Scientific
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Killing Operator for the Kerr Metric

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Abstract

When $\mathcal{D}: E \to F$ is a linear differential operator of order q between the sections of vector bundles over a manifold X of dimension n, it is defined by a bundle map $\Phi: J_q(E) \to F = F_0$ that may depend, explicitly or implicitly, on constant parameters $a, b, c, \cdots$. A "direct problem" is to find the generating compatibility conditions (CC) in the form of an operator $\mathcal{D}_1: F_0 \to F_1$. When $\mathcal{D}$ is involutive, that is when the corresponding system $R_q = ker(\Phi)$ is involutive, this procedure provides successive first order involutive operators $\mathcal{D}_1, \cdots, \mathcal{D}_n$. Though $\mathcal{D}_1 \circ \mathcal{D} = 0$ implies $ad(\mathcal{D}) \circ ad(\mathcal{D}_1) = 0$ by taking the respective adjoint operators, then $ad(\mathcal{D})$ may not generate the CC of $ad(\mathcal{D}_1)$ and measuring such "gaps" led to introduce extension modules in differential homological algebra. They may also depend on the parameters and such a situation is well known in ordinary or partial control theory. When R_q is not involutive, a standard *prolongation/projection* (PP) procedure allows in general to find integers r, s such that the image $R_{q+r}^{(s)}$ of the projection at order $q+r$ of the prolongation

$\rho_{r+s}(R_q) = J_{r+s}(R_q) \cap J_{q+r+s}(E) \subset J_{r+s}(J_q(E))$ is involutive but it may highly depend on the parameters. However, sometimes the resulting system no longer depends on the parameters and the extension modules do not depend on the parameters because it is known that they do not depend on the differential sequence used for their definition. The purpose of this paper is to study the above problems for the Kerr (m, a), Schwarzschild $(m, 0)$ and Minkowski $(0, 0)$ parameters while computing the dimensions of the inclusions $R_1^{(3)} \subset R_1^{(2)} \subset R_1^{(1)} = R_1 \subset J_1(T(X))$ for the respective Killing operators. Other striking motivating examples are also presented.

Equations, Kerr Metric, Differential Module, Extension Module, Contact Structure

1. Introduction

With standard notations of differential geometry, let $(E, F, \cdots)$ be vector bundles over a manifold X with dimension n with sections $(\xi, \eta, \cdots)$, tangent bundle T and cotangent bundle T^*. We shall denote by $J_q(E)$ the q-jet bundle of E with sections ξ_q transforming like the q-derivatives $j_q(\xi)$. If $\Phi: J_q(E) \to F$ is a bundle morphism, we shall consider the system $R_q = ker(\Phi) \subset J_q(E)$ of order q on E. The r-prolongation $\rho_r(R_q) = J_r(R_q) \cap J_{q+r}(E) \subset J_r(J_q(E))$ obtained by differentiating formally r times the given ordinary (OD) or partial (PD) defining equations of R_q will be the kernel of the composite morphism $\rho_r(\Phi): J_{q+r}(E) \to J_r(J_q(E)) \to J_r(F)$. The symbol $g_{q+r} = R_{q+r} \cap S_{q+r}T^* \otimes E \subset J_{q+r}(E)$ of R_{q+r} is the r-prolongation of the symbol g_q of R_q and the kernel of the composite morphism $\sigma_r(\Phi): S_{q+r}T^* \otimes E \to S_r T^* \otimes F$ obtained by restriction. The Spencer operator $d: R_{q+1} \to T^* \otimes R_q: \xi_{q+1} \to j_1(\xi_q) - \xi_{q+1}$ is obtained by using the fact that $R_{q+1} = J_1(R_q) \cap J_{q+1}(E)$ and that $J_1(R_q)$ is an affine vector bundle over R_q modelled on $T^* \otimes R_q$. We shall always suppose that Φ is an epimorphism and introduce the vector bundle $F_0 = J_q(E)/R_q$. The system R_q is aid to be *formally integrable* (FI) if $r+1$ prolongations do not bring new equations of order $q+r$ other than the ones obtained after only r prolongations, for any $r \geq 0$, that is all the equations of order $q+r$ can be obtained by differentiating r times *only* the given equations of order q for any $r \geq 0$. The system is said to be "*involutive*" if it is FI and the symbol g_q is involutive, a purely algebraic property.

The next problem is to define the CC operator $\mathcal{D}_1: F_0 \to F_1: \eta \to \zeta$ in such a way that the CC of $\mathcal{D}\xi = \eta$ is of the form $\mathcal{D}_1\eta = 0$. As shown in many books ([1]-[7]) and papers ([8] [9] [10] [11]), such a problem may be quite difficult because the order of the generating CC may be quite high. Proceeding in this way, we may construct the CC $\mathcal{D}_2: F_1 \to F_2$ of $\mathcal{D}_1$ and so on. When the system R_q or the operator $\mathcal{D}$ are involutive, the successive CC operators can only be *at most* $\mathcal{D}_1, \cdots, \mathcal{D}_n$ which are first order and involutive operators.

A second problem shown on the motivating examples is that "*jumps*" in the successive orders may appear, even on elementary examples. Now, if the map Φ depends on constant (or variable) parameters $(a, b, c, \cdots)$, then the study of the two previous problems becomes much harder because the ranks of the matrices $\rho_r(\Phi)$ and/or $\sigma_r(\Phi)$ may also highly depend on the parameters as we shall see. The case of the Killing operator for the Minkowski metric is well known but the study of the Killing operator for the Schwarzschild and Kerr metrics is rather recent and striking as it proves that both the numbers of generating second order CC and the numbers of generating third order CC may change

[12] [13] [14] [15]. We provide a few reasons for which we totally disagree with these publications, in particular with ([13]), as we already explained in the Introduction of ([11]).

1) Our first doubts came after a look at the core of ([12]) containing 10 pages (plus 2 pages in Appendix A) of *awfully technical computational notations* that we have never seen after more than 40 years dealing with systems of PDE. Looking at the details it is clear that the authors always try to bring any differential complex back to the de Rham (or Poincaré in France) complex for the exterior derivative in order to avoid dealing with the Spencer δ-cohomology.

2) Our second doubts came from the *systematic* reference to certain *specific* purely physical tools such as Teukolsky scalars, Cartan/Penrose/Weyl complex spinors, Newman-Penrose formalism or Killing Yano tensors, despite the claim that the search for CC has nothing to do with GR and is a purely mathematical problem of *formal integrability* that we explained to the authors when lecturing at the AEI (Albert Einstein Institute, Potsdam) in 2017.

3) Our third doubts came from an elementary example (See [11], end of Introduction) that we shall revisit in the following motivating examples. Indeed, a (difficult) theorem of homological algebra is saying that the only *intrinsic concept* that can be attached to a module is made by the *extension modules* that do not depend on the resolution used for their computation. In the present situation, *the two (m, a) parameters in the Kerr metric just disappear in the final system provided by the Theorem 1.1 below*, exactly like in the example just quoted. Hence we could not get more information than by using the tensor product of the Poincaré sequence by the Lie algebra of 2-dimensional translations, that is *nothing* because the parameters no longer appear.

4) Last but not least and "cherry on the cake", if we agree with the numbers of second order CC, we disagree with the numbers of third order CC. It was thus a challenge for computing these numbers independently of any relativistic argument. It has also been a surprise to discover that the long recent preprint (arXiv: 2207.12959) was again taking for granted the results of ([13]).

The purpose of this paper is thus to revisit these works by using new homological techniques ([4] [5] [7] [16]). It is a matter of fact that they do not agree with the previous ones for the third order CC. In order to escape from such an unpleasant situation, we have written this paper in such a way that we are only using elementary combinatorics and diagram chasing. However, an equally important second purpose is to question the proper target of the quoted publications. Indeed, important concepts such as *extension modules* have been introduced in *homological algebra* and can be extended to the differential framework. Thanks to a quite difficult theorem ([4] [17]), they are the *only* intrinsic results that could be obtained independently of the differential sequence that could be used, provided that $\Theta = ker(\mathcal{D})$ is the same, that is even if one is using another system with the same solutions. Equivalently, this amounts to say, in a few words but a more advanced language, if *we are keeping the same differential module M*

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but changing its presentation.

However, as we shall see, there are even simple academic systems depending on parameters but such that a convenient system (say involutive) may no longer depend on the parameters. Also, in a totally independent way still not acknowledged, E. Vessiot has shown in ([18]) that certain operators may depend on geometric objects satisfying non-linear *structure equations* that are depending on certain *Vessiot structure constants c*. The simplest example is the condition of constant Riemannian curvature ([1] [19]) which is necessary in order that the Killing system becomes FI but we shall prove that the case of contact structures is similar. In such situations, we shall prove that the extension modules only depend on these constants.

We now recall the main results and definitions that are *absolutely needed* for the applications.

With canonical epimorphism $\Phi_0 = \Phi : J_q(E) \Rightarrow J_q(E)/R_q = F = F_0$, the various prolongations are described by the following commutative and exact "*introductory diagram*" used in the sequel:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & g_{q+r+1} & \to & S_{q+r+1}T^* \otimes E & \overset{\sigma_{r+1}(\Phi)}{\to} & S_{r+1}T^* \otimes F_0 & \to & h_{r+1} & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & R_{q+r+1} & \to & J_{q+r+1}(E) & \overset{\rho_{r+1}(\Phi)}{\to} & J_{r+1}(F_0) & \to & Q_{r+1} & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & R_{q+r} & \to & J_{q+r}(E) & \overset{\rho_r(\Phi)}{\to} & J_r(F_0) & \to & Q_r & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Chasing along the diagonal of this diagram while applying the standard "*snake*" lemma, we obtain the useful "*long exact connecting sequence*" also often used in the sequel:

$$0 \to g_{q+r+1} \to R_{q+r+1} \to R_{q+r} \to h_{r+1} \to Q_{r+1} \to Q_r \to 0$$

which is thus connecting in a tricky way FI (*lower left*) with CC (*upper right*).

A key step in the procedure for constructing differential sequences will be to use the following (difficult) theorems and corollary (For Spencer cohomology and acyclicity or involutivity (See [1] [3] [5] for details and compare to [20] [21]) (See [7] [8] [11] for more details):

THEOREM 1.1: Let a system $R_q \subset J_q(E)$ be given and set $R_{q+r}^{(s)} = \pi_{q+r}^{q+r+s}(R_{q+r+s}), \forall r,s \ge 0$. If g_{q+1} is a vector bundle and g_q is 2-acyclic, then we have $\rho_r\left(R_q^{(1)}\right) = R_{q+r}^{(1)}, \forall r \ge 0$. Accordingly, there is a finite *Prolongation/Projection* (PP) algorithm providing two integers $r,s \ge 0$ by successive increase of each of them such that the new system $R_{q+r}^{(s)}$ has the same solutions as R_q but is FI with a 2-acyclic or involutive symbol and thus first order CC. It follows that the maximum order of $\mathcal{D}_1$ is thus equal to $r+s+1$ as we used

$r+s$ prolongations but it may be lower because certain CC may generate the higher order ones as will be seen in the motivating examples.

DEFINITION 1.2: A differential sequence is said to be *formally exact* if it is exact on the jet level composition of the prolongations involved. A formally exact sequence is said to be *strictly exact* if all the operators/systems involved are FI (See [22] for more details). A strictly exact sequence is called *canonical* if all the operators/systems are involutive.

When $d : J_{q+1}(E) \to T^* \otimes J_q(E) : \xi_{q+1} \to j_1(\xi_q) - \xi_{q+1}$ is the Spencer operator, we have:

PROPOSITION 1.3: If $R_q \subset J_q(E)$ and $R_{q+1} \subset J_{q+1}(E)$ are two systems of respective orders q and $q+1$, then $R_{q+1} \subset \rho_1(R_q)$ if and onlty if $\pi_q^{q+1}(R_{q+1}) \subset R_q$ *and* $dR_{q+1} \subset T^* \otimes R_q$.

DEFINITION 1.4: Let us "cut" the preceding introductory diagram by means of a central vertical line and define $R'_r = im(\rho_r(\Phi)) \subseteq J_r(F_0)$ with $R'_0 = F_0$. Chasing in this diagram, we notice that $\pi_r^{r+1} : J_{r+1}(F_0) \to J_r(F_0)$ induces an epimorphism $\pi_r^{r+1} : R'_{r+1} \to R'_r, \forall r \geq 0$. However, a chase in this diagram proves that *the kernel of this epimorphism is not* $im(\sigma_{r+1}(\Phi))$ unless R_q is FI (*care*). For this reason, we shall define it to be *exactly* g'_{r+1}.

THEOREM 1.5: $R'_{r+1} \subseteq \rho_1(R'_r)$ and $dim(\rho_1(R'_r)) - dim(R'_{r+1})$ is the number of new generating CC of order $r+1$.

COROLLARY 1.6: The system $R'_r \subset J_r(F_0)$ becomes FI with a 2-acyclic or involutive symbol and $R'_{r+1} = \rho_1(R'_r) \subset J_{r+1}(F_0)$ when r is large enough.

2. Motivating Examples

EXAMPLE 2.1: Let $n = 2, m = 1$ and introduce the trivial vector bundle E with local coordinates (x^1, x^2, ξ) for a section over the base manifold X with local coordinates (x^1, x^2). Let us consider the linear second order system $R_2 \subset J_2(E)$ defined by the two linearly independent equations $d_{22}\xi = 0$, $d_{12}\xi + ad_1\xi = 0$ where a is an arbitrary constant parameter. Using crossed derivatives, we get the second order system $R_2^{(1)} \subset R_2$ defined by the PD equations $d_{22}\xi = 0$, $d_{12}\xi + ad_1\xi = 0$, $a^2 d_1\xi = 0$ which is easily seen not to be involutive. Framing essential results and Janet tabulars, we have two possibilities:

- $a = 0$: We obtain the following second order homogeneous involutive system:

$$R_2^{(1)} = R_2 \subset J_2(E) \quad \begin{cases} d_{22}\xi = \eta^2 & \boxed{\begin{matrix} 1 & 2 \end{matrix}} \\ d_{12}\xi = \eta^1 & \boxed{\begin{matrix} 1 & \bullet \end{matrix}} \end{cases}$$

with the only first order homogeneous involutive CC $d_1\eta^2 - d_2\eta^1 = 0$ leading to the Janet sequence:

$$\boxed{0 \to \Theta \to E \xrightarrow[2]{\mathcal{D}} F_0 \xrightarrow[1]{\mathcal{D}_1} F_1 \to 0}$$

We let the reader check as an easy exercise that $ad(\mathcal{D})$ which is of order 2 does not generate the CC of $ad(\mathcal{D}_1)$ which is of order 1 and thus $ext^1(M) \neq 0$.

- $a \neq 0$: We obtain the second order system $R_2^{(1)}$ defined by $d_{22}\xi = 0$, $d_{12}\xi = 0$, $d_1\xi = 0$ with a strict inclusion $R_2^{(1)} \subset R_2$ because $3 < 4$. We may define $\eta = d_1\eta^2 - d_2\eta^1 + a\eta^1$ and obtain the involutive and finite type system in δ-regular coordinates:

$$R_2^{(2)} \subset J_2(E) \quad \begin{cases} d_{22}\xi = \eta^2 \\ d_{12}\xi = \eta^1 - \dfrac{1}{a}\eta \\ d_{11}\xi = \dfrac{1}{a^2}d_1\eta \\ d_1\xi = \dfrac{1}{a^2}\eta \end{cases} \quad \begin{array}{|cc|} \hline 1 & 2 \\ 1 & \bullet \\ 1 & \bullet \\ \bullet & \bullet \\ \hline \end{array}$$

Counting the dimensions, we have the following strict inclusions by comparing the dimensions:

$$R_2^{(2)} \subset R_2^{(1)} \subset R_2 \subset J_2(E), \quad 2 < 3 < 4 < 6$$

LEMMA 2.2: The symbol g_{2+r} is involutive with $dim(g_{2+r}) = 1, \forall r \geq 0$. Moreover, we have $dim(R_{2+r}) = 4, \forall r \geq 0$.

Proof: Using jet notation, the 4 parametric jets of R_2 are $(\xi, \xi_1, \xi_2, \xi_{11})$. The 4 parametric jets of R_3 are now $(\xi, \xi_2, \xi_{11}, \xi_{111})$ and so on. Accordingly, the dimension of g_{2+r} is 1 because the only parametric jet is $\xi_{1\ldots1}$. We have the short exact sequence $0 \to g_{r+2} \to R_{r+2} \to R_{r+1}^{(1)} \to 0$ and the symbol of R_2 is involutive. It follows from the (difficult) Theorem 1.1 (See [1], Theorem 2.4.5 p 70 and proposition 2.5.1 p 76 with $q = 2$) that $\rho_r\left(R_2^{(1)}\right) = R_{2+r}^{(1)}, \forall r \geq 0$ with $dim\left(R_{r+1}^{(1)}\right) = 3$, a result leading to $dim(R_{r+2}) = 3+1 = 4$ by counting the dimensions. As $R_2^{(1)}$ does not depend any longer on the parameter, the general solution is easily seen to be of the form $\xi = cx^2 + d$ and is thus only depending on two arbitrary constants, contrary to what could be imagined from this lemma but in a coherent way with the fact that $dim\left(R_{2+r}^{(2)}\right) = 2, \forall r \geq 0$. $\square$

After differentiating twice, we could be waiting for CC of order 3. However, we obtain the 4 CC:

$$d_2\eta^1 - \frac{1}{a}d_2\eta - d_1\eta^2 = 0, \frac{1}{a^2}d_{12}\eta - d_1\eta^1 + \frac{1}{a}d_1\eta = 0,$$

$$\frac{1}{a^2}d_2\eta - \eta^1 + \frac{1}{a}\eta = 0, \frac{1}{a^2}\left(d_1\eta - d_1\eta\right) = 0$$

The last CC that we shall call "*identity to zero*" must not be taking into account. The second CC is just the derivative with respect to x^1 of the third CC which amounts to

$$\left(d_{12}\eta^2 - d_{22}\eta^1 + ad_2\eta^1\right) - a^2\eta^1 + a\left(d_1\eta^2 - d_2\eta^1 + a\eta^1\right) = 0$$

$$\Leftrightarrow d_{12}\eta^2 - d_{22}\eta^1 + ad_1\eta^2 = 0$$

which is a second order CC amounting to the first. Hence we get the only generating CC operator $\mathcal{D}_1 : \left(\eta^1, \eta^2\right) \to d_{12}\eta^2 - d_{22}\eta^1 + ad_1\eta^2 = \zeta$ which is thus formally surjective.

For helping the reader, we recall that basic elementary combinatorics arguments are giving $dim\left(S_q T^*\right) = q+1$ while $dim\left(J_q\left(E\right)\right) = (q+1)(q+2)/2$ because $n = 2$ and $m = dim\left(E\right) = 1$.

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & g_2 & \to & S_2 T^* \otimes E & \to & F_0 & \to & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 \to & & R_2 & \to & J_2\left(E\right) & \to & F_0 & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & R_1 & \to & J_1\left(E\right) & \to & 0 & & \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0 & & & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
& 0 & & 0 & & 0 & & \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \to & 1 & \to & 3 & \to & 2 & \to & 0 \\
& \downarrow & & \downarrow & & \| & & \\
0 \to & 4 & \to & 6 & \to & 2 & \to & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \to & 3 & \to & 3 & \to & 0 & & \\
& \downarrow & & \downarrow & & & & \\
& 0 & & 0 & & & &
\end{array}
$$

$$
\begin{array}{ccccccccccc}
& 0 & & 0 & & 0 & & & & \\
& \downarrow & & \downarrow & & \downarrow & & & & \\
0 \to & g_3 & \to & S_3 T^* \otimes E & \to & T^* \otimes F_0 & \to & h_1 & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & R_3 & \to & J_3\left(E\right) & \to & J_1\left(F_0\right) & \to & Q_1 & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & R_2 & \to & J_2\left(E\right) & \to & F_0 & \to & 0 & \\
& & & \downarrow & & \downarrow & & & \\
& & & 0 & & 0 & & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
& 0 & & 0 & & 0 & & \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \to & 1 & \to & 4 & \to & 4 & \to & 1 & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & 4 & \to & 10 & \to & 6 & \to & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \to & 4 & \to & 6 & \to & 2 & \to & 0 \\
& & & \downarrow & & \downarrow & & \\
& & & 0 & & 0 & &
\end{array}
$$

$$\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & g_4 & \to & S_4 T^* \otimes E & \to & S_2 T^* \otimes F_0 & \to & h_2 & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & R_4 & \to & J_4(E) & \to & J_2(F_0) & \to & Q_2 & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & R_3 & \to & J_3(E) & \to & J_1(F_0) & \to & 0 & \\
& & & \downarrow & & \downarrow & & & \\
& & & 0 & & 0 & & &
\end{array}$$

$$\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & 1 & \to & 5 & \to & 6 & \to & 2 & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & 4 & \to & 15 & \to & 12 & \to & 1 & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & 4 & \to & 10 & \to & 6 & \to & 0 & \\
& & & \downarrow & & \downarrow & & & \\
& & & 0 & & 0 & & &
\end{array}$$

Using these diagrams, we obtain successively till we stop:

$$R'_0 = F_0 \Rightarrow R'_1 = J_1(F_0) \Rightarrow \rho_1(R'_1) = J_2(F_0) \Rightarrow R'_2 \subset \rho_1(R'_1) \Rightarrow R'_3 = \rho_1(R'_2)$$

Hence, the number of generating CC of order 1 is zero and the number of generating CC of strict order 2 is $dim(\rho_1(R'_1)) - dim(R'_2) = 12 - (15 - 4) = 12 - 11 = 1$ in a coherent way.

Setting $F_1 = Q_2$ with $dim(Q_2) = 1$, we obtain the commutative diagram:

$$\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & g_5 & \to & S_5 T^* \otimes E & \to & \boxed{S_3 T^* \otimes F_0} & \to & T^* \otimes F_1 & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & R_5 & \to & J_5(E) & \to & J_3(F_0) & \to & J_1(F_1) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & R_4 & \to & J_4(E) & \to & J_2(F_0) & \to & F_1 & \to 0 \\
& & & \downarrow & & \downarrow & & \downarrow & \\
& & & 0 & & 0 & & 0 &
\end{array}$$

with dimensions:

$$\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & 1 & \to & 6 & \to & \boxed{8} & \to & 2 & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & 4 & \to & 21 & \to & 20 & \to & 3 & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & 4 & \to & 15 & \to & 12 & \to & 1 & \to 0 \\
& & & \downarrow & & \downarrow & & \downarrow & \\
& & & 0 & & 0 & & 0 &
\end{array}$$

The upper symbol sequence is not exact at $S_3 T^* \otimes F_0$ even though the two other sequences are exact on the jet level. As a byproduct we have the exact sequences $\forall r \geq 0$:

$$0 \to R_{r+4} \to J_{r+4}(E) \to J_{r+2}(F_0) \to J_r(F_1) \to 0$$

Such a result can be checked directly through the identity:

$$4 - (r+5)(r+6)/2 + 2(r+3)(r+4)/2 - (r+1)(r+2)/2 = 0$$

We obtain therefore the formally exact sequence we were looking for, namely:

$$\boxed{0 \to \Theta \to E \underset{2}{\overset{\mathcal{D}}{\to}} F_0 \underset{2}{\overset{\mathcal{D}_1}{\to}} F_1 \to 0}$$

The *surprising fact* is that, *in this case*, $ad(\mathcal{D})$ generates the CC of $ad(\mathcal{D}_1)$. Indeed, multiplying by the Lagrange multiplier test function λ and integrating by parts, we obtain the second order operator $\lambda \to \left(-d_{22}\lambda = \mu^1, d_{12}\lambda - ad_1\lambda = \mu^2 \right)$ and thus $-a^2 d_1 \lambda = d_1 \mu^1 + d_2 \mu^2 + a\mu^2$. Substituting, we finally get the only second order CC $d_{12}\mu^1 + d_{22}\mu^2 - ad_1\mu^1 = 0$.

In the differential module framework over the commutative ring $D = K[d_1, d_2]$ of differential operators with coefficients in the trivially differential field $K = \mathbb{Q}(a)$, we have the free resolution:

$$0 \to D \underset{2}{\overset{\mathcal{D}_1}{\to}} D^2 \underset{2}{\overset{\mathcal{D}}{\to}} D \to M \to 0$$

of the differential module M with $ext^1(M) = 0$ and Euler-Poincaré characteristic $rk_D(M) = 1 - 2 + 1 = 0$. We recall that $R = R_\infty = hom_K(M, K)$ is a differential module for the Spencer operator $d : R \to T^* \otimes R : R_{q+1} \to T^* \otimes R_q$ (See [7] [23] for more details). *Only "fingers" could have been used*!

The two following examples will show how the differential extension modules may depend on the Vessiot structure constants.

EXAMPLE 2.3: With $m = n = 2, q = 1, K = \mathbb{Q}\langle \omega \rangle$ and $\omega = (\alpha, \beta)$ with $\alpha \in T^*$, $\beta \in \wedge^2 T^*$, let us consider the Lie operator $\mathcal{D} : T \to \Omega : \xi \to \mathcal{L}(\xi)\omega = \left(A = \mathcal{L}(\xi)\alpha, B = \mathcal{L}(\xi)\beta \right)$. The corresponding first order system:

$$A_i \equiv \alpha_r \partial_i \xi^r + \xi^r \partial_r \alpha_i = 0, B \equiv \beta \partial_r \xi^r + \xi^r \partial_r \beta = 0$$

is involutive whenever $\beta \neq 0$ and $\boxed{d\alpha = c\beta}$ where now d is the exterior derivative and $c = cst$.

We notice that $\bar{\omega}$ and ω provide the same system of Lie equations if and only if $\bar{\alpha} = a\alpha$, $\bar{\beta} = b\beta$ with $a, b \neq 0$ and thus $\bar{c} = \dfrac{a}{b}c$, a result showing that the only critical value of c is $c = 0$.

We have the formally exact differential sequence:

$$\boxed{0 \to \Theta \to T \overset{\mathcal{D}}{\to} T^* \times_X \wedge^2 T^* \overset{\mathcal{D}_1}{\to} \wedge^2 T^* \to 0}$$

or the resolution:

$$\boxed{0 \to D \xrightarrow{\mathcal{D}_1} D^3 \xrightarrow{\mathcal{D}} D^2 \xrightarrow{p} M \to 0}$$

Multiplying (A_1, A_2, B) respectively by (μ^1, μ^2, μ^3), we obtain $ad(\mathcal{D})$ in the form:

$$-\alpha_1 \left(\partial_1 \mu^1 + \partial_2 \mu^2\right) - \beta \left(\partial_1 \mu^3 - c\mu^2\right) = \nu^1,$$
$$-\alpha_2 \left(\partial_1 \mu^1 + \partial_2 \mu^2\right) - \beta \left(\partial_2 \mu^3 + c\mu^1\right) = \nu^2$$

Then, multiplying $\partial_1 A_2 - \partial_2 A_1 - cB$ by λ, we obtain $ad(\mathcal{D}_1)$ as:

$$\partial_2 \lambda = \mu^1, -\partial_1 \lambda = \mu^2, -c\lambda = \mu^3$$

We have therefore to consider the two cases:

- $c = 0$: We have the new CC $\partial_1 \mu^1 + \partial_2 \mu^2 = 0$ and $\mu^3 = 0$. It follows that the torsion module $ext^1(M) \neq 0$ is generated by the residue of $\mu^3 = \nu'$ because $\alpha \neq 0$ and we may thus suppose that $\alpha_1 \neq 0$. As for $ext^2(M)$, this torsion module is just defined by the system $\partial_2 \lambda = 0, \partial_1 \lambda = 0$ for λ and thus $ext^2(M) \neq 0$.

- $c \neq 0$: We must have the new CC:

$$\partial_1 \mu^3 - c\mu^2 = 0, \partial_2 \mu^3 + c\mu^1 = 0 \Rightarrow \partial_1 \mu^1 + \partial_2 \mu^2 = 0$$

It follows that $ext^1(M)$ is now generated by the residue of $\partial_1 \mu^1 + \partial_2 \mu^2 = \nu'$. Finally, $ker(ad(\mathcal{D}_1))$ is defined by $\lambda = 0$ and thus $ext^2(M) = 0$.

Hence, both $ext^1(M)$ and $ext^2(M)$ highly depend on the Vessiot structure constant c.

EXAMPLE 2.4: (*Contact transformations*)

With $m = n = 3, q = 1, K = \mathbb{Q}(x^1, x^2, x^3)$ or simply $\mathbb{Q}(x)$, we may introduce the 1-form $\alpha = dx^1 - x^3 dx^2 \in T^*$ and consider the system of finite Lie equations defined by $j_1(f)^{-1}(\alpha) = \rho(x)\alpha$. Eliminating the factor ρ and linearizing at the q-jet of the identity, we obtain a first order system $R_1 \subset J_1(T)$ made by η^1 and η^2 below which is not even formally integrable (using ∂ instead of d):

$$\begin{cases} \partial_3 \xi^1 - x^3 \partial_3 \xi^2 = \eta^2 \\ \partial_2 \xi^1 - x^3 \partial_2 \xi^2 + x^3 \partial_1 \xi^1 - \left(x^3\right)^2 \partial_1 \xi^2 - \xi^3 = \eta^1 \end{cases} \qquad \begin{array}{|ccc|} \hline 1 & 2 & 3 \\ 1 & 2 & \bullet \\ \hline \end{array}$$

with one equation of class 3 and one equation of class 2. Strikingly, the symbol g_1 is involutive with characters $\alpha_1^1 = 3, \alpha_1^2 = 2, \alpha_1^3 = 2$, a result leading to $dim(g_1) = 3 + 2 + 2 = 7$, $dim(g_2) = 3 + (2 \times 2) + (3 \times 2) = 13$ and more generally $dim(g_{r+1}) = 3 + 2(r+1) + (r+1)(r+2) = r^2 + 5r + 7, \forall r \geq 0$. However, the system R_1 is not formally integrable and thus not involutive. The PP procedure brings $\eta^3 = \partial_2 \eta^2 - \partial_3 \eta^1 + x^3 \partial_1 \eta^2$ below and we get the involutive system $R_1^{(1)} \subset R_1$ of infinitesimal Lie equations having the same solutions, which is already in δ-regular coordinates:

$$\begin{cases} \partial_3 \xi^3 + \partial_2 \xi^2 - \partial_1 \xi^1 + 2x^3 \partial_1 \xi^2 = \eta^3 \\ \partial_3 \xi^1 - x^3 \partial_3 \xi^2 = \eta^2 \\ \partial_2 \xi^1 - x^3 \partial_2 \xi^2 + x^3 \partial_1 \xi^1 - \left(x^3\right)^2 \partial_1 \xi^2 - \xi^3 = \eta^1 \end{cases} \qquad \begin{array}{|ccc|} \hline 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \\ \hline \end{array}$$

with two equations of class 3, one equation of class 2 and thus one CC of order 1 described by the trivially involutive first order operator

$$\mathcal{D}_1 : \eta \to \partial_3 \eta^1 - \partial_2 \eta^2 - x^3 \partial_1 \eta^2 + \eta^3 = \zeta .$$

The link existing between these two approaches is not evident as it highly depends on the fact that $\rho_r\left(R_1^{(1)}\right) = R_{r+1}^{(1)}$, according to the (quite difficult) theorem 2.4.5 of (See [7], p 70) that must be compared to [5] [24]. Setting $\Phi : J_1(T) \to J_1(T)\big/ R_1^{(1)} = F_0$ with $dim(F_0) = 3$ and $\Phi' : J_1(T) \to J_1(T)\big/ R_1 = F_0'$ with $dim(F_0') = 2$, the inclusion $R_1^{(1)} \subset R_1$ induces an epimorphism $F_0 \to F_0' \to 0$ and we have the commutative diagram when $r \geq 2$ and $F_1 = coker(\rho_1(\Phi))$:

$$
\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 \to & g_{r+1} & \to & S_{r+1}T^* \otimes T & \xrightarrow{\sigma_r(\Phi')} & S_r T^* \otimes F_0' & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & R_{r+1} & \to & J_{r+1}(T) & \xrightarrow{\rho_r(\Phi')} & J_r(F_0') & \to & 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & R_r^{(1)} & \to & J_r(T) & \xrightarrow{\rho_{r-1}(\Phi)} & J_{r-1}(F_0) & \to & J_{r-2}(F_1) & \to 0 \\
& \downarrow & & \downarrow & & & \\
& 0 & & 0 & & &
\end{array}
$$

The operator $\mathcal{D}' = \Phi' \circ j_1$ is formally surjective, a snake chase proves that the columns are exact and the bottom sequence is exact as it is just providing the formally exact Janet sequence:

$$0 \to \Theta \to T \xrightarrow[1]{\mathcal{D}} F_0 \xrightarrow[1]{\mathcal{D}_1} F_1 \to 0$$

Hence, it remains to check that the central row is also exact. For this, we notice that:

$$dim\left(R_r^{(1)}\right) = (r+1)(r+2)(r+3)/2 - r(r+1)(r+2)/2 + r(r-1)(r+1)/6$$
$$= (r+1)\left(r^2 + 8r + 18\right)$$

$$dim(R_{r+1}) = (r+2)(r+3)(r+4)/2 - (r+1)(r+2)(r+3)/3$$
$$= (r+2)(r+3)(r+10)/6$$

and check that $dim(R_{r+1}) - dim\left(R_r^{(1)}\right) = dim(g_{r+1})$, a result leading to the sequence:

$$0 \to \Theta \to T \xrightarrow[1]{\mathcal{D}'} F_0' \to 0$$

$$\left(\xi^1, \xi^2, \xi^3\right) \xrightarrow{\mathcal{D}'} \left(\eta^1, \eta^2\right) \to 0$$

Now, it is well known that this *contact* operator $\mathcal{D}$ allowing to define the contact module *M* admits an injective parametrization by one arbitrary potential function ϕ as follows:

$$\mathcal{D}_{-1} : \phi \rightarrow \left(-x^3 \partial_3 \phi + \phi = \xi^1, -\partial_3 \phi = \xi^2, \partial_2 \phi + x^3 \partial_1 \phi = \xi^3 \right) \Rightarrow \xi^1 - x^3 \xi^2 = \phi \left(\textit{lift} \right)$$

and thus $M \simeq D$. We have obtained the following locally exact differential sequence ([8]):

$$0 \rightarrow \phi \overset{\mathcal{D}_{-1}}{\rightarrow} \left(\xi^1, \xi^2, \xi^3 \right) \overset{\mathcal{D}}{\rightarrow} \left(\eta^1, \eta^2, \eta^3 \right) \overset{\mathcal{D}_1}{\rightarrow} \zeta \rightarrow 0$$

which is *not* a Janet sequence because the first operator is *not* involutive.

We obtain therefore a resolution of M ([22]):

$$\boxed{0 \rightarrow D \overset{\mathcal{D}_1}{\rightarrow} D^3 \overset{\mathcal{D}}{\rightarrow} D^3 \rightarrow M \rightarrow 0}$$

but also a resolution which is *not* strictly exact because R_1 is not even formally integrable ([22]):

$$0 \rightarrow D^2 \overset{\mathcal{D}'}{\rightarrow} D^3 \rightarrow M \rightarrow 0 \quad \Rightarrow \quad M \oplus D^2 \simeq D^3$$

As M is free and thus projective, the respective adjoint sequences are also exact. Applying $hom_D \left(\bullet, D \right)$, we obtain a split exact sequence of free *right* (care) differential modules. Passing from right to left differential modules by using the *side changing procedure* $N_D \rightarrow N =_D N = hom_K \left(\wedge^n T^*, N_D \right)$, it follows that the adjoint sequence is exact too, though not strictly exact. As such a result does not depend on the differential sequence used, according to a (difficult) general theorem in homological algebra, we shall prove it directly through a (delicate) technical explicit computation for the first of the two previous ones and let the reader prove it similarly for the other.

For this, let us multiply ζ by a test function λ and integrate by parts in order to obtain:

$$ad \left(\mathcal{D}_1 \right) : \lambda \rightarrow \begin{cases} -\partial_3 \lambda = \mu^1 \\ \partial_2 \lambda + x^3 \partial_1 \lambda = \mu^2 \\ \lambda = \mu^3 \end{cases}$$

This operator is injective but not involutive as the corresponding system $R_1 \subset J_1 \left(\lambda \right)$ is far from being even formally integrable. In a coherent way with the standard second Spencer sequence ([1] [2]), the corresponding involutive operator is j_1 as follows, with 6 first order CC for itself but first *and* second order for μ after substitution, that is a second order system which is not FI.

$$j_1 : \lambda \rightarrow \begin{cases} \partial_3 \lambda = -\mu^1 \\ \partial_2 \lambda = \mu^2 - x^3 \partial_1 \mu^3 \\ \partial_1 \lambda = \partial_1 \mu^3 \\ \lambda = \mu^3 \end{cases} \qquad \begin{array}{|ccc|} \hline 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \hline \end{array}$$

- 1 *identity to zero* $\partial_1 \mu^3 - \partial_1 \mu^3 = 0$.
- 2 first order CC, namely $-v^3 \equiv \partial_3 \mu^3 + \mu^1 = 0$,

  $-\left(v^2 + x^3 v^1 \right) \equiv \partial_2 \mu^3 + x^3 \partial_1 \mu^3 - \mu^2 = 0$ which are differentially independent because, when μ^3 is given arbitrarily, the first is providing μ^1 while the second is providing μ^2.

- 2 second order CC that are just prolongations of the two previous first order ones.

- 1 second order CC $\partial_3 \mu^2 - x^3 \partial_{13} \mu^3 - \partial_1 \mu^3 + \partial_2 \mu^1 = 0$.

Using the previous second order CC $\partial_{13} \mu^3 + \partial_1 \mu^1 = 0$, we finally get the *new* first order CC $-v^1 \equiv \partial_3 \mu^2 + x^3 \partial_1 \mu^1 + \partial_2 \mu^1 - \partial_1 \mu^3 = 0$ which is a differential consequence of the two previous first order CC because we have (*by chance*!) the involutive system:

$$\begin{cases} \partial_3 \mu^3 + \mu^1 = -v^3 \\ \partial_3 \mu^2 + x^3 \partial_1 \mu^1 + \partial_2 \mu^1 - \partial_1 \mu^3 = -v^1 \\ \partial_2 \mu^3 + x^3 \partial_1 \mu^3 - \mu^2 = -(v^2 + x^3 v^1) \end{cases} \quad \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \end{matrix}$$

providing the only first order CC:

$$ad(\mathcal{D}_{-1}) : (v^1, v^2, v^3) \rightarrow x^3 \partial_3 v^1 + 2v^1 + \partial_3 v^2 - \partial_2 v^3 - x^3 \partial_1 v^3 = \theta$$

As a byproduct, we can find the formally exact differential sequence:

$$0 \leftarrow \theta \overset{ad(\mathcal{D}_{-1})}{\leftarrow} (v^1, v^2, v^3) \overset{ad(\mathcal{D})}{\leftarrow} (\mu^1, \mu^2, \mu^3) \overset{ad(\mathcal{D}_1)}{\leftarrow} \lambda \leftarrow 0$$

which is *exactly* the adjoint of the first differential sequence we have provided.

Coming back to the Vessiot structure equations, we notice that α is not invariant by the contact Lie pseudogroup and cannot be considered as an associated geometric object. We have shown in ([2], pp. 684-691) that the corresponding geometric object is a 1-form density ω leading to the system of infinitesimal Lie equations in Medolaghi form:

$$\Omega_i \equiv (\mathcal{L}(\xi)\omega)_i \equiv \omega_r \partial_i \xi^r - \frac{1}{2} \omega_i \partial_r \xi^r + \xi^r \partial_r \omega_i = 0$$

that may be also written as:

$$\partial_i (\omega_r \xi^r) - \frac{1}{2} \omega_i \partial_r \xi^r + \xi^r (\partial_r \omega_i - \partial_i \omega_r) = \Omega_i$$

and to the *only* Vessiot structure equation, *still not known today*:

$$\boxed{\omega_1 (\partial_2 \omega_3 - \partial_3 \omega_2) + \omega_2 (\partial_3 \omega_1 - \partial_1 \omega_3) + \omega_3 (\partial_1 \omega_2 - \partial_2 \omega_1) = c}$$

with the *only* structure constant *c*. In the present contact situation, we may choose $\omega = (1, -x^3, 0)$ and get $c = 1$ but we may also choose $\omega = (1, 0, 0)$ and get $c = 0$, these two choices both bringing an involutive system. Let us prove that the situation becomes completely different with the new system:

$$-2\Omega_1 \equiv \partial_3 \xi^3 + \partial_2 \xi^2 - \partial_1 \xi^1 = 0, \quad \Omega_2 \equiv \partial_2 \xi^1 = 0, \quad \Omega_3 \equiv \partial_3 \xi^1 = 0$$

having the only CC $d_2 \Omega_3 - d_3 \Omega_2 = 0$.

Multilying the three previous equations by the three test functions μ, the only CC by the test function λ and integrating by parts, we get the adjoint operators:

$$0 = \mu^1, \quad \partial_3 \lambda = \mu^2, \quad -\partial_2 \lambda = \mu^3$$

$$\partial_1 \mu^1 - \partial_2 \mu^2 - \partial_3 \mu^3 = v^1, \quad -\partial_2 \mu^1 = v^2, \quad -\partial_3 \mu^1 = v^3$$

It follows that $0 \neq D\xi^1 = t(M) \subset M$ with a strict inclusion and $ext^1(M) \neq 0$. Similarly, $ker(ad(\mathcal{D}_1))$ is defined by $\partial_2 \lambda = 0, \partial_3 \lambda = 0$ and thus $ext^2(M) \neq 0$.

Our problem will be now to construct and compare the differential sequences:

$$\phi \xrightarrow{\mathcal{D}_{-1}} \xi \xrightarrow{\mathcal{D}} \Omega \xrightarrow{\mathcal{D}_1} C$$

$$\theta \xleftarrow{ad(\mathcal{D}_{-1})} v \xleftarrow{ad(\mathcal{D})} \mu \xleftarrow{ad(\mathcal{D}_1)} \lambda$$

For this, linearizing the only Vessiot structure equation, we get the CC operator $\mathcal{D}_1$ and the corresponding system $\mathcal{D}_1 \Omega = C$ in the form:

$$\omega_1 (\partial_2 \Omega_3 - \partial_3 \Omega_2) + \omega_2 (\partial_3 \Omega_1 - \partial_1 \Omega_3) + \omega_3 (\partial_1 \Omega_2 - \partial_2 \Omega_1)$$
$$+ (\partial_2 \omega_3 - \partial_3 \omega_2) \Omega_1 + (\partial_3 \omega_1 - \partial_1 \omega_3) \Omega_2 + (\partial_1 \omega_2 - \partial_2 \omega_1) \Omega_3 = C$$

Multiplying on the left by the test function λ and integrating by parts, we get the operator $ad(\mathcal{D}_1)$ in the form:

$$\begin{cases} \Omega_1 \rightarrow \omega_3 \partial_2 \lambda - \omega_2 \partial_3 \lambda + 2(\partial_2 \omega_3 - \partial_3 \omega_2) \lambda = \mu^1 \\ \Omega_2 \rightarrow \omega_1 \partial_3 \lambda - \omega_3 \partial_1 \lambda + 2(\partial_3 \omega_1 - \partial_1 \omega_3) \lambda = \mu^2 \\ \Omega_3 \rightarrow \omega_2 \partial_1 \lambda - \omega_1 \partial_2 \lambda + 2(\partial_1 \omega_2 - \partial_2 \omega_1) \lambda = \mu^3 \end{cases}$$

We obtain the *crucial formula* $2c\lambda = \omega_i \mu^i$ showing how the previous sequences are *essentially* depending on the Vessiot structure constant c. Indeed, if $c \neq 0$, then $\mu = 0 \Rightarrow \lambda = 0$ and the operator $ad(\mathcal{D}_1)$ is injective. This is the case when $\omega = (1, -x^3, 0) \Rightarrow c = 1 \Rightarrow \lambda = 0$. On the contrary, if $c = 0$, then the operator $ad(\mathcal{D}_1)$ may not be injective as can be seen by choosing $\omega = (1, 0, 0)$. Indeed, in this case we get a kernel defined by $\partial_3 \lambda = 0, \partial_2 \lambda = 0$.

Finally, in order to exhibit the generating CC of $ad(\mathcal{D}_1)$ when $c \neq 0$, we just need to substitute $\lambda = (1/2c) \omega_i \mu^i$ in the previous equations $ad(\mathcal{D}_1)\lambda = \mu$. On the other side, multiplying the equations $\mathcal{D}\xi = \Omega$ by test functions μ^i and integrating by parts, we get $ad(\mathcal{D})\mu = v$ in the form:

$$\begin{cases} \xi^1 \rightarrow -\partial_i (\omega_1 \mu^i) + \frac{1}{2}\partial_1 (\omega_i \mu^i) + (\partial_1 \omega_i) \mu^i = v^1 \\ \xi^1 \rightarrow -\partial_i (\omega_2 \mu^i) + \frac{1}{2}\partial_2 (\omega_i \mu^i) + (\partial_2 \omega_i) \mu^i = v^2 \\ \xi^1 \rightarrow -\partial_i (\omega_3 \mu^i) + \frac{1}{2}\partial_3 (\omega_i \mu^i) + (\partial_3 \omega_i) \mu^i = v^3 \end{cases}$$

We let the reader check, as a tricky exercise, that we have indeed $ad(\mathcal{D}) \circ ad(\mathcal{D}_1) \equiv 0$ and it remains to prove that $ad(\mathcal{D})$ generates the CC of $ad(\mathcal{D}_1)$. It is in such a situation that we can measure the usefulness of homological algebra and we only prove this result directly when $\omega = (1, -x^3, 0) \Rightarrow c = 1$. In this case, the kernel of $ad(\mathcal{D})$ is easily seen to be defined by:

$$\begin{cases} \frac{1}{2}\partial_1 \mu^1 + \partial_2 \mu^2 + \partial_3 \mu^3 + \frac{1}{2}x^3 \partial_1 \mu^2 = 0 \\ -(x^3)^2 \partial_1 \mu^2 + x^3 \partial_1 \mu^1 + \partial_2 \mu^1 - x^3 \partial_2 \mu^2 + 2\mu^3 = 0 \\ \partial_3 \mu^1 - x^3 \partial_3 \mu^2 - 3\mu^2 = 0 \end{cases}$$

while, setting now $2\lambda = \mu^1 - x^3\mu^2$ and substituting, the CC of $ad(\mathcal{D}_1)$ seem to be only defined by the two PD equations:

$$\begin{cases} \partial_3\mu^1 - x^3\partial_3\mu^2 - 3\mu^2 = 0 \\ \partial_2\mu^1 - x^3\partial_2\mu^2 - (x^3)^2\partial_1\mu^2 + x^3\partial_1\mu^1 + 2\mu^3 = 0 \end{cases}$$

The strange fact is that *such a system is not formally integrable* and one has to differentiate the second PD equation with respect to x^3 and substract the first PD equation differentiated with respect to x^2 in order to get the additional PD equation:

$$\partial_3\mu^3 + \partial_2\mu^2 + \frac{1}{2}x^3\partial_1\mu^2 + \frac{1}{2}\partial_1\mu^1 = 0$$

and find an isomorphic involutive system, a result showing that the differential sequence and its formal adjoint are both formally exact though *not* strictly exact. We conclude this example with the following striking result ([arXiv: 1803.09610]):

THEOREM 2.5: The *contact differential sequence* and its formal adjoint are both split long exact sequences of free and thus projective modules, if and only if $c \neq 0$.

Proof: As we have just proved that $ad(\mathcal{D})$ was generating the CC of $ad(\mathcal{D}_1)$, we may look for the CC of $ad(\mathcal{D})$ in order to recover the parametrization of $\mathcal{D}$ given at the beginning of this example.

First of all, as proved in [2], two geometric objects ω and $\bar{\omega}$ provide the same system $R_1 \subset J_1(T)$ if and only if $\bar{\omega} = a\omega$ for a constant parameter $a = cst$, a result providing $\bar{c} = a^2 c$. Accordingly, the only possible critical value of the Vessiot structure constant is $c = 0$.

Now, as already noticed, we have $\omega_i\mu^i = 2c\lambda$ and $ad(\mathcal{D}_1)$ is injective if and only if $c \neq 0$. It follows that the differential module defined by $\mathcal{D}_1$ is projective and the sequences split if we are able to construct $\mathcal{D}_{-1}$ and to prove that it is an injective operator. For this, changing slightly our previous notations, we notice that the symbol map of $ad(\mathcal{D})$ is (using d instead of ∂)

$$-\omega_i d_r\mu^r + \frac{1}{2}\omega_r d_i\mu^r + \cdots = v_i$$

Using standard notations of classical geometry, we may rewrite it as:

$$-A\vec{\omega} + \vec{\nabla}\cdot B + \cdots = \vec{v} \Rightarrow \vec{\omega}\wedge(\vec{\nabla}.A) + \cdots = \vec{\nabla}\wedge\vec{v} \Rightarrow \vec{\omega}\cdot(\vec{\nabla}\wedge\vec{v}) + \cdots = 0$$

Hence, after one prolongation on the symbol level, we get the only CC:

$$\omega_1(d_3v_2 - d_2v_3) + \omega_2(d_1v_3 - d_3v_1) + \omega_3(d_2v_1 - d_1v_2) + \cdots = 0$$

After substitution and a tedious computation, one finally obtains:

$$\begin{aligned} &\omega_1(d_3v_2 - d_2v_3) + \omega_2(d_1v_3 - d_3v_1) + \omega_3(d_2v_1 - d_1v_2) \\ &+ 2(\partial_2\omega_3 - \partial_3\omega_2)v_1 + 2(\partial_3\omega_1 - \partial_1\omega_3)v_2 + 2(\partial_1\omega_2 - \partial_2\omega_1)v_3 = \theta \end{aligned}$$

We may thus obtain $\mathcal{D}_{-1}$ in the form:

$$\begin{cases} \omega_2\partial_3\phi - \omega_3\partial_2\phi + (\partial_2\omega_3 - \partial_3\omega_2)\phi = \xi^1 \\ \omega_3\partial_1\phi - \omega_1\partial_3\phi + (\partial_3\omega_1 - \partial_1\omega_3)\phi = \xi^2 \\ \omega_1\partial_2\phi - \omega_2\partial_1\phi + (\partial_1\omega_2 - \partial_2\omega_1)\phi = \xi^3 \end{cases}$$

and this operator is injective whenever $c \neq 0$ because we have the lift $\omega_r \xi^r = c\phi$.

The situation is similar in arbitrary dimension $n = 2p+1$ with 1-form $\alpha = dx^n - \sum_{\alpha=1}^{p} x^{\bar{\alpha}} dx^\alpha$ as we have again one Vessiot structure constant c and the injective parametrization:

$$\phi - x^{\bar{\beta}}\frac{\partial\phi}{\partial x^\beta} = \xi^n, \quad -\frac{\partial\phi}{\partial x^{\bar{\alpha}}} = \xi^\alpha, \quad \frac{\partial\phi}{\partial x^\alpha} + x^{\bar{\alpha}}\frac{\partial\phi}{\partial x^n} = \xi^{\bar{\alpha}} \quad \Rightarrow \quad \phi = \alpha(\xi)$$

We have the locally exact split differential sequence:

$$0 \to \wedge^0 T^* \xrightarrow{\mathcal{D}_{-1}} T \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} \cdots \xrightarrow{\mathcal{D}_{n-2}} F_{n-2} \to 0$$

with $dim(F_r) = n!/(r+2)!(n-r-2)!$ and we refer again to ([2]) because, when $n \geq 5$, we have to use a 2-contravariant skewsymmetric tensor density.

It follows that contact geometry must be entirely revisited in the light of these new results. $\square$

EXAMPLE 2.6: (*Macaulay examples revisited*).

With $n = 3, m = 1, q = 2, K = \mathbb{Q}, D = K[d_1, d_2, d_3]$, let us consider the linear homogeneous second order operator $\mathcal{D} : \xi \to (d_{33}\xi = \eta^1, d_{23}\xi - d_{11}\xi = \eta^2, d_{22}\xi = \eta^3)$ with corresponding system $R_2 \subset J_2(E)$ which is trivially formally integrable (FI). It is easy to check that $dim(g_2) = 3$, $dim(g_3) = 1$, $g_4 = 0$. Hence, R_2 and thus R_3 are FI but not involutive. Then R_4 is trivially involutive with $dim(R_{3+r}) = 2^n = 8, \forall r \geq 0$ and a full basis of parametric jets is $(\xi, \xi_1, \xi_2, \xi_3, \xi_{11}, \xi_{12}, \xi_{13}, \xi_{111})$. We have proved in ([6]) that there are three second order CC with only one second order CC. The corresponding formally exact differential sequence is:

$$0 \to \Theta \to \xi \underset{2}{\xrightarrow{\mathcal{D}}} \eta \underset{2}{\xrightarrow{\mathcal{D}_1}} \zeta \underset{2}{\xrightarrow{\mathcal{D}_2}} \tau \to 0$$

and the minimum free differential resolution of the corresponding differential module M is:

$$0 \to D \xrightarrow{\mathcal{D}_2} D^3 \xrightarrow{\mathcal{D}_1} D^3 \xrightarrow{\mathcal{D}} D \xrightarrow{p} M \to 0$$

where p is the canonical projection. The situation is *much more tricky* if we consider now the new third order differential operator $\mathcal{D}' = \Phi \circ j_3 : E \xrightarrow{j_3} J_3(E) \xrightarrow{\Phi} J_3(E)/R_3$ with $R_3 = \rho_1(R_2)$. Indeed, it is easy to check that the sequence:

$$0 \to \wedge^2 T^* \otimes g_3 \xrightarrow{\delta} \wedge^3 T^* \otimes g_2 \to 0$$

is exact because both bundles have the same dimension equal to 3. It follows that g_3 is 3-acyclic and it follows that the 21 CC are described by an operator of order 1. However, the corresponding system has a symbol which is not 2-acyclic. We have proved in ([6]) that the next CC are of order 2, and so on. By chance, it

happens that the next operator $\mathcal{D}_2'$ is involutive and we may start a Janet sequence with first order involutive operators $\mathcal{D}_3', \mathcal{D}_4', \mathcal{D}_6'$ in the differential sequence:

$$0 \to \Theta \to 1 \underset{3}{\overset{\mathcal{D}'}{\to}} 12 \underset{1}{\overset{D_1'}{\to}} 21 \underset{2}{\overset{\mathcal{D}_2'}{\to}} 46 \underset{1}{\overset{\mathcal{D}_3'}{\to}} 72 \underset{1}{\overset{\mathcal{D}_4'}{\to}} 48 \underset{1}{\overset{\mathcal{D}_5'}{\to}} 12 \to 0$$

in which we have specified the dimensions of the bundles and the orders of the successive operators. We finally notice again that

$rk_D(M) = 1 - 12 + 21 - 46 + 72 - 48 + 12 = 0$.

We invite the reader to consider the similar example when $n = 2$. Indeed, if we consider the second order system $d_{22}\xi = 0$, $d_{12}\xi - d_{11}\xi = 0$, we have $dim(g_2) = 1$, $g_3 = 0$ and $dim(R_2) = dim(R_3) = 2^n = 4$ with the basis of parametric jets $(\xi, \xi_1, \xi_2, \xi_{11})$. In this case, R_2 is FI but not involutive while R_3 is trivially involutive with Janet sequence $0 \to \Theta \to 1 \underset{3}{\to} 6 \underset{1}{\to} 7 \underset{1}{\to} 2 \to 0$ and $rk_D(M) = 1 - 6 + 7 - 2 = 0$.

3. Kerr Metric

Let us recall a few facts from Riemannian geometry. A metric $\omega = (\omega_{ij}) \in S_2 T^*$ with $det(\omega) \neq 0$ is providing the Christoffel symbols $\gamma = (\gamma_{ij}^k)$ as geometric objects according to the forgotten work of E. Vessiot in 1903 ([1] [18] [19]). The Riemann tensor $\rho_{l,ij}^k \in \wedge^2 T^* \otimes g_1 \subset \wedge^2 T^* \otimes T^* \otimes T$ is a section of the Spencer δ-cohomology vector bundle $H^2(g_1)$ because it is also killed by the Spencer surjective map $\delta : \wedge^2 T^* \otimes g_1 \to \wedge^3 T^* \otimes T^* \otimes T$ and has thus $n^2(n-1)^2/4 - n^2(n-1)(n-2)/6 = n^2(n^2-1)/12$ components because $dim(g_1) = n(n-1)/2$ and $g_2 = 0$. Needless to say that these definitions are far from the ones that can be found in any place (books or papers) dealing with GR ([7] [8]).

Now, the linearization $\Omega = (\Omega_{ij}) \in S_2 T^*$ of ω induces a linearization $\Gamma = \Gamma_{ij}^k \in S_2 T^* \otimes T$ and a linearization $R = (R_{l,ij}^k) \in \wedge^2 T^* \otimes T$. With ∂ for objects and d for their linearization, we have:

$$\rho_{l,ij}^k = \partial_i \gamma_{lj}^k - \partial_j \gamma_{li}^k + \gamma_{lj}^r \gamma_{ri}^k - \gamma_{li}^r \gamma_{rj}^k$$

and thus, because $\Gamma \in S_2 T^* \otimes T$ is a tensor::

$$\begin{aligned} R_{l,ij}^k &= d_i \Gamma_{lj}^k - d_j \Gamma_{li}^k + \gamma_{lj}^r \Gamma_{ri}^k - \gamma_{li}^r \Gamma_{rj}^k + \gamma_{ri}^k \Gamma_{lj}^r - \gamma_{rj}^k \Gamma_{li}^r \\ &= \left(d_i \Gamma_{lj}^k - \gamma_{li}^r \Gamma_{rj}^k + \gamma_{ri}^k \Gamma_{lj}^r \right) - \left(d_j \Gamma_{li}^k - \gamma_{lj}^r \Gamma_{ri}^k + \gamma_{rj}^k \Gamma_{li}^r \right) \\ &= \nabla_i \Gamma_{lj}^k - \nabla_j \Gamma_{li}^k \end{aligned}$$

by introducing the covariant derivative ∇. We recall that $\nabla_r \omega_{ij} = 0, \forall r, i, j$ or, equivalently, that $(id, -\gamma) : \xi \in T \to (\xi^k, \xi_i^k = -\gamma_{ir}^k \xi^r) \in R_1$ is a R_1-connection with $\omega_{sj}\gamma_{ir}^s + \omega_{is}\gamma_{jr}^s = \partial_r \omega_{ij}$, a result allowing to move down the index k in the previous formulas (See [17] [18] [22] for more details).

We may thus take into account the Bianchi identities implied by the cyclic

sums on (ijr)

$$\beta_{kl,ijr} \equiv \nabla_r \rho_{kl,ij} + \nabla_i \rho_{kl,jr} + \nabla_j \rho_{kl,ri} = 0 \quad \Leftrightarrow \quad \beta \equiv \underset{cycl}{\Sigma}\left(\partial \rho - \gamma \rho\right) = 0$$

and their respective linearizations $B_{kl,ijr} = 0$ as described below. Both β and B are sections of the Spencer cohomogy vector bundle $H^3(g_1)$ defined by the short exact sequence:

$$0 \to H^3(g_1) \to \wedge^3 T^* \otimes g_1 \overset{\delta}{\to} \wedge^4 T^* \otimes T \to 0$$

$$\begin{aligned} dim\left(H^3(g_1)\right) \\ = \left(n(n-1)(n-2)/6\right)\left(n(n-1)/2\right) - \left(n(n-1)(n-2)(n-3)/24\right)n \\ = n^2\left(n^2-1\right)(n-2)/24 \end{aligned}$$

that is 20 when $n = 4$ because $dim(g_1) = n(n-1)/2$ and the coboundary bundle is $B^3(g_1) = 0$.

Such results cannot be even imagined by somebody not aware of the δ-acyclicity ([9] [10] [11]).

We have the linearized cyclic sums of covariant derivatives both with their respective symbolic descriptions, not to be confused with the non-linear corresponding ones:

$$\begin{aligned} B_{kl,rij} \equiv \nabla_r R_{kl,ij} + \nabla_i R_{kl,jr} + \nabla_j R_{kl,ri} = 0 \quad mod\,(\Gamma) \\ \Leftrightarrow \underset{cycl}{\Sigma}\left(dR - \gamma R - \rho\Gamma\right) = 0 \\ \Leftrightarrow B \equiv \underset{cycl}{\Sigma}\left(\nabla R\right) = \underset{cycl}{\Sigma}\left(\rho\Gamma\right) \end{aligned}$$

In order to recapitulate these new concepts obtained after one, two or three prolongations, we have successively $\omega \to \gamma \to \rho \to \beta$ and the respective linearizations $\Omega \to \Gamma \to R \to B$.

The purpose of this section will be to consider the Killing operator $\mathcal{D}: T \to S_2 T^* : \xi \to \mathcal{L}(\xi)\omega$ and the corresponding Killing system $R_1 \subset J_1(T)$ when ω is the Kerr metric. In particular, we shall exhibit the successive inclusions $R_1^{(3)} \subset R_1^{(2)} \subset R_1^{(1)} = R_1 \subset J_1(T)$ with dimensions $2 < 4 < 10 = 10 < 20$ by means of elementary combinatorics and diagram chasing exactly like in the previous motivating examples.

We now write the Kerr metric in Boyer-Lindquist coordinates:

$$ds^2 = \frac{\rho^2 - mr}{\rho^2}dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \frac{2amr\sin^2(\theta)}{\rho^2}dtd\phi$$
$$- \left(r^2 + a^2 + \frac{ma^2 r\sin^2(\theta)}{\rho^2}\right)\sin^2(\theta)d\phi^2$$

where $\Delta = r^2 - mr + a^2, \rho^2 = r^2 + a^2\cos^2(\theta)$ as usual and we recover the Schwarschild metric when $a = 0$. We notice that t or ϕ do not appear in the coefficients of the metric. We shall simplify the coordinate system by using the so-called "*rational polynomial*" coefficients as follows:

$$\left(x^0 = t, x^1 = r, x^2 = c = \cos(\theta), x^3 = \phi\right) \Rightarrow \mathrm{d}x^2 = -\sin(\theta)\mathrm{d}\theta$$

$$\Rightarrow \left(\mathrm{d}x^2\right)^2 = \left(1 - c^2\right)\mathrm{d}\theta^2$$

We obtain over the differential field $K = \mathbb{Q}(a, m)(t, r, c, \phi) = \mathbb{Q}(a, m)(x)$:

$$\mathrm{d}s^2 = \frac{\rho^2 - mr}{\rho^2}\mathrm{d}t^2 - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \frac{\rho^2}{1 - c^2}\mathrm{d}c^2 - \frac{2amr\left(1 - c^2\right)}{\rho^2}\mathrm{d}t\mathrm{d}\phi$$

$$-\left(1 - c^2\right)\left(r^2 + a^2 + \frac{ma^2 r\left(1 - c^2\right)}{\rho^2}\right)\mathrm{d}\phi^2$$

with now $\Delta = r^2 - mr + a^2$ and $\rho^2 = r^2 + a^2 c^2$. For a later use, it is also possible to set $\omega_{33} = -\left(1 - c^2\right)\left(\left(r^2 + a^2\right)^2 - a^2\left(1 - c^2\right)\left(a^2 - mr + r^2\right)\right)\Big/\left(r^2 + a^2 c^2\right)$ and $det(\omega) = -\left(r^2 + a^2 c^2\right)^2$.

Framing the leading derivatives and setting $\xi\partial = \xi^i\partial_i$, we obtain:

$$R_1 \subset J_1(T) \begin{cases} \Omega_{33} \equiv 2\left(\omega_{33}\boxed{\xi_3^3} + \omega_{03}\xi_3^0\right) + \xi\partial\omega_{33} = 0 \\[4pt] \Omega_{23} \equiv \omega_{33}\boxed{\xi_2^3} + \omega_{03}\xi_2^0 + \omega_{22}\xi_3^2 = 0 \\[4pt] \Omega_{22} \equiv 2\omega_{22}\boxed{\xi_2^2} + \xi\partial\omega_{22} = 0 \\[4pt] \Omega_{13} \equiv \omega_{33}\boxed{\xi_1^3} + \omega_{03}\xi_1^0 + \omega_{11}\xi_3^1 = 0 \\[4pt] \Omega_{12} \equiv \omega_{22}\boxed{\xi_1^2} + \omega_{11}\xi_2^1 = 0 \\[4pt] \Omega_{11} \equiv 2\omega_{11}\boxed{\xi_1^1} + \xi\partial\omega_{11} = 0 \\[4pt] \Omega_{03} \equiv \omega_{33}\boxed{\xi_0^3} + \omega_{03}\left(\xi_0^0 + \xi_3^3\right) + \omega_{00}\xi_3^0 + \xi\partial\omega_{03} = 0 \\[4pt] \Omega_{02} \equiv \omega_{22}\boxed{\xi_0^2} + \omega_{00}\xi_2^0 + \omega_{03}\xi_2^3 = 0 \\[4pt] \Omega_{01} \equiv \omega_{11}\boxed{\xi_0^1} + \omega_{00}\xi_1^0 + \omega_{03}\xi_1^3 = 0 \\[4pt] \Omega_{00} \equiv 2\left(\omega_{00}\boxed{\xi_0^0} + \omega_{03}\xi_0^3\right) + \xi\partial\omega_{00} = 0 \end{cases}$$

Looking at the symbol $g_1 \subset T^* \otimes T$, elementary linear combinatorics allow to prove ([10] [11]):

$$\omega_{33}\boxed{\xi_3^3} + \omega_{03}\xi_3^0 = 0 \ mod(\xi),$$

$$\omega_{33}\boxed{\xi_0^3} + \omega_{00}\xi_3^0 = 0 \ mod(\xi),$$

$$\omega_{33}\boxed{\xi_0^0} - \omega_{03}\xi_3^0 = 0 \ mod(\xi)$$

Then, multiplying Ω_{22} by ω_{11}, Ω_{11} by ω_{22} and adding, we finally obtain:

$$2\left(\omega_{11}\omega_{22}\right)\left(\xi_1^1 + \xi_2^2\right) + \xi\partial\left(\omega_{11}\omega_{22}\right) = 0$$

Using the rational coefficients in the differential field $K = \mathbb{Q}(m, a)(r, c)$, the nonzero components of the corresponding Riemann tensor can be found in textbooks. Among them, we have:

$$\left\{\begin{array}{l}
\rho_{03,03} = \dfrac{mr\left(1-c^2\right)\left(r^2 - mr + a^2\right)\left(r^2 - 3a^2c^2\right)}{2\left(r^2 + a^2c^2\right)^3} \\[3ex]
\rho_{12,12} = -\dfrac{mr\left(r^2 - 3a^2c^2\right)}{2\left(1-c^2\right)\left(r^2 + a^2c^2\right)\left(r^2 - mr + a^2\right)} \\[3ex]
\rho_{13,13} = \dfrac{-\left(1-c^2\right)mr\left(r^4 - 2a^2c^2r^2 + 4a^2r^2 - 2a^4c^2 + 3a^4 - 2a^2mr\left(1-c^2\right)\right)\left(r^2 - 3a^2c^2\right)}{2\left(r^2 + a^2c^2\right)^3\left(r^2 - mr + a^2\right)} \\[3ex]
\rho_{23,23} = \dfrac{mr\left(2r^4 - a^2c^2r^2 + 5a^2r^2 - a^4c^2 + 3a^4 - a^2mr\left(1-c^2\right)\right)\left(r^2 - 3a^2c^2\right)}{2\left(r^2 + a^2c^2\right)^3} \\[3ex]
\rho_{01,23} = \dfrac{amc\left(2r^2 - a^2c^2 + 3a^2\right)\left(3r^2 - a^2c^2\right)}{2\left(r^2 + a^2c^2\right)^3} \\[3ex]
\rho_{02,31} = -\dfrac{amc\left(r^2 - 2a^2c^2 + 3a^2\right)\left(3r^2 - a^2c^2\right)}{2\left(r^2 + a^2c^2\right)^3} \\[3ex]
\rho_{03,12} = -\dfrac{amc\left(3r^2 - a^2c^2\right)}{2\left(r^2 + a^2c^2\right)^2}
\end{array}\right.$$

As they will be used in the sequel, we also provide a few components vanishing for the S-metric:

$$\left\{\begin{array}{l}
\rho_{02,10} = \dfrac{3a^2mc\left(3r^2 - a^2c^2\right)}{2\left(r^2 + a^2c^2\right)^3} \\[3ex]
\rho_{02,32} = \dfrac{amr\left(3r^2 - mr + 3a^2\right)\left(r^2 - 3a^2c^2\right)}{2\left(r^2 + a^2c^2\right)^3} \\[3ex]
\rho_{13,23} = -\dfrac{3a^2mc\left(1-c^2\right)\left(r^2 + a^2\right)\left(3r^2 - a^2c^2\right)}{2\left(r^2 + a^2c^2\right)^3} \\[3ex]
\rho_{01,13} = \dfrac{amr\left(1-c^2\right)\left(3r^2 + 3a^2 - 2mr\right)\left(r^2 - 3a^2c^2\right)}{2\left(r^2 + a^2c^2\right)^3\left(r^2 - mr + a^2\right)}
\end{array}\right.$$

We finally exhibit the 8 vanishing components of the Riemann tensor:

$$\rho_{01,03} = 0, \rho_{01,12} = 0, \rho_{02,03} = 0, \rho_{02,12} = 0, \rho_{03,13} = 0, \rho_{03,23} = 0, \rho_{12,13} = 0, \rho_{12,23} = 0$$

Introducing the *formal Lie derivative* $R = L\left(\xi_1\right)\rho$ and using the fact that $\rho \in \wedge^2 T^* \otimes T^* \otimes T^*$ is a tensor, the system $R_1^{(2)} \subset R_1^{(1)} = R_1 \subset J_1\left(T\right)$ contains the new equations:

$$R_{kl,ij} \equiv \rho_{rl,ij}\xi_k^r + \rho_{kr,ij}\xi_l^r + \rho_{kl,rj}\xi_i^r + \rho_{kl,ir}\xi_j^r + \xi^r \partial_r \rho_{kl,ij} = 0$$

but we have to take into account the ten identities to zero obtained because $\rho_{ij} = 0$, namely:

$$R_{ij} \equiv \rho_{rj}\xi_i^r + \rho_{ir}\xi_J^r + \xi^r \partial_r \rho_{ij} = 0$$

We check the identity $\rho_{01,23} + \rho_{02,31} + \rho_{03,12} = 0 \Rightarrow R_{01,23} + R_{02,31} + R_{03,12} = 0$ and we have:

$$\begin{cases} R_{03,03} \equiv 2\rho_{03,03}\left(\xi_0^0 + \xi_3^3\right) + \xi\partial\rho_{03,03} = 0 \\ R_{12,12} \equiv 2\rho_{12,12}\left(\xi_1^1 + \xi_2^2\right) + \xi\partial\rho_{12,12} = 0 \\ R_{01,23} \equiv \rho_{01,23}\left(\xi_0^0 + \xi_1^1 + \xi_2^2 + \xi_3^3\right) + \xi\partial\rho_{01,23} = 0 \end{cases}$$

We obtain therefore $\xi\partial\left(\rho_{12,12}/\left(\omega_{11}\omega_{22}\right)\right) = 0$ but we have also $\xi\partial\left(\rho_{03,03}\rho_{12,12}/det(\omega)\right) = 0$.

The following invariants are obtained successively in a coherent way:

$$\omega_{11}\omega_{22} = \frac{\left(r^2 + a^2 c^2\right)^2}{\left(1 - c^2\right)\left(r^2 - mr + a^2\right)} \Rightarrow \left|\rho_{12,12}\right|/\left(\omega_{11}\omega_{22}\right) = \frac{mr\left(r^2 - 3a^2 c^2\right)}{2\left(r^2 + a^2 c^2\right)^3}$$

Also, as $a \in K$, then $\rho_{01,23}$ and $\rho_{02,13}$ can be both divided by a and we get the new invariant:

$$\rho_{01,23}/\rho_{03,12} = \frac{2r^2 - a^2 c^2 + 3a^2}{r^2 + a^2 c^2}$$

These results are leading to $\boxed{\xi^1 = 0}$, $\boxed{\xi^2 = 0}$, thus to $\boxed{\xi_1^1 = 0}$, $\boxed{\xi_2^2 = 0}$ and $\xi_0^0 + \xi_3^3 = 0$ after substitution. In the case of the S-metric, we have only $\xi^1 = 0$ as an equation of zero order.

The following relations allow to keep only the four parametric jets $\left(\xi_0^1, \xi_0^2, \xi_3^1, \xi_3^2\right)$ *on the right side*:

$$\boxed{\begin{cases} \omega^{11}\xi_1^0 + \omega^{00}\xi_0^1 + \omega^{03}\xi_3^1 = 0, & \omega^{11}\xi_1^3 + \omega^{03}\xi_0^1 + \omega^{33}\xi_3^1 = 0 \\ \omega^{22}\xi_2^0 + \omega^{00}\xi_0^2 + \omega^{03}\xi_3^2 = 0, & \omega^{22}\xi_2^3 + \omega^{03}\xi_0^2 + \omega^{33}\xi_3^2 = 0 \end{cases}}$$

if we use the fact that $\omega^{03} = -\omega_{03}/\left(\omega_{00}\omega_{33} - \left(\omega_{03}\right)^2\right)$ in the inverse metric.

Taking now into account that $R_{ij} = \omega^{rs}R_{ri,sj}\, mod\,(\Omega)$, we have proved in ([10]) that the 6 second order equations $R_{ij,ij} = 0$ are only depending on the 3 equations $R_{01,01} = 0$, $R_{02,02} = 0$, $R_{01,13} = 0$ by using $R_{ii} = 0$ and $R_{03} = 0$. We obtain in particular the two equations:

$$\begin{cases} \frac{1}{2}R_{01,01} \equiv \rho_{01,01}\left(\xi_0^0 + \xi_1^1\right) + \rho_{01,31}\xi_0^3 + \rho_{01,02}\xi_1^2 = 0 \\ \frac{1}{2}R_{02,02} \equiv \rho_{02,02}\left(\xi_0^0 + \xi_2^2\right) + \rho_{02,32}\xi_0^3 + \rho_{01,02}\xi_2^1 = 0 \end{cases}$$

Using the fact that we have now:

$$\omega_{22}\xi_1^2 + \omega_{11}\xi_2^1 = 0 \Leftrightarrow \omega^{11}\xi_1^2 + \omega^{22}\xi_2^1 = 0$$

We may multiply the first equation by ω^{11}, the second by ω^{22} and sum in order to obtain:

$$\left(\omega^{11}\rho_{01,01} + \omega^{22}\rho_{02,02}\right)\xi_0^0 + \left(\omega^{11}\rho_{01,31} + \omega^{22}\rho_{02,32}\right)\xi_0^3 = 0$$

Using the previous identities for $R_{00} = 0$ and $R_{03} = 0$, we obtain therefore:

$$\omega^{33}\rho_{03,03}\xi_0^0 - \omega^{03}\rho_{03,03}\xi_0^3 = 0 \Rightarrow \omega^{33}\xi_0^0 - \omega^{03}\xi_0^3 = 0 \Leftrightarrow \omega_{00}\xi_0^0 + \omega_{03}\xi_0^3 = 0$$

As this relation was satisfied $mod(\xi)$, we must introduce the new equation:

$$R_{01,13} \equiv \rho_{01,13}\left(\xi_0^0 + 2\xi_1^1 + \xi_3^3\right) - \rho_{13,13}\xi_0^3 - \rho_{01,01}\xi_3^0 + \left(\rho_{01,23} + \rho_{02,13}\right)\xi_1^2 = 0$$

As we have $\xi_0^0 + \xi_3^3 = 0$ and $\xi_1^1 = 0$, we obtain therefore a linear equation of the form:

$$\boxed{\rho_{13,13}\xi_0^3 + \rho_{01,01}\xi_3^0 - \left(\rho_{01,23} + \rho_{02,13}\right)\xi_1^2 = 0}$$

Similarly, we have also:

$$R_{01,02} \equiv \rho_{01,02}\left(2\xi_0^0 + \xi_1^1 + \xi_2^2\right) - \left(\rho_{01,23} + \rho_{02,13}\right)\xi_0^3 + \rho_{01,02}\xi_2^1 + \rho_{01,02}\xi_1^2 = 0$$

and we obtain therefore a linear equation of the form:

$$\boxed{2\rho_{01,02}\xi_0^0 - \left(\rho_{01,23} + \rho_{02,13}\right)\xi_0^3 + \rho_{01,01}\xi_2^1 + \rho_{01,02}\xi_1^2 = 0}$$

In the case of the S-metric, that is when $a = 0$, we obtain respectively $\xi_3^0 = 0$ and $\xi_2^1 = 0$ as in [18] because $\xi_0^0 \simeq \xi_3^0$. The previous linear system has thus a rank equal to 2 and we obtain therefore because $\xi_0^3 \simeq \xi_3^0, \xi_1^2 \simeq \xi_2^1$:

$$\boxed{\xi_3^0 = 0}, \boxed{\xi_2^1 = 0} \Leftrightarrow \boxed{\xi_0^3 = 0}, \boxed{\xi_1^2 = 0}, \boxed{\xi_0^0 = 0}, \boxed{\xi_3^3 = 0}$$

It remains to study the following 4 linear equations, namely ([10]):

$$R_{01,03} = 0, R_{03,23} = 0, R_{03,13} = 0, R_{02,03} = 0$$

THEOREM 3.1: The rank of the previous system with respect to the 4 jet coordinates $\left(\xi_0^1, \xi_0^2, \xi_3^1, \xi_3^2\right)$ is equal to 2, for both the S and K-metrics. We have the two striking identities:

$$\boxed{R_{03,13} + a\left(1 - c^2\right)R_{01,03} = 0, \quad R_{02,03} + \frac{a}{r^2 + a^2}R_{03,23} = 0}$$

We may define the so-called *algebroid bracket* for sections $\xi_q, \eta_q \in J_q(T)$ by setting ([1] [2] [3] [6]):

$$\left[\xi_q, \eta_q\right] = \left\{\xi_{q+1}, \eta_{q+1}\right\} + i(\xi)d\eta_{q+1} - i(\eta)d\xi_{q+1} \in J_q(T)$$

In this formula, "$i(\ )$" is the interior multiplication of a 1-form by a vector and we let the reader check that the right member does not depend on the respective lifts when the *algebraic bracket* is defined by linearity from the standard bracket of vector fields by the formula:

$$\left\{j_{q+1}(\xi), j_{q+1}(\eta)\right\} = j_q\left(\left[\xi, \eta\right]\right) \in J_q(T)$$

Now, we know that if $R_q \subset J_q(T)$ is a system of infinitesimal Lie equations, then we have the algebroid bracket and its link with the *prolongation/projection* (PP) procedure ([1] [2] [6] [7]):

$$\left[R_q, R_q\right] \subset R_q \Rightarrow \left[R_{q+r}^{(s)}, R_{q+r}^{(s)}\right] \subset R_{q+r}^{(s)}, \forall q, r, s \geq 0$$

As $R_1^{(1)} = \pi_1^2(R_2) = R_1$, it follows that $R_1^{(2)} = \pi_1^3(R_3)$ is such that $\left[R_1^{(2)}, R_1^{(2)}\right] \subset R_1^{(2)}$ with $dim\left(R_1^{(2)}\right) = 20 - 16 = 4$ because we have obtained a total of 6 *new different* first order equations. Using the first general diagram of

the Introduction, we discover that the operator defining R_1 has $10 + 4 = 14$ CC of order 2, a result obtained totally independently of the specific GR technical objects (Teukolski scalars, Killing-Yano tensors) introduced in ([12] [13] [14] [15]).

We have *on sections* (*care*) the 16 (linear) equations $mod\left(j_2\left(\Omega\right)\right)$ of $R_1^{(2)}$ as follows ([11]):

$$R_1^{(2)} \subset R_1 \subset J_1(T) \begin{cases} \xi^1 = 0, \xi^2 = 0 & \Rightarrow \boxed{\omega_{00}\xi_1^0 + \omega_{03}\xi_1^3} + \omega_{11}\xi_0^1 = 0, \xi_1^1 = 0, \xi_2^2 = 0 \\ \xi_2^1 = 0 & \Rightarrow \xi_1^2 = 0 \\ \xi_3^1 + lin\left(\xi_0^1, \xi_0^2\right) = 0 & \Rightarrow \boxed{\omega_{03}\xi_1^0 + \omega_{33}\xi_1^3} + \omega_{11}\xi_3^1 = 0 \\ \xi_3^2 + lin\left(\xi_0^1, \xi_0^2\right) = 0 & \Rightarrow \boxed{\omega_{00}\xi_2^0 + \omega_{03}\xi_2^3} + \omega_{22}\xi_0^2 = 0, \\ & \qquad \boxed{\omega_{03}\xi_2^0 + \omega_{33}\xi_2^3} + \omega_{22}\xi_3^2 = 0 \\ \xi_3^0 = 0 & \Rightarrow \xi_2^0 = 0, \xi_0^0 = 0, \xi_3^3 = 0 \end{cases}$$

The coefficients of the linear equations lin involved depend on the Riemann tensor as in ([11]). Accordingly, we may choose only the 2 parametric jets $\left(\xi_0^1, \xi_0^2\right)$ among $\left(\xi_0^1, \xi_3^1, \xi_0^2, \xi_3^2\right)$ to which we must add $\left(\xi^0, \xi^3\right)$ *in any case* as they are not appearing in the Killing equations.

The system is *not* involutive because its symbol is finite type but non-zero.

Using *one more prolongation*, all the *sections* (*care again*) vanish but ξ^0 and ξ^3, a result leading to $dim\left(R_1^{(3)}\right) = 2$ in a coherent way with the only nonzero Killing vectors $\left\{\partial_t, \partial_\phi\right\}$. We have indeed:

$$\boxed{\xi_0^1 = 0}, \boxed{\xi_0^2 = 0} \Rightarrow \xi_3^1 = 0, \xi_3^2 = 0 \Rightarrow \xi_1^0 = 0, \xi_1^3 = 0, \xi_2^0 = 0, \xi_2^3 = 0$$

Taking therefore into account that the metric only depends on $\left(x^1 = r, x^2 = c = \cos(\theta)\right)$ we obtain *after three prolongations* the first order system $R_1^{(3)} \subset R_1^{(2)} \subset R_1^{(1)} = R_1 \subset J_1(T)$ which is trivially involutive because all the first order jets vanish but $\left(\xi^0, \xi^3\right)$.

Surprisingly and contrary to the situation found for the S metric, we have now an involutive first order system with only solutions $\left(\xi^0 = cst, \xi^1 = 0, \xi^2 = 0, \xi^3 = cst\right)$ and notice that $R_1^{(3)}$ does not depend any longer on the parameters $(m, a) \in K$. The difficulty is to know what second members must be used along the procedure met for all the motivating examples. In particular, we have again identities to zero like $d_0\xi^1 - \xi_0^1 = 0$, $d_0\xi^2 - \xi_0^2 = 0$ and thus *at least* 6 third order CC coming from the 6 following components of the Spencer operator, namely:

$$\boxed{d_1\xi^1 - \xi_1^1 = 0, d_2\xi^1 - \xi_2^1 = 0, d_3\xi^1 - \xi_3^1 = 0, d_1\xi^2 - \xi_1^2 = 0, d_2\xi^2 - \xi_2^2 = 0, d_3\xi^2 - \xi_3^2 = 0}$$

a result that cannot be even imagined from ([12] [13] [14] [15]). Of course, proceeding like in the motivating examples, we must substitute in the right members the values obtained from $j_2(\Omega)$ and set for example $\xi_1^1 = -\dfrac{1}{2\omega_{11}}\xi\partial\omega_{11}$ while replacing ξ^1 and ξ^2 by the corresponding linear combinations of the Riemann tensor already obtained for the right members of the two zero order equations.

Along with the examples, we recall below *all* the diagrams that *must* be intro-

duced successively:

$$
\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \to & S_2 T^* \otimes T & \overset{\sigma_1(\Phi)}{\to} & T^* \otimes S_2 T^* & \to 0 & \\
& & \downarrow & & \downarrow & & \\
0 \to & R_2 & \to \quad J_2(T) & \overset{\rho_1(\Phi)}{\to} & J_1(S_2 T^*) & \to 0 & \\
& \downarrow & \downarrow & & \downarrow & & \\
0 \to & R_1 & \to \quad J_1(T) & \overset{\Phi}{\to} & S_2 T^* & \to 0 & \\
& \downarrow & \downarrow & & \downarrow & & \\
& 0 & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \to & 40 & \to & 40 & \to 0 & \\
& & \downarrow & & \downarrow & & \\
0 \to & 10 & \to & 60 & \to 50 & \to 0 & \\
& \downarrow & & \downarrow & \downarrow & & \\
0 \to & 10 & \to & 20 & \to 10 & \to 0 & \\
& \downarrow & & \downarrow & \downarrow & & \\
& 0 & & 0 & 0 & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \to & S_3 T^* \otimes T & \overset{\sigma_2(\Phi)}{\to} & S_2 T^* \otimes S_2 T^* & \to & h_2 & \to 0 & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & R_3 & \to \quad J_3(T) & \overset{\rho_2(\Phi)}{\to} & J_2(S_2 T^*) & \to & Q_2 & \to 0 & \\
& \downarrow & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & R_2 & \to \quad J_2(T) & \overset{\rho_1(\Phi)}{\to} & J_1(S_2 T^*) & \to & 0 & & \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0 & & & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \to & 80 & \to & 100 & \to & 20 & \to 0 & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & 4 & \to & 140 & \to 150 & \to & 14 & \to 0 & \\
& \downarrow & & \downarrow & \downarrow & & \downarrow & & \\
0 \to & 10 & \to & 60 & \to 50 & \to & 0 & & \\
& & & \downarrow & \downarrow & & & & \\
& & & 0 & 0 & & & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & \\
& & \downarrow & & \downarrow & & & \\
0 & \to & S_4 T^* \otimes T & \xrightarrow{\sigma_3(\Phi)} & S_3 T^* \otimes S_2 T^* & \to & h_3 & \to 0 \\
& & \downarrow & & \downarrow & & \downarrow & \\
0 \to & R_4 & \to \quad J_4(T) & \xrightarrow{\rho_3(\Phi)} & J_3(S_2 T^*) & \to & Q_3 & \to 0 \\
& \downarrow & \downarrow & & \downarrow & & \downarrow & \\
0 \to & R_3 & \to \quad J_3(T) & \xrightarrow{\rho_2(\Phi)} & J_2(S_2 T^*) & \to & Q_2 & \to 0 \\
& & \downarrow & & \downarrow & & \downarrow & \\
& & 0 & & 0 & & 0 &
\end{array}
$$

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & \\
& & \downarrow & & \downarrow & & & \\
0 & \to & 140 & \to & 200 & \to & 60 & \to 0 \\
& & \downarrow & & \downarrow & & \downarrow & \\
0 \to & 2 & \to & 280 & \to & 350 & \to & 72 & \to 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & 4 & \to & 140 & \to & 150 & \to & 14 & \to 0 \\
& & \downarrow & & \downarrow & & \downarrow & \\
& & 0 & & 0 & & 0 &
\end{array}
$$

Applying the Spencer δ-map to the top sequence, we get the commutative diagram in which the two central columns are exact:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & \\
& & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & S_4 T^* \otimes T & \to & S_3 T^* \otimes F_0 & \to & h_3 & \to 0 \\
& & \downarrow & & \downarrow & \searrow & \downarrow & \\
0 & \to & T^* \otimes S_3 T^* \otimes T & \to & T^* \otimes S_2 T^* \otimes F_0 & \to & \boxed{T^* \otimes h_2} & \to 0 \\
& & \downarrow & & \downarrow & & & \\
0 & \to & \wedge^2 T^* \otimes S_2 T^* \otimes T & \to & \wedge^2 T^* \otimes T^* \otimes F_0 & \to & 0 & \\
& & \downarrow & & \downarrow & & & \\
0 \to & \boxed{\wedge^3 T^* \otimes g_1} & \to & \wedge^3 T^* \otimes T^* \otimes T & \to & \wedge^3 T^* \otimes F_0 & \to & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \to & \wedge^4 T^* \otimes T & = & \wedge^4 T^* \otimes T & \to & 0 & & \\
& \downarrow & & \downarrow & & & & \\
& 0 & & 0 & & & &
\end{array}
$$

Using a (delicate) snake chase, we obtain the long exact sequence:

$$\boxed{0 \to h_3 \to T^* \otimes h_2 \to \wedge^3 T^* \otimes g_1 \to \wedge^4 T^* \otimes T \to 0},$$

$$0 \to 60 \to 80 \to 24 \to 4 \to 0$$

and the long exact connecting diagram:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \to & S_4 T^* \otimes T & \to & S_3 T^* \otimes F_0 & \to & T^* \otimes h_2 & \to & H^3(g_1) & \to & 0 \\
& & \downarrow & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \rho_1(g_2') & \to & S_3 T^* \otimes F_0 & \to & T^* \otimes Q_2 & \to & Q_1' & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

allowing to compare the geometry of the Kerr metric (lower sequence) to the classical geometry of the Minkowski metric (upper sequence). In the later case we recall that the Bianchi indentities provide a section $B \in H^3(g_1)$. Moreover, we have $dim(Q_1') \le dim(H^3(g_1)) = 20$ if we define Q_1' by the following commutative and exact diagram obtained by cutting the fundamental diagram and going one step further on in the sequences in order to replace the R and Q by R' and Q':

$$
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to \rho_1(g_2') & \to & S_3 T^* \otimes F_0 & \overset{\sigma_1(\Psi)}{\to} & T^* \otimes Q_2 & \to & h_1' & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \searrow \downarrow & \\
0 \to \rho_1(R_2') & \to & J_3(F_0) & \overset{\rho_1(\Psi)}{\to} & J_1(Q_2) & \to & Q_1' & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to R_2' & \to & J_2(F_0) & \overset{\Psi}{\to} & Q_2 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

with dimensions:

$$
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to 144 + x & \to & 200 & \to & 56 & \to & x & \to 0 \\
\downarrow & & \downarrow & & \downarrow \searrow & \| & \\
0 \to 280 + x & \to & 350 & \to & 70 & \to & x & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to 136 & \to & 150 & \to & 14 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

Accordingly, the number 14 of second order CC plus the number of differentially independent third order CC obtained by one prolongation of these second order CC is equal to $70 - x$. However, if we denote by y the number of new generating CC of third order that are not differential consequences of the CC of second order already found, we must have $72 - y = 70 - x$ and thus $y = x + 2$.

As we have 4 divergence identities for the Einstein operator, we obtain therefore $y = 4 + 2 = 6$, contrary to ([12] [13] [14] [15]). We have already provided these new third order CC as 6 components of the Spencer operator.

Similarly, in the case of the S-metric, we have $R_1^{(3)} \subset R_1^{(2)} \subset R_1^{(1)} = R_1 \subset J_1(T)$ with respective dimensions $4 < 5 < 10 = 10 < 20$ and we obtain:

$$Q_1 = 0, dim(Q_2) = 15, dim(Q_3) = 74 \Rightarrow 74 - y = 75 - x \Rightarrow y = x - 1 = 3$$

$$\boxed{d_1\xi^1 - \xi_1^1 = 0, \, d_2\xi^1 - \xi_2^1 = 0, \, d_3\xi^1 - \xi_3^1 = 0}$$

In the case of the M-metric (See [1] [6] [7] for more details), we have $R_1^{(3)} = R_1^{(2)} = R_1^{(1)} = R_1 \subset J_1(T)$ with respective dimensions $10 = 10 = 10 = 10 < 20$ and we obtain:

$$Q_1 = 0, dim(Q_2) = 20, dim(Q_3) = 80, dim(Q_1') = 20 \Rightarrow x = 20, y = 0$$
$$\Rightarrow 80 - y = 100 - x$$

in a coherent way with the number of components of the Riemann tensor (20) and the number of Bianchi identities (20) when $n = 4$. The corresponding Killing system is FI if and only if the following Vessiot structure equations $\rho_{l,ij}^k = c\left(\delta_i^k \omega_{lj} - \delta_j^k \omega_{li}\right)$ with the only constant c are satisfied, that is the well known constant Riemannian curvature condition ([19]).

4. Conclusion

E. Vessiot discovered the so-called *Vessiot structure equations* as early as in 1903 and, only a few years later, E. Cartan discovered the so-called *Maurer-Cartan structure equations*. Both are depending on a certain number of constants like the single *geometric structure constant* of the constant Riemannian curvature for the first and the many *algebraic structure constants* of Lie algebra for the second. However, Cartan and followers never acknowledged the existence of another approach which is therefore still totally ignored today, in particular by physicists ([11] [19]). Now, it is well known that the structure constants of a Lie algebra play a fundamental part in the Chevalley-Eilenberg cohomology of Lie algebras and their deformation theory ([6]). It was thus a challenge to associate the Vessiot structure constants with other homological properties related to systems of Lie equations, namely the extension modules determined by Lie operators. As a striking consequence, such a possibility opens a new way to understand and revisit the various contradictory works done during the last fifty years or so by different groups of researchers, using respectively Cartan, Gröbner or Janet bases while looking for a modern interpretation of the work done by C. Lanczos from 1938 to 1962. However, the reader must not forget that the Weyl tensor was not known by Lanczos, even as late as in 1967, and that it was not possible to discover any solution of the parametrization problem by potentials through double duality before 1990/1995, that is too late for the many people already engaged in this type of research ([16]). As for the study of the Killing operator, the intrinsic understanding of such a technical problem that we achieved has strictly nothing to do with GR. Indeed, thanks to a difficult theorem of homological algebra, the intrinsic properties of the resolution of a differential module *M* only depend on *M* itself but not on the resolution. Roughly, *the only important thing is the group, not the metric*. In actual practice, the resolutions *by themselves* no

longer depend on the respective Kerr, Schwarzschild or Minkowski parameters. The same comment can be done on the link existing between electromagnetism and the conformal group ([24] [25] [26]) because the conformal sequence must be entirely revisited in order to agree with computer algebra results obtained by A. Quadrat (See [6], Appendix 2 and [27]). We finally hope that this paper will open a new domain for applying computer algebra and we are offering a collection of useful but tricky test examples.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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