Single Charged Particle Motion in a Flat Surface with Static Electromagnetic Field and Quantum Hall Effect

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Abstract

Taking into account the non separable solution for the quantum problem of the motion of a charged particle in a flat surface of lengths $x_L$ and $y_L$ with transversal static magnetic field $B$ and longitudinal static electric field $E$, the quantum current, the transverse (Hall) and longitudinal resistivities are calculated for the state $n=0$ and $j=0$. We found that the transverse resistivity is proportional to an integer number, due to the quantization of the magnetic flux, and longitudinal resistivity can be zero for times $t \gg L_xB/cE$. In addition, using a modified periodicity of the solution, a modified quantization of the magnetic flux is found which allows to have IQHE and FQHE of any filling factor of the form $\nu = k/l$, with $k,l \in \mathbb{Z}$.

Keywords

Landau’s Gauge, Quantum Hall Effect, Degeneration

1. Introduction

There are a lot of literature dealing with the phenomenon of Quantum Hall Effect [1]-[8], and most of them use the Landau’s solution of the eigenvalue problem associated to the charged particle motion in a flat surface with static transversal magnetic field to the surface. This brings about the known Landau’s levels for the energies and a separable variable solution for the eigenfunctions [9]. However, it has been shown that a non separable of variables solution exists for this problem with the same Landau’s levels [10] [11], and these levels are numerable degenerated [12], determining the operators which causes this degeneration. In addition, the quantization of the magnetic flux appears naturally [10],
where \( m \) is the mass of the charge \( q \), \( c \) is the speed of light, \( \omega_c \) is the so called cyclotron frequency, \( B \) is the magnitude of the static magnetic field, \( A = L_x L_y \) is the area of the sample, and \( 2\pi\hbar = h \) is the Planck’s constant. As we mentioned before, Landau’s separable solution is normally used to try to explain the so called Integer Quantum and Fractional Quantum Hall Effects (IQHE and FQHE) [4] [5] [6] [7], which were first discovered experimentally [1] [2] [3]. The IQHE is normally explained as a single particle phenomenon; meanwhile, the FQHE is explained as a many particle event [4] [5] [6]. Experimentally, both of them occur in highly impure samples, where these impurities have the effect of extending the range of magnetic field intensity where the resistivity is quantized [2] [3] [7]. The main characteristic of the IQHE or FQHE is the resistivity (or voltage) which appears on the transverse motion of the charges, so called Hall’s resistivity \( \rho_H \). This Hall’s resistivity acquires a constant value on certain regions of the magnetic field, and within these regions, the longitudinal resistivity is zero. The values of these constant \( \rho_H \) turn out to be inverse to an integer number (IQHE) or proportional to an integer number (FQHE) multiplied by the constant \( h/q^2 \), called von Klitzing constant [2] [3] (\( h/q^2 \approx 25812.80745 \Omega \)). In this paper, we calculate the quantum current and the expected value of the transverse and longitudinal resistivities for a single charged particle motion on a flat surface using the non separable solution in the lowest Landau level (\( n = 0 \)) and using the first wave function (\( j = 0 \)).

2. Quantum Current

The Hamiltonian associated to the motion of a charge particle \( q \) with mass \( m \) on a flat surface of lengths \( L_x \) and \( L_y \) with transverse magnetic field \( B = (0,0,B) \) and longitudinal electric field \( E = (0,E,0) \) is given by

\[
\hat{H} = \frac{1}{2m} \left( \mathbf{p} - \frac{q}{c} \mathbf{A} \right)^2 + qV,
\]

where \( \mathbf{A} \) is the vector potential, \( \mathbf{B} = \nabla \times \mathbf{A} \), and \( V \) is the scalar potential, \( E = -\nabla V \). The Schrödinger’s equation,

\[
\frac{i\hbar}{\partial t} \Psi = \hat{H}\Psi,
\]

can be written, using the operator \( \mathbf{p} = -i\hbar \nabla \), as

\[
\frac{i\hbar}{\partial t} \Psi = \frac{1}{2m} \left[ -\hbar^2 \nabla^2 + i\frac{\hbar q}{c} (\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla) + \frac{q^2 A^2}{c^2} \right] \Psi + qV \Psi.
\]

Taking the usual complex conjugated to this expression, a similar equation is gotten for the function \( \Psi^* \). Multiplying this one by \( \Psi \), (4) by \( \Psi^* \) and subtracting both, the following continuity equation is obtained
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot J = 0, \]  

(5)

where \( \rho \) and \( J \) are defined as

\[ \rho = \Psi \cdot \Psi^* \]  

(6)

and

\[ J = \frac{i\hbar}{2m} (\Psi \nabla \Psi^* - \Psi^* \nabla \Psi) - \frac{q}{mc} \rho A. \]  

(7)

Since \( \Psi \) is a scalar complex function, it can be written as \( \Psi = |\Psi|e^{i\phi} \), where \( |\Psi| \) and \( \phi \) are real functions, and \( \phi \) is the argument of the function. Then, the current is given by

\[ J = \left( \frac{\hbar}{m} \nabla \phi - \frac{q}{mc} A \right) |\Psi|^2. \]  

(8)

For the general solution of (3), the function \( \phi \) can be very complicated expression of all variables. However, for a particular state solution of the system, say

\[ \psi_n(x,t) = e^{i\phi_n(x,t)}f_n(x), \]  

(9)

the argument is just \( \phi = \phi_n(x,t) \), and the current associated to this state of the system is given by

\[ J_n = \left( \frac{\hbar}{m} \nabla \phi_n - \frac{q}{mc} A \right) |f_n|^2. \]  

(10)

3. Single Charged Particle Current

The non separable solution of (3) using the Landau’s gauge \( A = B(-y,0,0) \) and the longitudinal constant electric field \( E = (0,E,0) \) was given as

\[ f_n^0 = \frac{1}{\sqrt{2^{n+1}n!L_z}} \left( \frac{m\omega_c}{\pi \hbar} \right)^{1/4} e^{i\phi_n} e^{-\frac{m\omega_c}{2\hbar}(x-cE t/B)^2} H_n \left( \sqrt{\frac{m\omega_c}{\hbar}}(x-cE t/B) \right), \]  

(11a)

where \( E = qE, \omega_c \) is the cyclotron frequency (1), and \( \phi_n \) is given by

\[ \phi_n = -\left[ \hbar \omega_c \left( n + \frac{1}{2} \right) \frac{mc^2E}{2B^2} - \frac{m\omega_c}{\hbar} \left( x - \frac{cEt}{B} \right) \right] \left( y - \frac{mc^2E}{qB^2} \right). \]  

(11b)

These functions are degenerated in the sense that for each Landau’s level \( (\hbar \omega_c (n+1/2)) \), one has a numerable solutions \( f_n' = \{ \hat{p}_j \} f_n^0, j \in \mathbb{Z} \). Thus, the expressions (11a) define the state of the system. Using this function \( \phi_n \) in (10) and for the index of degeneration \( j = 0 \), we have

\[ J_n = \left[ \frac{cE}{B} \hat{i} - \omega_c \left( x - \frac{cEt}{B} \right) \hat{j} \right] |f_n^0|^2. \]  

(12)

In particular, for the ground state of Landau’s energy, it follows that the components of the current are
\[ J_o^x = \frac{e}{B} |f_o^x|^2, \]

and

\[ J_o^y = -\omega \left( x - \frac{cE_t}{B} \right) |f_o^y|^2. \]

The electric conductivity along the x-axis is called Hall’s conductivity and is given by

\[ \sigma_H = \frac{q}{e} J_o^x = \frac{q_c}{B} |f_o^x|^2. \]

Thus, the Hall’s resistivity is \( \rho_H = 1/\sigma_H \), and the expected value of the resistivity in the state \( f_o^x \) is

\[ \langle f_o^x | \rho_H | f_o^x \rangle = \int_0^{L_y} \int_0^{L_y} \frac{|f_o^x|^2}{\sigma_H} \, dx \, dy = \frac{BA}{qc}. \]

Now, multiplying and dividing this quantity by \( m_{ao}/\hbar \) and making some rearrangements, one gets

\[ \langle f_o^x | \rho_H | f_o^x \rangle = \frac{\hbar}{q^2} \left( \frac{m_{ao}}{\hbar} A \right), \]

and taking into consideration the magnetic field flux quantization (1), it follows that

\[ \langle f_o^x | \rho_H | f_o^x \rangle = \frac{\hbar}{q^2} l, \quad l \in \mathbb{Z}. \]

The expected value in the state \( f_o^x \) of the longitudinal resistivity \( \rho_y \) is

\[ \langle f_o^x | \rho_y | f_o^x \rangle = \int_0^{L_x} \int_0^{L_y} \frac{|f_o^x|^2}{\sigma_y} \, dx \, dy = \frac{E}{q^2} \int_0^{L_x} \int_0^{L_y} \frac{|f_o^x|^2}{J_o^y} \, dx \, dy \]

\[ = \frac{E}{q^2 \omega_c} \int_0^{L_x} \int_0^{L_y} \frac{dx \, dy}{c \frac{E_t}{B}} = -\frac{E}{q^2 \omega_c} \ln \left( 1 - \frac{L_x \cdot B}{cE_t} \right) \approx 0 \]

since one has normally in the experiments that \( L_x \cdot B/cE_t \ll 1 \), that is, the time in the experiments are such that

\[ t \gg \frac{L_x \cdot B}{cE} . \]

For example, on the reference [2] and with respect the voltage gate \( V_g \), one has that \( BL_x/cE = BA/cV_g \approx 4.5 \times 10^{-9} \text{ sec} \). So, the condition (21) is well satisfied in this experiment.

Note that the expression (18) implies a filling factor \( \nu = 1/l \), which correspond to the IQHE phenomenon for \( l=1 \) and to the FQHE phenomenon for \( l>1 \). However, this result is valid for an analysis of a single charged particle, and both QHE phenomena appear due to the quantization of the magnetic flux (1). In addition, one must note that this analysis is still valid for any \( n>0 \) and \( j=0 \).
4. Full IQHE and FQHE

The quantization of the magnetic flux (1) arises from the periodicity of the solutions of the Hamiltonian [10], which can be expressed using (11a) for \( \mathcal{E} = 0 \) as

\[
f^0_n(L_x, y + L_y, t) = f^0_n(L_x, y, t).
\]

(22)

However (and also for \( \mathcal{E} = 0 \)), let us assume that \( L_x = N l_y \) where \( l_y \ll L_y \) and \( N \in \mathbb{Z}^+ \), that is, the total area \( L_x L_y \) is covered with slices of area \( L_y l_y \), with horizontal length \( L_x \) and width \( l_y \). Let us impose the periodicity condition of the form

\[
f^0_n(L_x, y + k l_y, t) = f^0_n(L_x, y, t), \quad k \in \mathbb{Z},
\]

(23)
such that with the phase (11b), one gets

\[
\frac{m o}{\hbar} L_y k l_y = 2\pi l, \quad l \in \mathbb{Z}
\]

(24)

which brings about the relation

\[
\frac{m o}{\hbar} a = 2\pi \frac{l}{k}, \quad \text{with } a = L_y l_y.
\]

(25)

Using (1) and making some rearrangements, the magnetic field can be given by

\[
B = \alpha \frac{l}{k}, \quad \text{with } \alpha = \frac{\hbar c}{qa}
\]

(26)

and using (25) in (17), the expected value of the Hall resistivity would be

\[
\left< f^0_n \left| \rho_y \right| f^0_n \right> = \frac{\hbar}{q^2} \frac{l}{k}, \quad k, l \in \mathbb{Z},
\]

(27)

implying now a filling factor of \( \nu = k/l \), which represents the full IQHE (for \( l = 1 \)) and FQHE (for \( l > 1 \)). To determine the magnetic values \( B \) where these phenomena occur, one looks for the value \( B_0 \) where the first IQHE \( (l = k = 1) \) appears, which intersect the normal linear dependence behavior straight line, and this defines \( \alpha = B_0 \). Then, one uses the resulting expression

\[
B = B_0 \frac{l}{k}
\]

(28)

to find the other quantized magnetic fields which correspond to IQHE or FQHE.

For example, on the experimental data shown on the reference [3], one sees that \( B_0 \approx 5.5 \text{ T} \) for \( l = k = 1 \) (corresponding to an area \( a \approx 8.27 \times 10^{-4} \mu \text{m}^2 \)), and the other FQHE are matched quite well for \( l = 3 \) and \( k = 1 \), that is \( B \approx 15 \text{ T} \). Another example is shown on the reference [8] page 886, one sees that \( B_0 \approx 9.8 \text{ T} \) for \( l = k = 1 \) (corresponding to an area \( a \approx 4.22 \times 10^{-4} \mu \text{m}^2 \)), and the other IQHE and FQHE magnetic fields are matched quite well for \( l > 1 \) and \( k > 1 \). In addition, on reference [13] page 207, one sees that \( B_0 \approx 4.2 \text{ T} \) for \( l = k = 1 \) (corresponding to an area \( a \approx 9.85 \times 10^{-4} \mu \text{m}^2 \)), and the other IQHE and FQHE magnetic fields are matched quite well for \( l > 1 \) and \( k > 1 \). Finally, on reference [14] page 156801-2, one sees that \( B_0 \approx 5.3 \text{ T} \) for \( l = k = 1 \) (correspond-
ing to an area \( a = 7.8 \times 10^{-4} \text{ \mu m}^2 \), and for the filling factor \( \nu = 3/4 \) one gets \( B = 4B_0/3 = 7.06 \text{ T} \), which is approximately the experimental value reported.

5. Conclusion

Using the known non-separable solution for the quantum motion of a charged particle in a flat surface with static fields, in the state \( n = 0 \) and \( j = 0 \), the Hall and the longitudinal resistivities were calculated. For the quantization of the magnetic flux, which can appear from the simple periodicity on the \( y \)-direction, the results bring about the IQHE and FQHE phenomena since from the expression (18) it appears a filling factor of \( 1/l \) for a single charged particle due to the quantization of the magnetic flux. If \( l = 1 \), one gets the IQHE phenomenon, and if \( l > 1 \), one gets the FQHE phenomenon. However, it is not possible to say anything about filling factors of the form \( \nu = k/l \). For a more extended quantization of the magnetic flux (25), which appears of the extended periodicity (23), one gets also IQHE and FQHE but with a filling factor of \( \nu = k/l \).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References


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