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How the Redshift of Gravitons Explains Dark Matter and Dark Energy

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Abstract

The theory that gravitons lose energy by way of gravitational redshift while traveling in a gravitational field is applied to the expansion of the universe and to spiral and dwarf galaxy rotation curves using General Relativity. This is a graviton self interaction model which derives an expansion equation which is identical in form to the standard Lambda Cold Dark Matter model. In the domain of galaxies, spiral and dwarf galaxy rotation curves are matched using only baryonic mass. Thus, the requirement for dark matter and dark energy in the universe is replaced by this paradigm.

Keywords

Gravitons, Gravitational Redshift, Baryon Mass Density, Supernovae, Spiral Galaxies

1. Introduction

This paper describes a theory of gravitons acting in the universe and it superscedes the theory expressed in [1] in regards to the expansion of the universe. We will also describe the action of gravitons in galaxies [2]. Assuming that gravitons are the agents of interaction in a gravitational field, then our goal is to describe how the gravitational redshift of gravitons shows up as what has traditionally been labeled as dark matter and dark energy. We will show how the redshift of gravitons can explain both of these phenomena using high precision data. Throughout this paper, all reference to sources of mass and test particles are assumed to be of baryonic nature, unless otherwise specified. When we speak of the graviton mass m_g, it is a relativistic mass.

We assume gravitons are bosonic particles which travel at constant speed c in vacuum, where c equals the speed of light. Gravitons traveling in a gravitational field between a source mass M and a test mass m, modeled by the equivalence principle as an accelerating system, should experience an average energy loss of $\delta\xi$ due to motion in that field, over a short time period $\delta t = \delta r/c$, where the acceleration a at a point r in the field is given by $a = -GM/r^2$. The energy loss is expressed differentially as

$$\delta\xi = -\left(m_g c^2\right)\frac{\delta v}{c} = -\left(m_g c\right)a\delta t = -\left(\frac{GMm_g}{r^2}\right)\delta r, \tag{1}$$

where $m_g = m/n$ is the average relativistic graviton mass, n is the number of gravitons, m is the test mass, δv is the change in the free fall velocity of the system observed from an inertial reference frame, G is Newton's gravitational constant, M is the baryonic mass of the field source, r is the distance between the center of the source and the location of the moving gravitons, δt is the short travel time of the gravitons at speed c over distance δr. The energy change is a loss (negative), because the velocity change δv is in the same direction as the motion of the gravitons, so that for an inertial observer moving in the same direction as the graviton, the energy of the graviton is redshifted. We call this effect of energy loss a gravitational redshift which, as we have described it resembles a Doppler effect. Since gravitons are agents of the gravitational field, our model is essentially an attempt to describe the graviton-graviton interaction as in the self interaction of a quantum gravity theory [3].

Gravitons exist and travel in a gravitational field, which in principle is equivalent to an accelerating system. Assume that the total graviton energy for a system of two masses is expressed by,

$$\Xi = \frac{GMm}{r}, \tag{2}$$

where $m = nm_g$ is the total graviton mass associated with the test mass m, where m_g is the average graviton mass and n is the number of gravitons. The total graviton energy decrease $\delta\Xi$ due to its freefall in the gravitational field of mass M, when viewed from an inertial system, is expressed by

$$\delta\Xi = -\Xi\frac{\delta v}{c} = -\left(\frac{GMm}{r}\right)\frac{\delta v}{c} = -\left(\frac{GMm}{r}\right)\left(\frac{GM}{c^2 r^2}\right)\delta r, \tag{3}$$

where, as in (1), $\delta v = \left(GM/cr^2\right)\delta r$ is the velocity increase in the accelerated reference frame equivalent, according to the principle of equivalence, to the gravitational field of mass M at the position r.

2. Gravitons in an Expanding Universe

Consider the universe as a sphere of interior baryonic mass M with a thin spherical shell of mass m. The masses M and m are constants. The thin shell has a radius $r(t)$ at time t. Only the mass interior to the shell has an effect on the shell. The total graviton energy $\Xi(t)$ within the shell at time t is given by (2), where $r = r(t)$. Assuming isotropic uniformity, the mass density $\rho(t)$ within the shell at time t is

$$\rho(r(t)) = \frac{3M}{4\pi r^3(t)}. \tag{4}$$

At the present epoch of time t_0 the baryon mass density is ρ_b, which is given by,

$$\rho_b = \frac{3M}{4\pi \bar{a}^3}, \tag{5}$$

where $\bar{a} = r(t_0)$ is the radius of the universe at the present epoch.

2.1. Energy Loss Due to Gravitational Redshift

Since the potential function $\Phi(r) = GMm/r$ is defined as having zero energy at infinity and having a negative energy at position *r*, likewise we define the graviton energy loss to be zero at infinity and negative at position *r*. Applying (1) to the *n* gravitons in free fall in the expanding universe we have the energy loss,

$$\Delta\Xi_{dm} = K_{dm} \int_0^{\Delta\Xi} n\delta\xi = -K_{dm} \int_\infty^r -\frac{GMm}{r^2} \mathrm{d}r = -K_{dm}\left(\frac{GMm}{r}\right), \tag{6}$$

where K_{dm} is a coupling constant to be determined by observation and the extra minus sign accounts for an energy loss because the gravitons travel in the same direction as the free fall velocity change and appear redshifted to an inertial observer also moving in the same direction. Substituting for *M* from (5) and simplifying yields,

$$\Delta\Xi_{dm} = -K_{dm}\left(\frac{4\pi G\bar{a}^3}{3}\right)\frac{m\rho_b}{r}. \tag{7}$$

This component is the energy loss of the so called dark matter.

2.2. Energy Loss Due to Expansion

Gravitons traveling at speed *c* in the vacuum of the expanding universe undergo cosmological redshift in three dimensions on the way to interaction with the masses. We express this redshift by applying the 3-D velocity differential ($\delta v_x \delta v_y \delta v_z / c^3$) to the total graviton energy Ξ from (2), and applying (3) in the three spatial dimensions, given in the form,

$$\delta\xi = -\frac{GMm}{r}\left(\frac{\delta v_x \delta v_y \delta v_z}{c^3}\right), \tag{8}$$

where the negative sign is applied because the motion of the gravitons is in the same direction as the freefall in the field. We can convert the 3-D velocity differential to a ratio of 3-D volume differential by the construction,

$$\frac{\delta v_x \delta v_y \delta v_z}{c^3} = \left(\frac{\delta x}{c\delta t_x}\right)\left(\frac{\delta y}{c\delta t_y}\right)\left(\frac{\delta z}{c\delta t_z}\right) = \frac{\delta x \delta y \delta z}{c^3 \delta t_x \delta t_y \delta t_z}, \tag{9}$$

where *x*, *y* and *z* are Cartesian co-ordinates, t_x, t_y and t_z are independent times and where $\delta v_x = \delta x/\delta t_x$, $\delta v_y = \delta y/\delta t_y$ and $\delta v_z = \delta z/\delta t_z$. Furthermore, we convert the volume differential in Cartesian co-ordianes to radial co-ordiates, in the form

$$\delta x\delta y\delta z=4\pi r^{2}\delta r. \tag{10}$$

Now, applying the transformations (9) and (10) to (8), while also moving the volume differential $c^{3}\delta t_{x}\delta t_{y}\delta t_{z}$ to the left hand side of the equation we get,

$$\delta\xi\left(c^{3}\delta t_{x}\delta t_{y}\delta t_{z}\right)=-\frac{GMm}{r}\left(4\pi r^{2}\delta r\right). \tag{11}$$

The left hand side of (11) is a quadruple differential whilst the right hand side is a single differential. Integrating both sides of (11) yields,

$$\begin{aligned}\left(\frac{\bar{a}^{3}}{\sigma_{de}}\right)\Delta\Xi_{de}&=\int_{0}^{T}\int_{0}^{T}\int_{0}^{T}\int_{0}^{\Delta\Xi}\delta\xi\left(c^{3}\delta t_{x}\delta t_{y}\delta t_{z}\right)=\int_{r_0}^{r}\left(\frac{-mGM}{r}\right)4\pi r^{2}\delta r\\&=\int_{r_0}^{r}-4\pi mGMr\delta r=-\frac{8\pi^{2}G}{3}\bar{a}^{3}m\rho_{b}r^{2},\end{aligned} \tag{12}$$

where the expansion time is taken to be

$$T=\bar{a}/c, \tag{13}$$

where $\bar{a}$ is the present radius of the universe and σ_{de} is a dimensionless coupling constant and where, in the final line we substituted for M from (5). Rearranging (12) we get the graviton energy loss due to the expansion of the universe, the so called dark energy component,

$$\Delta\Xi_{de}=-\sigma_{de}\left(\frac{8\pi^{2}G}{3}\right)m\rho_{b}r^{2}. \tag{14}$$

We will subsequently show that the graviton energy losses $\Delta\Xi_{dm}$ and $\Delta\Xi_{de}$ can account for the expansion rate of the universe without dark matter or dark energy which is required by the Lamda Cold Dark Matter cosmological model [4].

2.3. Equation of the Expanding Universe

The total energy of the shell of mass *m*, where $m\ll M$, having kinetic energy, gravitational potential energy and energy loss due to the cosmological redshift of gravitons (14), is expressed by

$$\begin{aligned}&\frac{1}{2}mv^{2}-\frac{GMm}{r}+\Delta\Xi_{dm}+\Delta\Xi_{de}\\&=\frac{1}{2}mv^{2}-\left(\frac{4\pi G\bar{a}^{3}}{3}\right)\frac{m\rho_{b}}{r}-K_{dm}\left(\frac{4\pi G\bar{a}^{3}}{3}\right)\frac{m\rho_{b}}{r}-\sigma_{de}\left(\frac{8\pi^{2}G}{3}\right)m\rho_{b}r^{2}\\&=-\frac{1}{2}mc^{2}k\bar{a}^{2},\end{aligned} \tag{15}$$

where the term on the far right is the total energy, where k is a constant (curvature) with dimensionality [length]$^{-2}$, $\bar{a}$ is the present radius of the universe and M is the mass of the universe. Multiplying (15) by $2/mr^{2}$ and simplifying, we get the expression for the expansion of the shell,

$$\frac{v^{2}}{r^{2}}=\frac{8\pi G\bar{a}^{3}\rho_{b}}{3}\frac{1}{r^{3}}+K_{dm}\left(\frac{8\pi G\bar{a}^{3}\rho_{b}}{3}\right)\frac{1}{r^{3}}+\sigma_{de}\left(\frac{16\pi^{2}G\rho_{b}}{3\bar{a}^{3}}\right)-\frac{kc^{2}\bar{a}^{2}}{r^{2}}. \tag{16}$$

Note that the shell mass m can be made arbitrarily small compared to the universe mass M.

Define the distance r by

$$r = \bar{a}a, \tag{17}$$

where the time varying scale factor a is dimensionless with $0 < a \le 1$. Using (17), the velocity v takes the form

$$v = \frac{\mathrm{d}r}{\mathrm{d}t} = \bar{a}\frac{\mathrm{d}a}{\mathrm{d}t}. \tag{18}$$

Substituting (18) into (16) gives us,

$$\left(\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = \frac{8\pi G\left(1+K_{dm}\right)\rho_b}{3a^3} + \frac{16\pi^2 G\sigma_{de}\rho_b}{3} - \frac{kc^2}{a^2}. \tag{19}$$

Putting (19) in terms of the mass densities we have,

$$\left(\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = \frac{8\pi G}{3}\left(\rho_m(a) + \rho_{dm}(a) + \rho_{de}(a) + \rho_k(a)\right), \tag{20}$$

where

$$\rho_m(a) = \frac{\left(1+K_{dm}\right)\rho_b}{a^3} \tag{21}$$

is the mass density composed of baryons and graviton energy loss mass due to gravitational redshift (an apparent dark matter mass density),

$$\rho_{de}(a) = 2\pi\sigma_{de}\rho_b \tag{22}$$

is the graviton energy loss due to the expanding universe (an apparent dark energy mass density) and

$$\rho_k(a) = \frac{-3kc^2}{8\pi G a^2} \tag{23}$$

is the curvature mass density.

Define the Hubble parameter $H(t)$ by

$$H(t) = \frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}t}, \tag{24}$$

where, by (17), $r = \bar{a}a$. Equation (24) can also be written as

$$v = \frac{\mathrm{d}r}{\mathrm{d}t} = H(t)r(t), \tag{25}$$

which is the Hubble law for $H_0 = H(t_0)$ where t_0 is the present epoch of cosmic time. Thus, $H(t)$ defined by (24) is the general form of Hubble's law [5]. Substituting $H(t)$ for $\mathrm{d}a/a\mathrm{d}t$ in (20), with some manipulation, we get

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G} = \rho_m(t) + \rho_{de}(t) + \rho_k(t), \tag{26}$$

where $\rho_c(t)$ is called the critical mass density at time t. Dividing (26) by $\rho_c(t)$ yields the parametric equation

$$\frac{\rho_c(t)}{\rho_c(t)} = \Omega_c = 1 = \Omega_m(t) + \Omega_{de}(t) + \Omega_k(t), \tag{27}$$

where $\Omega_m(t) = \rho_m(t)/\rho_c(t)$, $\Omega_{de}(t) = \rho_{de}(t)/\rho_c(t)$ and $\Omega_k(t) = \rho_k(t)/\rho_c(t)$. At the present epoch t_0, the mass density parameter is

$$\Omega_m(t_0) = (1 + K_{dm})\Omega_b, \tag{28}$$

where Ω_b is the baryon mass density parameter, the graviton expansion energy loss mass density parameter is

$$\Omega_{de}(t_0) = 2\pi\sigma_{de}\Omega_b \tag{29}$$

and the curvature density parameter is

$$\Omega_k(t_0) = -\frac{kc^2}{H_0^2}. \tag{30}$$

3. General Relativity for an Expanding Universe with Graviton Interaction

The Friedmann-Lemaître-Robertson-Walker (FLRW) metric line element [6] [7] [8] [9] in terms of the scale factor $a(t)$ is

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}^2\theta + r^2\sin^2(\theta)\mathrm{d}^2\phi\right), \tag{31}$$

where the scale factor $0 < a(t) \le 1$ and the curvature k has units of $[length]^{-2}$ where $k < 0$, $k > 0$ or $k = 0$. The Einstein equations [10] [11] in trace reverse form is

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right), \tag{32}$$

where $R_{\mu\nu}$ is the Ricci tensor, $T_{\mu\nu}$ is the energy-momentum tensor, T is the contracted energy-momentum tensor, $g_{\mu\nu}$ is the metric tensor defined by (31) and $\kappa = 8\pi G/c^4$. Define the energy-momentum tensor $T_{\mu\nu}$ of a perfect fluid,

$$T_{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{33}$$

where, referring to (21) to (23) for the mass densities, the total mass density ρ is given by

$$\rho = \rho_m + \rho_{de} + \rho_k, \tag{34}$$

where

$$\rho_m = \frac{(1+K_{dm})\rho_b\bar{a}^3}{r^3}, \tag{35}$$

$$\rho_{de} = 2\pi\sigma_{de}\rho_b, \tag{36}$$

$$\rho_k = \frac{-3k\bar{a}^2c^2}{8\pi G r^2}, \tag{37}$$

and where the current baryon mass density is

$$\rho_b = \rho_c \Omega_b, \tag{38}$$

where Ω_b is the baryon mass density parameter. We have assumed the equation of state for the relativistic particles $p_{de} = \omega_{de} \rho c^2$, with $\omega_{de} = -1$. We are neglecting the radiation density (photon and neutrinos) and we will justify this when we fit the model to SNe Ia data.

Solving the Einstein Equations (32) given the mass-energy tensor (33), and simplifying the results, yields the equation for the rate of change of the scale factor,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \tag{39}$$

Assuming $k = 0$ for no curvature, and the total mass density given by (34) - (38), the Hubble parameter $H(a)$ is

$$H(a) = H_0 \sqrt{\frac{(1+K_{dm})\Omega_b}{a^3} + 2\pi\sigma_{de}\Omega_b}, \tag{40}$$

where $\Omega_b = 8\pi G \rho_b / 3H_0^2$.

4. Fits to Type Ia Supernova Data and Comparison with the Standard Model

The scale factor a has the relation to the cosmological redshift z expressed by

$$a = \frac{1}{1+z}. \tag{41}$$

An increment of comoving distance δd_c defined in terms of the scale factor is

$$\delta d_c = c\frac{\mathrm{d}t}{a} = c\frac{\mathrm{d}a}{a^2 H(a)}, \tag{42}$$

where we used the definition (24) of the Hubble parameter. In terms of the cosmological redshift z, the comoving distance is

$$\delta d_c = c\frac{\mathrm{d}z}{H(z)}, \tag{43}$$

where we used the fact that $\mathrm{d}a = -\mathrm{d}z/(1+z)^2 = -a^2\mathrm{d}z$ to transform (42), dropping the minus sign.

The flux Φ_0 from a distant light source at redshift z is defined in terms of the observed luminosity $L_o = L/(1+z)^2$, where L is the luminosity of the emitting source,

$$\Phi_0 = \frac{L}{4\pi(1+z)^2 d_p^2}, \tag{44}$$

where d_p is the proper distance. The luminosity distance, from (44) is

$$(1+z)d_p = \sqrt{\frac{L}{4\pi\Phi_0}}. \tag{45}$$

The luminosity distance, from (43), is

$$D_L(z) = (1+z)d_c = (1+z)\int_0^z \frac{c\mathrm{d}z}{H(z)}, \tag{46}$$

where d_c is the co-moving distance and where

$$H(z) = H_0\sqrt{(1+K_{dm})\Omega_b(1+z)^3 + 2\pi\sigma_{de}\Omega_b}, \tag{47}$$

and we changed the negative sign to positive by inverting the limits of integration. The form (47) of the Hubble parameter $H(z)$ is identical to that of the standard model. Therefore, fitting to the SNe Ia data will be identical. The magnitude is defined, in the standard way,

$$Mu(z) = 5\log(D_L(z)) - \mu_B + a_{off}, \tag{48}$$

where μ_B is the source magnitude and a_{off} is an offset. Generally, the source magnitude is combined into a_{off}.

We applied (48) in a fit to 580 Type Ia supernovae (SNe Ia) magnitude data from the Super Nova Cosmology Project Union 2.1 data set [12]. With Hubble constant $H_0 = 70\ \mathrm{km}\cdot\mathrm{s}^{-1}\cdot\mathrm{Mpc}^{-1}$, densities $\Omega_m = 0.271$, $\Omega_{de} = 0.729$ [12] and offset $a_{off} = -87.441$, the fit to the SNe Ia data set obtained a two parameter $\chi^2 = 0.9769$. **Figure 1** shows the fit to the data. Regarding neglecting the radiation density in our model, the radiation density parameter of photons, with $h = 0.7$ is given by [13], $\Omega_r = (2.47\times10^{-5})/h^2 = 5.04\times10^{-5}$. At the maximum redshift

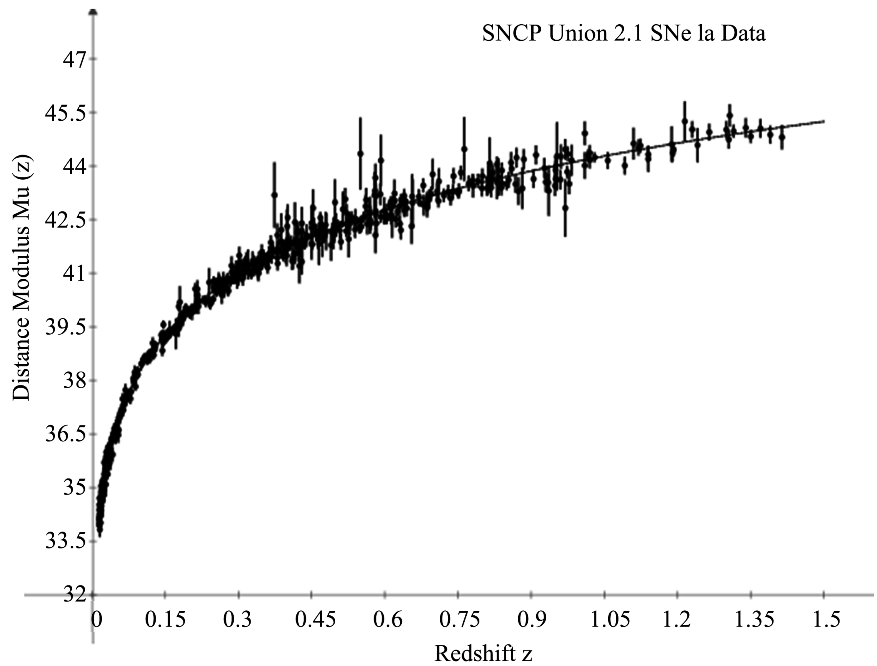


Figure 1. Supernova cosmology project union 2.1 SNe Ia magnitude vs redshift data points with error bars. The solid line is the fit for the graviton model with $\Omega_m = 0.271$ and $\Omega_{de} = 0.729$ with a two parameter $\chi^2 = 0.9769$. The dotted line is the fit for the LCDM model with the same parameters and the same χ^2. The graviton model and the LCDM model fits are (obviously) identical.

of $z_{\max} = 1.5$ for the SNe Ia data, the relative magnitude of the error ε_{rad} in comparing the radiation and matter densities is

$$\varepsilon_{rad} = \frac{\Omega_r (1+z_{\max})^4}{\Omega_m (1+z_{\max})^3} = 4.6\times 10^{-4}, \tag{49}$$

which justifies ignoring the radiation density in the fitting.

Assuming that the curvature $k = 0$, so that $\Omega_k = 0$, then from (27) we have for the present epoch

$$1 = (1+K_{dm})\Omega_b + 2\pi\sigma_{de}\Omega_b, \tag{50}$$

and from (29),

$$\sigma_{de} = \frac{\Omega_{de}}{2\pi\Omega_b}. \tag{51}$$

For example, for $\Omega_{de} = 0.721$ and $\Omega_b = 0.049$ which is in the big bang nucleosynthesis (BBN) allowable range [14] we get a value of the coupling constant $\sigma_{de} = 2.342$. Also, from (28) we get $K_{dm} = (\Omega_m/\Omega_b) - 1 = 4.694$. The expansion time, from (13) has the value $T = 1/H_0 = 13.97\times 10^9$ yr.

5. Gravitons in Galaxies

Integrating (1) up to radial distance r we obtain the average energy change per graviton $\Delta\Xi_g$ expressed by

$$\Delta\Xi_g = -\int_0^r (m_g c^2)\frac{\mathrm{d}u}{c} = -\int_0^r m_g\left(\frac{GM_b}{r^2}\right)\mathrm{d}r. \tag{52}$$

Equation (52) describes the gravitational redshift of the energy of the average graviton as it travels from a lower, more negative potential to a position r of higher, less negative potential and is consistent with energy conservation.

Now, consider the energy for a galaxy of mass M interior of a small mass m in a circular orbit of radius r. The gravitons traversing the distance at lightspeed from the interior mass to the orbiting mass will experience a decrease in energy as described by (52), $\Delta\Xi = n\Delta\Xi_g$, where n is the number of gravitons. Taking the energy loss of the gravitons into account, the total orbital energy of the orbiting mass m is

$$\frac{1}{2}mv^2 - \frac{GMm}{r} + K_g n\Delta\Xi_g = E, \tag{53}$$

where v is the rotational velocity of the orbiting mass, K_g is a coupling coefficient, a constant for each galaxy, and the total energy $E = -GM/2r$. Using (53) by expanding $\Delta\Xi_g$ using (52), with $m = nm_g$, multiplying by $2/m$ and moving all terms except v^2 to the right hand side, we obtain the expression for the orbital velocity,

$$v^2 = \frac{GM(r)}{r} + 2K_g\int_0^r\left(\frac{GM(r)}{r^2}\right)\mathrm{d}r. \tag{54}$$

As an approximation, we model the mass distribution of a spiral galaxy by a spherically symmetric distribution $\rho(r)$, even though a mass density distribution consisting of a spherically symmetric central bulge surrounded by an axially symmetric thin disk would be more realistic. Then the mass $M(r)$ of the galaxy within the radial distance *r* from the galaxy center is given by,

$$M(r) = \int_0^r 4\pi\rho(r) r^2 \mathrm{d}r. \tag{55}$$

Coupling Coefficient K_g

Under our assumption that the coupling coefficient is constant for each galaxy, (54) can be solved for K_g at the galaxy edge, where $r = r_f$ and $v = v_f$, giving

$$K_g = \frac{v_f^2 - \dfrac{GM(r_f)}{r_f}}{2\int_0^{r_f} \left(\dfrac{GM(r)}{r^2}\right) \mathrm{d}r}, \tag{56}$$

where $M(r_f) = M_b$ is the total baryonic mass in the galaxy. By Newton's gravitational law, at the galaxy edge, the final velocity v_f is related to the total mass M_g contained within the radial distance r_f by,

$$v_f^2 = \frac{GM_g}{r_f} = \frac{G(M_b + M_d)}{r_f}, \tag{57}$$

where $M_g = M_b + M_d$, where M_d is the total apparent mass due to the graviton energy loss, the mass of the so called "dark matter". The total apparent dark matter in a galaxy is given at the galaxy edge by,

$$M_{d\,obs} = \frac{v_f^2 r_f}{G} - M_b. \tag{58}$$

Using the results from analyzing galaxy rotational data we can estimate the dark matter by (58) as M_d, and substituting that and (57) into (56) we arrive at a formula for estimating K_g as,

$$K_g = \frac{M_{d\,obs}}{2r_f \int_0^{r_f} \left(\dfrac{M(r)}{r^2}\right) \mathrm{d}r}. \tag{59}$$

6. The Einstein Equations for Galaxies with Gravitons

Assuming a spherically symmetric mass distribution for a galaxy we will use the Schwarzschild metric to describe the motions within the galaxy. The Einstein equations for the region outside of a spherical mass distribution is

$$R_{\mu\nu} = 0 \tag{60}$$

where $R_{\mu\nu}$ is the Ricci tensor. The metric element we use is defined in spherical coordinates,

$$\mathrm{d}s^2 = -\mathrm{e}^{\nu} c^2 \mathrm{d}t^2 + \mathrm{e}^{\lambda} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin(\theta) \mathrm{d}\phi^2\right), \tag{61}$$

where

$$\mathrm{e}^{\nu} = 1 - \frac{2GM(r)}{c^2 r}, \tag{62}$$

$$\mathrm{e}^{\lambda} = \left(1 - \frac{2GM(r)}{c^2 r}\right)^{-1}, \tag{63}$$

and where $M(r)$ is the mass within radius *r*. From the metric element (61) the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} -\mathrm{e}^{\nu} & 0 & 0 & 0 \\ 0 & \mathrm{e}^{\lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\theta) \end{pmatrix}. \tag{64}$$

Since the metric (64) does not depend on time t and angles θ and ϕ, it follows from Hamilton's principle and the Lagrange equations[1] that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathrm{e}^{\nu} \frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = 0, \tag{65}$$

implying that the total energy E is

$$\mathrm{e}^{\nu} \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{E}{mc^2}, \tag{66}$$

from which we get

$$\mathrm{e}^{\nu}\left(\frac{c^2 \mathrm{d}t^2}{\mathrm{d}\tau^2}\right) = \mathrm{e}^{-\nu}\left(\frac{E^2}{m^2 c^2}\right). \tag{67}$$

And similarly for the ϕ component, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2 \sin^2(\theta) \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right) = 0, \tag{68}$$

implying that the specific angular momentum is

$$r^2 \sin^2(\theta) \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = h, \tag{69}$$

from which we get,

$$r^2 \sin^2(\theta) \frac{\mathrm{d}\phi^2}{\mathrm{d}\tau^2} = \frac{h^2}{r^2}. \tag{70}$$

For motion in the plane defined by r and ϕ, $\mathrm{d}\theta/\mathrm{d}\tau = 0$, where θ is defined as

$$\theta = \frac{\pi}{2}. \tag{71}$$

Equation of Motion

We derive the equation of motion using the metric element (61), where $\mathrm{d}s^2 = -c^2 \mathrm{d}\tau^2$ where $\mathrm{d}\tau$ is the differential of proper time τ. Dividing the metric element by $\mathrm{d}\tau^2$ and substituting from (66)-(71) with the metric element

[1]Wikipedia: Schwarzshild geodesics, https://en.wikipedia.org/Schwarzschild_geodesics.

we get, dropping the θ term since $\mathrm{d}\theta = 0$,

$$\frac{\mathrm{d}s^2}{\mathrm{d}\tau^2} = -c^2 = -\mathrm{e}^{-\nu}\frac{E^2}{m^2c^2} + \mathrm{e}^{\lambda}\frac{\mathrm{d}r^2}{\mathrm{d}\tau^2} + \frac{h^2}{r^2}, \tag{72}$$

Multiplying (72) by $\mathrm{e}^{-\lambda}$ and simplifying yields

$$\frac{\mathrm{d}r^2}{\mathrm{d}\tau^2} = \mathrm{e}^{-(\nu+\lambda)}\left(\frac{E^2}{m^2c^2}\right) - \mathrm{e}^{-\lambda}\left(c^2 + \frac{h^2}{r^2}\right). \tag{73}$$

Substituting for $\mathrm{e}^{-\lambda}$ from (63) into (73) and, since $\mathrm{e}^{\lambda} = \mathrm{e}^{-\nu}$ from (62) and (63) so that $\mathrm{e}^{-(\nu+\lambda)} = \mathrm{e}^{0} = 1$, we get after simplification,

$$\frac{\mathrm{d}r^2}{\mathrm{d}\tau^2} = \frac{E^2}{m^2c^2} - c^2 + \frac{2GM(r)}{r} - \frac{h^2}{r^2} + \frac{2GM(r)h^2}{c^2r^3}. \tag{74}$$

We assume that the mass $M(r)$ within distance r from the galaxy center is given by the sum of the baryonic mass $M_b(r)$ within r and the mass $M_g(r)$ equivalent to the graviton gravitational redshift energy loss mass within r. The mass density of the baryons is given $\rho_b(r)$. The total mass within r, which is the sum of baryonic mass and gravitonic energy loss mass, is expressed by

$$M(r) = M_b(r) + M_g(r), \tag{75}$$

where the baryonic mass is

$$M_b(r) = \int_0^r 4\pi s^2 \rho_b(s)\,\mathrm{d}s, \tag{76}$$

where ρ_b is the baryonic mass density and from (54) we express the gravitonic energy loss mass in terms of the baryonic mass density by,

$$M_g(r) = 2K_g r \int_0^r \frac{M_b(s)}{s^2}\,\mathrm{d}s, \tag{77}$$

where K_g is a constant coupling coefficient which is peculiar to each galaxy. For circular motion the specific angular momentum term in (74) is

$$\frac{h^2}{r^2} = \frac{GM(r)}{r}. \tag{78}$$

Dropping the constant terms in (74) and substituting with (78), the velocity of a particle in orbit in the galaxy is expressed by

$$v^2 = \frac{\mathrm{d}r^2}{\mathrm{d}\tau^2} = \frac{GM(r)}{r} + 2c^2\left(\frac{GM(r)}{c^2r}\right)^2. \tag{79}$$

7. Results for SPARC Galaxies

We use the velocities from the Spitzer Photometry and Accurate Rotation Curves (SPARC) data base [15] [16], derived from near-infrared (NIR) surface photometry at 3.6 μm. We select the spiral galaxies NGC 2403, NGC 2841 and dwarf galaxies DDO 154 and NGC 2915. From the photometric data which has been reduced to the equivalent velocities for the galaxy bulge, disk and gas mass content, we approximate the baryonic mass as due to a spherically symmetric distribution, which is given by the Newtonian relation for the velocity to mass contained within the radius r from the galaxy center, expressed by

$$M_b(r) = \left(\frac{r}{G}\right)\Big(\left|v_{gas}(r)\right| v_{gas}(r) + \Upsilon_{dsk}(r)\left|v_{dsk}(r)\right| v_{dsk}(r) + \Upsilon_{bul}(r)\left|v_{bul}(r)\right| v_{bul}(r)\Big), \tag{80}$$

where $r = r_i$, $i = 1, 2, \cdots, N$, $N > 1$, N the number of radial distances observed, and the absolute values of the velocities are needed because they can sometimes be negative (Ref. [16], p. 5). The velocities for the disk and gas from **Table 2** of [15] are taken with $\Upsilon_* = 1$. In our **Table 2**, Υ_{dsk} and Υ_{bul} are the $M_\odot/L_\odot$ used in (80) to make the fits. Using the SPARC results for the mass at r in terms of the gas, disk and bulge velocities, where the mass internal to r is given by $M_b(r)$ of (80), the equivalent graviton energy loss mass is

$$M_g(r) = (2K_g r)\sum_{j=1}^{n}\left(\int_{r_{j-(j>1)}}^{r_{j+(j<2)}}\left(\frac{M_b(s)}{s^2}\right)\mathrm{d}s\right). \tag{81}$$

The predicted velocity (79) is expressed in the form

$$v^2(r_n) = \frac{G\big(M_b(r_n) + M_g(r_n)\big)}{r_n} + 2c^2\left(\frac{G\big(M_b(r_n) + M_g(r_n)\big)}{c^2 r_n}\right)^2, \tag{82}$$

where $n = 1, 2, \cdots, N$, $N > 1$.

We strived to obtain good fits to the velocity $v_{obs}(r_k)$ at each radial distance r_k by minimizing the mean absolute error MAE between $v(r_k)$ and $v_{obs}(r_k)$ while iterating $\Upsilon_{dsk}(r_k)$ for the disk and, when available, $\Upsilon_{bul}(r_k)$ for the bulge. We constrained the mass to agree with the baryonic Tully-Fisher relation (BTFR) [17] [18] mass for each galaxy. **Table 1** lists for each galaxy the baryonic mass M_b determined by the velocity profiles used in (80), the BTFR estimated galaxy mass, the mean data rotation velocity error V_{err}, the mean absolute fitted error MAE and the coefficient K_g.

Table 2 lists the average values for disk Υ_{dsk} and bulge Υ_{bul} mass to light ratios which were used in making each galaxy fit. Also listed in the table are minimum and maximum Υ_{dsk} taken from Table 4 and Table 5 of [19] for

Table 1. Results of fits to SPARC galaxy data using the graviton model (82) with masses from (80) and (81). The V_{err} are the mean error of the reported stellar velocities. The MAE errors are the average absolute error for the fits.

Galaxy	M_b ($M_\odot \times 10^{10}$)	† M_b BTFR ($M_\odot \times 10^{10}$)	V_{err} (km·s^{-1})	MAE (km·s^{-1})	K_g
NGC 2403	1.612	1.612	2.421	0.528	0.424
NGC 2841	37.277	37.356	7.67	1.476	0.189
DDO 154	0.06474	‡ 0.02143	0.625	0.225	0.578
NGC 2915	0.2810	0.2799	8.064	0.814	0.653

† Using final velocity as flat velocity. ‡ Rotation curve does not flatten.

Table 2. Results of fits to SPARC galaxy data using the graviton model (82). The columns for Υ_{dsk} and Υ_{bul} are the averages for the disk and bulge mass to light ratios determined by modeling the rotation velocity curve.

Galaxy	Avg Υ_{dsk}	Avg Υ_{bul}	† min Υ_{dsk}	† max Υ_{dsk}
NGC 2403	0.579	0	1.3	1.8
NGC 2841	1.039	0.833	2.0	5.1
DDO 154	6.183	0	1.1	1.2
NGC 2915	1.874	0	na	na

0 in the Υ_{bul} column means the bulge velocity is zero in the data. † Min and max Υ_{dsk} values for high surface mass galaxies from [19]. NA means the result is not available.

comparison. The fitted spiral galaxy disk mass to light ratios are $\Upsilon_{dsk} < 1.0$ except for NGC 2841 which fitted with $\Upsilon_{dsk} < 2.0$. A good reference for fits to SPARC data can also be found in [20], especially for spiral galaxy NGC 2841, where from our **Table 2** we have an average of $\Upsilon_{dsk} = 0.856\ \mathrm{M}_\odot/\mathrm{L}_\odot$ and $\Upsilon_{bul} = 1.254\ \mathrm{M}_\odot/\mathrm{L}_\odot$, each of which is of the same order of magnitude as the fitted values $\Upsilon_{disk} = 0.81 \pm 0.05\ \mathrm{M}_\odot/\mathrm{L}_\odot$ and $\Upsilon_{bul} = 0.93 \pm 0.05\ \mathrm{M}_\odot/\mathrm{L}_\odot$, respectively, from **Figure 1** of [20]. For each plot of the galaxy rotation velocity, the Newtonian velocity curve is also displayed. Plots for the results of the SPARC galaxies can be found in **Figure 2** and **Figure 3**.

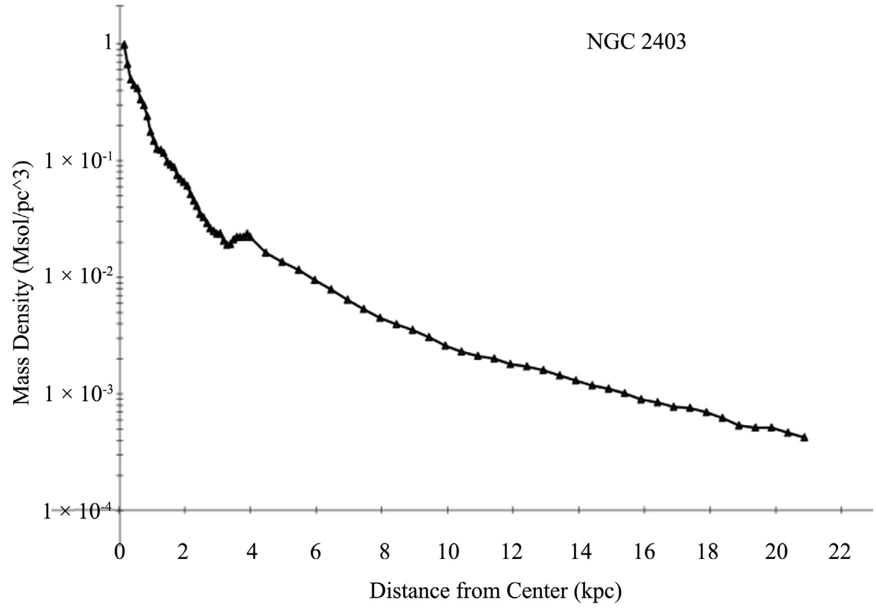
SPARC Galaxies DDO 154 and NGC 2915

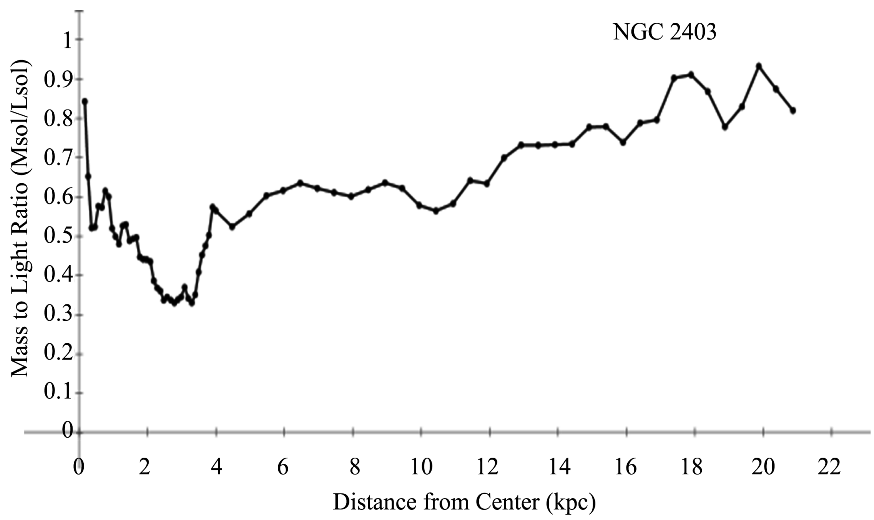
Two dwarf galaxies that have been difficult to understand in terms of missing mass are DDO 154 and NGC 2915. **Figure 3** shows the results for the fits using the graviton model. The upper curve shows the mass density, the middle curve displays the M/L ratios and the lower plot shows the observed and predicted rotation velocities. With the graviton model for DDO 154 we obtained a total baryonic mass of $M_{bary} = 6.47 \times 10^8\,\mathrm{M}_\odot$, which is less than 2× the detected luminous mass [21] of $M_* + M_{HI+He} = 3.65 \times 10^8\,\mathrm{M}_\odot$. Significantly, it is only one-fifth of the estimated dark plus luminous matter of $M_{dark+lum} = 3.1 \times 10^9\,\mathrm{M}_\odot$. Our fitted disk mass to light ratios have an average of $\Upsilon_{dsk} = 6.183\ \mathrm{M}_\odot/\mathrm{L}_\odot$ which is 3× the published value of $\Upsilon_{dsk} \approx 2\ \mathrm{M}_\odot/\mathrm{L}_\odot$.

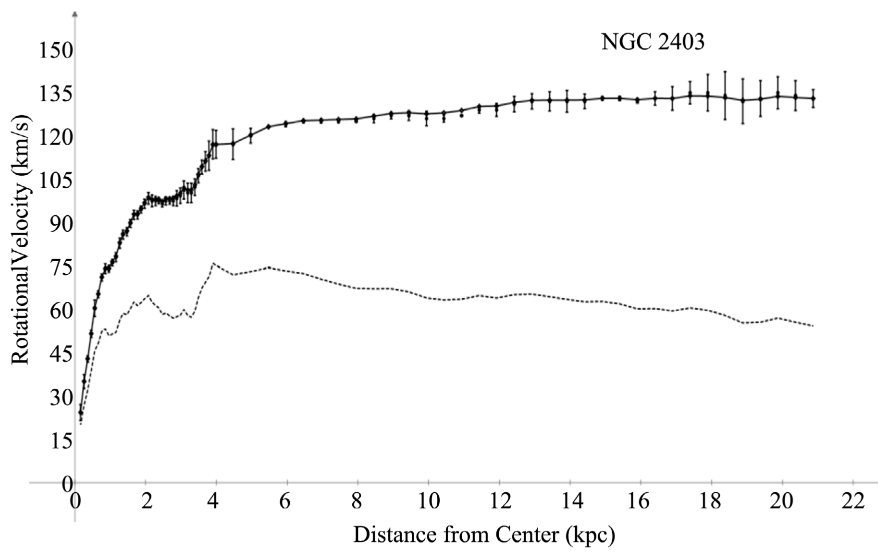
For NGC 2915 we obtained a total baryonic mass of $M_{bary} = 2.81 \times 10^9\,\mathrm{M}_\odot$, where the stellar mass and luminous HI mass [22] [23] $M_{HI} + M_* = 1.06 \times 10^9\,\mathrm{M}_\odot$. This compares with the total dynamical mass $M_T = 21 \times 10^9\,\mathrm{M}_\odot$ which is 7× the total baryonic mass derived by the graviton model.

8. Discussion

Regarding the baryon acoustic oscillations (BAO) in the primordial plasma and the resulting cosmic microwave background radiation (CMB), if the energy loss in the gravitational field due to the gravitational redshift of graviton energy is the cause of the apparent dark matter in the universe, implying that dark matter


NGC 2403
Mass Density (Msol/pc^3)
1
1 × 10-1
1 × 10-2
1 × 10-3
1 × 10-4
0 2 4 6 8 10 12 14 16 18 20 22
Distance from Center (kpc)


NGC 2403
Mass to Light Ratio (Msol/Lsol)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 2 4 6 8 10 12 14 16 18 20 22
Distance from Center (kpc)


NGC 2403
RotationalVelocity (km/s)
150
135
120
105
90
75
60
45
30
15
0
0 2 4 6 8 10 12 14 16 18 20 22
Distance from Center (kpc)

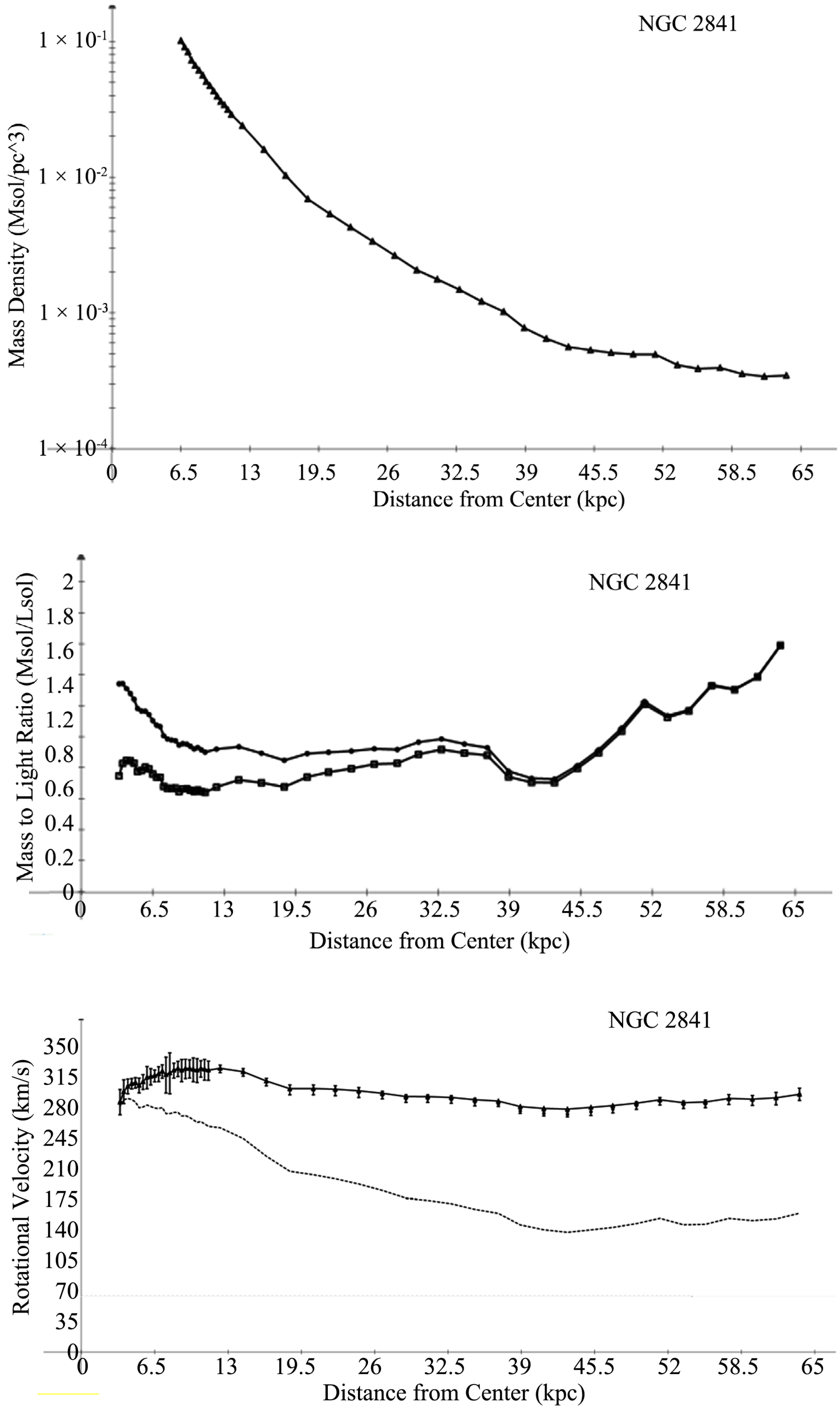
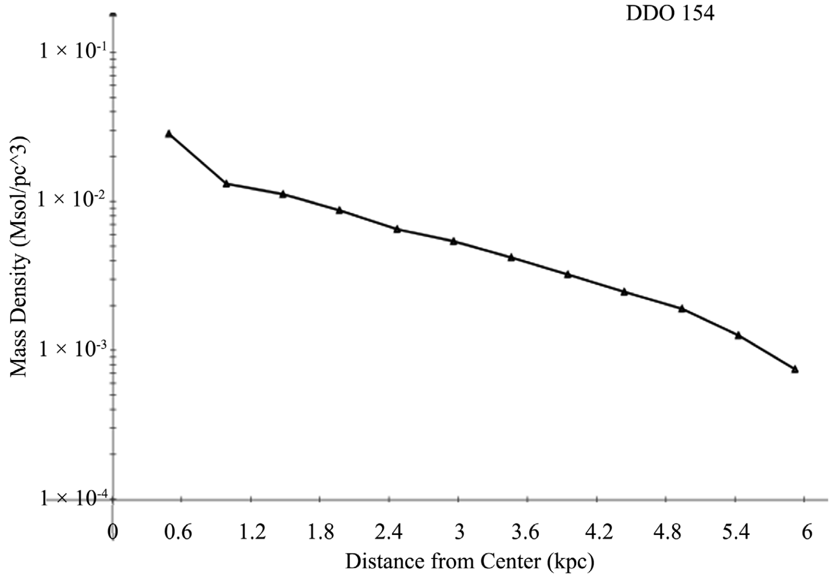
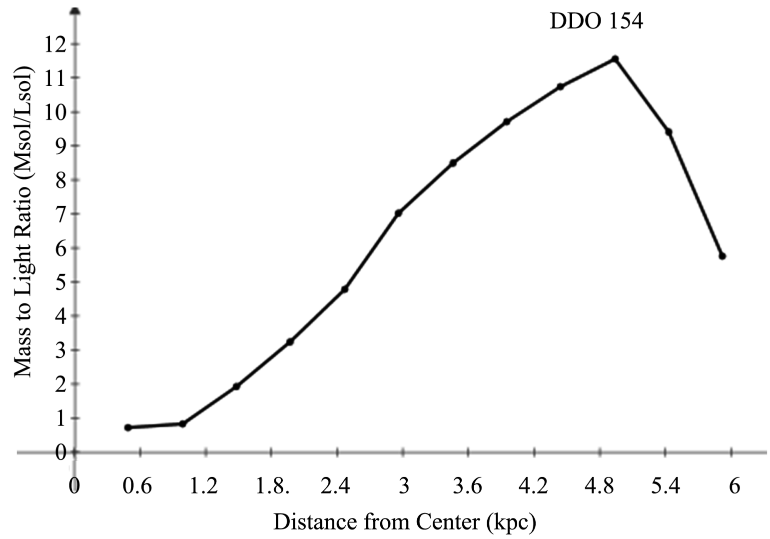
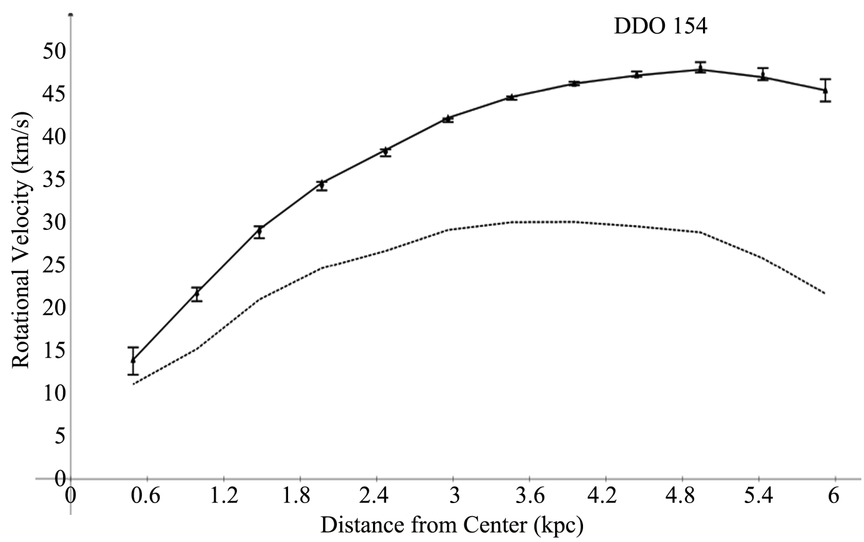


Figure 2. NGC 2403 and NGC 2841. Fits with SPARC data, the masses derived from (80) and (81) with velocity profiles for gas, disk and bulge. For mass to light curves, Υ_{dsk} is the solid line and filled circles, Υ_{bul} is the solid line and open squares. For rotation velocity curves, the solid line is the minimised fit to the data with the graviton model (82) and the dashed line is the Newtonian velocity. Top Half: Upper: Mass density. Middle: Mass to light ratio. Lower: The model velocity vs. radial distance from the galactic center and the Newtonian velocity. Bottom Half: Upper: Mass density. Middle: Mass to light ratio. Lower: The model velocity vs. radial distance from the galactic center and the Newtonian velocity.


DDO 154
Mass Density (Msol/pc^3)
1 × 10-1
1 × 10-2
1 × 10-3
1 × 10-4
0
0.6
1.2
1.8
2.4
3
3.6
4.2
4.8
5.4
6
Distance from Center (kpc)


DDO 154
Mass to Light Ratio (Msol/Lsol)
12
11
10
9
8
7
6
5
4
3
2
1
0
0
0.6
1.2
1.8.
2.4
3
3.6
4.2
4.8
5.4
6
Distance from Center (kpc)


DDO 154
Rotational Velocity (km/s)
50
45
40
35
30
25
20
15
10
5
0
0
0.6
1.2
1.8
2.4
3
3.6
4.2
4.8
5.4
6
Distance from Center (kpc)

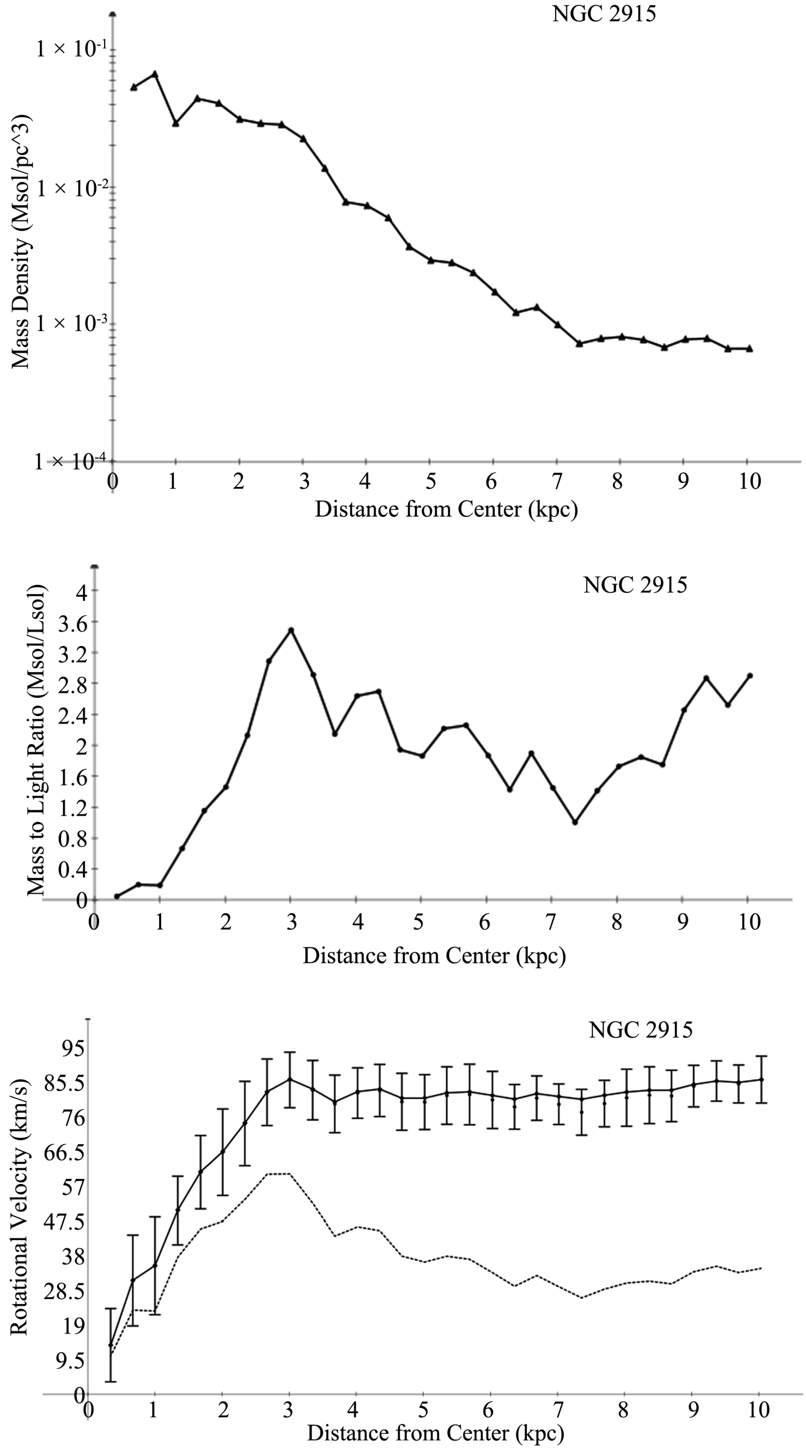


Figure 3. DDO 154 and NGC 2915. Fits with SPARC data, the masses derived from (80) and (81) with velocity profiles for gas, disk and bulge. For mass to light curves, Υ_{dsk} is the solid line and filled circles. For rotation velocity curves, the solid line is the minimised fit to the data with the graviton model (82) and the dashed line is the Newtonian velocity. Top Half: Upper: Mass density. Middle: Mass to light ratio. Lower: The model velocity vs. radial distance from the galactic center and the Newtonian velocity. Bottom Half: Upper: Mass density. Middle: Mass to light ratio. Lower: The model velocity vs. radial distance from the galactic center and the Newtonian velocity.

particles do not exist, then in density perturbations of the primordial plasma just before recombination, the baryons, photons and gravitons are synchronized and there is not a central region of dark matter particles as theorized in the standard model [24], so that when recombination begins the photons dissociate from the electrons and the resultant hydrogen atoms begin to attract and form dense regions in the perturbation. Just how the graviton redshift energy loss affects the density formations in the primordial plasma and how this affects the determination of the Hubble constant from the BAO and CMB modeling [25] is an area which needs to be studied.

Regarding galaxy dynamics, with the modified Newtonian dynamics (MOND) [26] [27] there may be a way to compare it with the graviton gravitational redshift theory. MOND takes effect in galaxies at a distance where the central acceleration is around 1.2×10^{-10} m·s^{-2} by effecting a transition in the gravitational attractive force from a r^{-2} to a r^{-1} form. On the other hand, the graviton gravitational redshift theory augments the Newtonian rotational velocity beginning from the galactic center, which allows to distinguish it from MOND. Referring to **Figure 2**, at a radius of $r = 24.59\ \text{kpc}$ the spiral galaxy NGC 2841 has an observed velocity of $V = 298.0\ \text{km}\cdot\text{s}^{-1}$ and we calculate the Newtonian velocity of $V_N = 191.5\ \text{km}\cdot\text{s}^{-1}$ using the SPARC data. The radial acceleration at this distance is $Acc = V^2/r = 1.171\times10^{-10}\ \text{m}\cdot\text{s}^{-2}$ which is at the beginning of the MOND regime and, assuming we have the correct baryon mass within that radius, MOND would predict an orbital velocity smoothly connected to the Newtonian value V_N but we see that it would be smaller than V by 106.5 km·s^{-1}.

Another interesting alternative approach to modified Newtonian dynamics is given in [28] which, by way of an additive inverse Yukawa-like term to Newtonian gravitation, purports to account for gravitational dynamics from solar systems to galaxies and galaxy clusters and to the large scale universe expansion. There is an analogy between (Ref. [29] Equation (19)) and our (54) which can to be further explored.

9. Conclusions

The graviton self interaction model describes the effect of gravitons in free fall in the gravitational field, losing energy by way of gravitational redshift and cosmological redshift without emitting any radiation. General Relativity was applied for both the universe expansion and for spiral galaxy rotation curves. By assuming a coupling coefficient K_{dm} for the graviton redshift in free fall, the graviton model can account for the apparent dark matter in the universe being related to the baryon density. Likewise, the dark energy depends on the coupling constant σ_{de} and the baryon density. Thus, the apparent dark matter and dark energy are replaced by two constants and the hypothesized redshift of graviton energy.

In the case of the rotational characteristics of spiral galaxies, the graviton theory well explains the greater than expected galaxy rotational velocities in the SPARC data with only the baryonic mass derived from the gas, disk and bulge velocity data with fitted Υ_* ratios, with a galaxy dependent coupling coefficient K_g and with the total baryon mass conforming to the BTFR.

It is apparent that the fundamental aspect of graviton redshift points to the need to include this in the General Relativity field equations.

Conflicts of Interest

The author declares no conflict of interest.

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