(1+\frac{g_{14}^2}{g_{44}}\right)\mathrm{d}r^2-g_{44}\left(c-\frac{g_{14}}{g_{44}}\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\mathrm{d}t^2$$

Substitution of Equations (40) and (42) yields

$$\mathrm{d}s_e^2=g_{11}^e\mathrm{d}r^2-g_{44}^e c^2\mathrm{d}t^2,\quad g_{11}^e=\frac{r}{r-r_g},\quad g_{44}^e=\frac{r\left(r-r_g\right)}{\left(r+r_g\right)^2}$$

As can be seen, the expression for g_{11}^e corresponds to the Schwarzschild solution in Equations (26). Finally, we arrive at the following equation for the escape velocity:

$$v_e=\frac{\mathrm{d}r}{\mathrm{d}t}\sqrt{\frac{g_{11}^e}{g_{44}^e}}=c\sqrt{\frac{r_g}{r}}$$

This result coincides with Equation (28) for the Schwarzschild solution and means that the spherical object with radius R becomes invisible if $R=r_g$. However, there is a principal difference between the obtained and the Schwarzschild solutions—the metric coefficients in Equations (40) are not singular and r_g is not the radius of the horizon of events.

To determine the angle for the light beam deviation in the vicinity of Sun for the space with the line element in Equation (41), we follow the procedure used for the Schwarzschild solution [3]. Consider the light beam trajectory in the equatorial plane ($\theta=\pi/2$) of the sphere with radius R. The deviation angle can be found as [3]

$$\Delta\varphi=2\left[\varphi\left(R\right)-\varphi\left(r\to\infty\right)\right]-\pi \tag{43}$$

Take $\theta=\pi/2$ in Equation (41) and apply identical transformations to present it in the following form:

$$\mathrm{d}s^2=\alpha\left(r\right)\mathrm{d}r^2+r^2\mathrm{d}\varphi^2-\beta\left(r\right)\mathrm{d}\tau^2 \tag{44}$$

where

$$\alpha\left(r\right)=1+\frac{r_g}{r\left(1-r_g/r\right)},\quad \beta\left(r\right)=1-\frac{r_g}{r},\quad \mathrm{d}\tau=\mathrm{d}t-\frac{\sqrt{r_g/r}}{1-r_g/r}\mathrm{d}r \tag{45}$$

Geodesic line is specified by the following equations:

$$\frac{\mathrm{d}^2r}{\mathrm{d}p^2}+\frac{\alpha'}{2\alpha}\left(\frac{\mathrm{d}r}{\mathrm{d}p}\right)^2-\frac{r}{\alpha}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}p}\right)^2+\frac{\beta'}{2\alpha}\left(\frac{\mathrm{d}\tau}{\mathrm{d}p}\right)^2=0 \tag{46}$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}p^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}p}\frac{\mathrm{d}\varphi}{\mathrm{d}p}=0 \tag{47}$$

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}p^2}+\frac{\beta'}{\beta}\frac{\mathrm{d}r}{\mathrm{d}p}\frac{\mathrm{d}\tau}{\mathrm{d}p}=0 \tag{48}$$

Here p is some parameter that is counted along the trajectory and can be arbitrary chosen. The solution of Equation (48) is

$$\beta \frac{\mathrm{d}\tau}{\mathrm{d}p} = C_1$$

We can take the parameter p such that $C_1 = 1$ and get

$$\frac{\mathrm{d}\tau}{\mathrm{d}p} = \frac{1}{\beta} \tag{49}$$

The first integral of Equation (47) is

$$\frac{\mathrm{d}\varphi}{\mathrm{d}p} = \frac{C_2}{r^2} \tag{50}$$

Consider Equation (46). Using Equations (49) and (50), we can reduce it to

$$\frac{\mathrm{d}^2 r}{\mathrm{d}p^2} + \frac{\alpha'}{2\alpha}\left(\frac{\mathrm{d}r}{\mathrm{d}p}\right)^2 - \frac{r}{\alpha}\left(\frac{C_2}{r}\right)^2 + \frac{\beta'}{2\alpha\beta^2} = 0$$

Integration yields

$$\alpha\left(\frac{\mathrm{d}r}{\mathrm{d}p}\right)^2 + \left(\frac{C_2}{r}\right)^2 - \frac{1}{\beta} = C_3 \tag{51}$$

We can prove that $C_3 = 0$. Indeed, transform Equation (44) with the aid of Equations (49)-(51) as

$$\begin{aligned} \mathrm{d}s^2 &= \left[\alpha\left(\frac{\mathrm{d}r}{\mathrm{d}p}\right)^2 + r^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}p}\right)^2 - \beta\left(\frac{\mathrm{d}\tau}{\mathrm{d}p}\right)^2\right]\mathrm{d}p^2 \\ &= \left[C_3 - \left(\frac{C_2}{r}\right)^2 + \frac{1}{\beta} + r^2\left(\frac{C_2}{r^2}\right)^2 - \beta\left(\frac{1}{\beta}\right)^2\right]\mathrm{d}p^2 = C_3 \mathrm{d}p^2 \end{aligned}$$

However, for the light beam $\mathrm{d}s^2 = 0$, so $C_3 = 0$. Using Equation (50) to change $\mathrm{d}p$ to $\mathrm{d}\varphi$ in Equation (51), we arrive at

$$\frac{\alpha}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + \frac{1}{r^2} - \frac{1}{\beta C_2^2} = 0 \tag{52}$$

Now, take into account that the light beam trajectory has a symmetric configuration and passes the sphere in the vicinity of its surface $r = R$. Then, at $r = R$ we have $\mathrm{d}\varphi/\mathrm{d}r = 0$. This condition allows us to find $C_2^2 = R/\beta(R)$ from Equation (52) and to present the result in the following form:

$$\mathrm{d}\varphi = \frac{\mathrm{d}r}{r}\sqrt{\frac{\alpha}{\frac{r^2\beta(R)}{R^2\beta(r)} - 1}}$$

Finally, the corresponding term in Equation (43) becomes

$$\varphi(r) - \varphi(r \to \infty) = \int_r^\infty \frac{\mathrm{d}r}{r}\sqrt{\frac{\alpha}{\frac{r^2\beta(R)}{R^2\beta(r)} - 1}}$$

As for the Schwarzschild solution [3], this integral can be reduced to elliptic integrals. However, integration can be considerably simplified if we take into account that the ratio $\bar{r}_g = r_g/R$ is small in comparison with unity. Particularly for Sun, $\bar{r}_g = 4.3\times10^{-6}$ and we can linearize the functions $\alpha(r)$ and $\beta(r)$ in Equations (45) with respect to the ratio r_g/r. Neglecting relatively small terms, we finally get

$$\varphi(r)-\varphi(r\to\infty)\cong\int_r^\infty \frac{\mathrm{d}r}{r\sqrt{\frac{r^2}{R^2}-1}}\left[1+\frac{r_g}{2r}+\frac{r_g r}{2R(r+R)}\right]$$
$$=\sin^{-1}\left(\frac{R}{r}\right)+\frac{r_g}{2R}\left(2-\sqrt{1-\frac{R^2}{r^2}}-\sqrt{\frac{r-R}{r+R}}\right)$$

Then, Equation (43) yields $\Delta\varphi = 2\bar{r}_g$. For Sun, the numerical result is $\Delta\varphi = 1.75''$ which is in agreement with the experiment.

Now, assume that the metric coefficients g_{14} and g_{44} in Equation (32) depend on the radial coordinate and time. The field equations that generalize Equations (33)-(36) are

$$E_D^{11} = E^{11} + \frac{1}{rg^2}\left(g_{14}\dot{g}_{44} - 2g_{44}\dot{g}_{14}\right) = 0 \quad (53)$$

$$E_D^{22} = E^{22} + \frac{1}{2r^3g^2}\left(g_{14}\dot{g}_{44} - 2g_{44}\dot{g}_{14} - 2rg_{14}^2\dot{g}'_{14} + 2rg_{14}g'_{14}\dot{g}_{14} - 2rg_{44}\dot{g}'_{14} + r\dot{g}_{14}\dot{g}_{44}\right) = 0 \quad (54)$$

$$E_D^{14} = E_D^{41} = E^{14} = E^{41} = 0,\ E_D^{44} = E^{44} = 0 \quad (55)$$

Here, E^{ij} are specified by Equations (33)-(36). As can be seen, equations (55) coincide with Equations (35) and (36) whose solution in Equations (40) does not depend on time. Thus, the spherically symmetric problem for the space with the line element in Equation (41) has only a static solution. This result is analogous to the Birkhoff theorem in the Schwarzschild solution (Section 2).

Consider the internal space ($0 \le r \le R$) of a solid sphere. Since the space with the line element in Equation (41) is Euclidean in coordinates r,θ,φ, the compatibility Equation (20) of the Newton theory is valid and, being added to the conservation equations, allows us to determine the stresses.

4. Conclusion

The special geometry of the four-dimensional Riemannian space induced by gravitation is proposed. This geometry is Euclidean with respect to the space coordinates and is Riemannian with respect to the time coordinate. Field equations for a spherically symmetric problem yield the solution known as Gullstrand-Painleve solution which, in contrast to the classical Schwarzschild solution, is not singular. The obtained solution allows us to suppose the existence of invisible spherical objects with finite level of gravitation.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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