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Non-Linear Effects in Optical Systems by Lie Algebra and Symplectic Mapping

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Abstract

The use of signals of different frequencies determines the geometrical deviation with respect to the optical axes of a given beam. This angle can be determined by Sympletic Map (SM), a powerful and simple mathematical tool for the characterization and construction of images in Geometrical Optics. The Sympletic Map constitutes a Lie Group, with an algebra associated: the Lie Algebra. In general, the SM can be expressed as an infinite series, where each term corresponds to different contributions produced by the optical devices that constitute the optical system (lenses, apertures, bandwidth cutoff, etc.). The level of correction to be performed on the image to recover the original object is clear and controllable by SM. This formalism can be extended easily to physical optics to describe diffraction and interference phenomena.

Keywords

Symplectic Mapping, Geometrical Optics, Non-Linear Effects

1. Introduction

Geometrical optics is perhaps one of the oldest branches of physics. The observation of propagation and refraction of light date back to the ancient Greek world. However, it was during the XVII century when Snell and Descartes turned those empirical observations into physical laws. The principles of geometrical optics can be stated as a variational principle (Fermat) and the corresponding equations, that describe the geometrical properties of the light beams when passing through different media, can be then stated. The dependence on frequency in a specific medium determines the tilting angle of the propagating beam with respect to the optical axis.

To determine that deviation from the optical axis, the standard approach is by using the Hamiltonian. In this function, the refraction index can be made dependent on the medium and signal frequency. In most standard cases, the medium is assumed isotropic and linear.

For a more complex system, which consists of a Hamiltonian function dependent on the frequency by means of the refractive index, there exist several transformations, namely the Canonical Transformations (CT) dependent on frequency, which, when applied to the Hamiltonian, leave this unchanged, that is, the Hamiltonian is invariant to canonical transformations [1] [2].

The CT can be written as a Lie series of the Lie Transformation, where the first term corresponds to the Hamiltonian's Poisson Brackets of the dynamical variable of the system under study [1] [2]. The CT, Poisson Brackets and Lie Algebra constitute what is called the Symplectic Map (SM), a concept which allows to solve a wide number of problems in science and technology.

2. Geometrical Optics

Let us assume a medium characterized by a local refraction index $n(q)$, being

$$\boldsymbol{q}(s)=\left(q_x(s),q_y(s),q_z(s)\right)$$

the position vector relative to an arbitrary frame of reference, and s the arc length, defined in the usual way:

$$\mathrm{d}s=\sqrt{\mathrm{d}q_x^2+\mathrm{d}q_x^2+\mathrm{d}q_x^2}$$

with the condition

$$\left|\mathrm{d}\boldsymbol{q}(s)/\mathrm{d}s\right|=1 \tag{1}$$

which corresponds to the inextensibility condition. From differential geometry, the tangent vector can be written as:

$$\boldsymbol{p}=n\left(\mathrm{d}\boldsymbol{q}(s)/\mathrm{d}s\right) \tag{2}$$

This vector is tangent to the light beam and it is used to indicate any deviation with respect to the optical axis.

$$|\boldsymbol{p}|=n(\boldsymbol{q})$$

Also, this tangent vector corresponds to the momentum vector p in the Descartes' sphere [3] [4] [5] where, in each point the relation. The Hamilton equations state [3] [4] [5]:

$$\mathrm{d}p_i/\mathrm{d}z=\dot{p}_i=-\partial H^{opt}/\partial q_i\ ;\ \ \mathrm{d}q_i/\mathrm{d}z=\dot{q}_i=-\partial H^{opt}/\partial p_i\ ;\ \ i=1,2 \tag{3}$$

is satisfied. This sphere is built in such a way that the Snell-Descartes Law means a conservation of the tangent component of the momentum of the light along the surface between two media characterized with two different refractive index, where the optical Hamiltonian takes the form:

$$H^{opt}=-\sqrt{n(\boldsymbol{q}z)^2-p^2} \tag{4}$$

Here "z" indicates the optical axis, then the derivatives respect to z indicates deviations from the optical axis.

These equations can also be obtained by using Fermat's principle, which is essentially a variational principle, where the distance traveled by a light beam to go, along the optical path, from one medium to another (with different refraction indexes) is minimized. By using Hamilton equations it is possible to obtain the deviation suffered by a light beam passing from one medium to another.

The Hamilton equations can be expanded in a power series in $\boldsymbol{p}$ to obtain:

$$H^{opt} = H^{paraxial} + O\left(\left(p^2\right)^2\right) = p^2/2n_0 - n(\boldsymbol{r}, z) + O\left(\left(p^2\right)^2\right) \tag{5}$$

where $n_0 = n(0,z) = \text{constant}$.

This approximation is good when $|\boldsymbol{q}| \ll 1$ and $p^2 \ll n^2$; this means that the light beam is near to the optical axis (z axis). As it can be noticed, these equations are very similar to the corresponding equation for mechanical systems, since by just exchanging z by the time t, one obtains Newton's equations [3].

3. Hamiltonian Systems, Poisson Brackets and Canonical Transformation

For a mechanical system, the Hamiltonian $H(\boldsymbol{p}, \boldsymbol{q}, t)$ describes the time evolution of the system. The Hamiltonian $H(q, p, t)$ is a $2n$ variables function q_i; p_i; $i = 1, 2, \cdots, n$. $H : R^{2n} \to R$ with the dynamical equations [1] [2]:

$$\dot{p}_i = -\partial H^{opt}/\partial q_i\ ;\ \ \dot{q}_i = -\partial H^{opt}/\partial p_i\ ;\ \ i = 1, 2, \cdots, n \tag{6}$$

The solution of Hamilton equations allows to describe the trajectory of the system in phase space ($\boldsymbol{q}(t)$, $\boldsymbol{p}(t)$). These trajectories provide the position and momentum of the system as a function of time [1] [2].

The Poisson Brackets (PB) allow a simple representation of the dynamical equations of the system. These are defined as:

$$\{u, v\} \equiv \sum_{j=1}^{n}\left(\left(\partial u/\partial q_j\right)\left(\partial v/\partial p_j\right) - \left(\partial u/\partial p_j\right)\left(\partial v/\partial q_j\right)\right) \tag{7}$$

being n the number of degrees of freedom. With this definition, two variables are called canonical conjugated, provided they satisfy:

$$\{q_i, q_j\} = 0\ ;\ \ \{p_i, p_j\} = 0\ ;\ \ \{q_i, p_j\} = \delta_{ij} \tag{8}$$

In this notation Hamilton's equations look like [1] [2]:

$$\{q_i, H\} = \dot{q}_i\ ;\ \ \{p_i, H\} = \dot{p}_i \tag{9}$$

In order to solve specific physical problems, a coordinate system has to be specified. The choice of the coordinate system is important because an appropriated election produce what is called "cyclic coordinates" (coordinates that do not appear explicitly in the Hamiltonian) [1] [2]. The existence of these cyclic coordinates assures symmetries and conservation laws for the corresponding variables. Even when any coordinate system can be used to solve the problem, the election of a one who possess the largest number of cyclic coordinates, reduces significantly the difficulty of finding the solution. The Hamilton-Jacobi method deals with this matter [1] [2].

It is possible to find a transformation between coordinate systems to obtain the adequate system to describe the problem. However, to find such transformation is not, generally, a simple task. From the large number of possible transformations, the canonical transformation deserves special attention because they leave invariant the Hamiltonian.

4. Lie Transformation and Lie Series

The Lie derivative of an analytical function f(q, p, t) associated with a Hamiltonian is defined as [6] [7]:

$\mathrm{d}f/\mathrm{d}t = \{f, H\}$ and, in general $\mathrm{d}^k f/\mathrm{d}t^k = \{\{\{\cdots k \text{ times} \cdots \{\{f, H\}\}\cdots\}\}\}$ (10)

The function $f(\boldsymbol{q}, \boldsymbol{p}, t)$ can be expanded in a Taylor series around $t = 0$ to obtain:

$$f(\boldsymbol{q}, \boldsymbol{p}, t) = f(\boldsymbol{q}, \boldsymbol{p}, 0) + t\{f, H\}_{t=0} + (t^2/2!)\{\{f, H\}, H\} + \cdots \quad (11)$$

This expansion is usually called "Lie Series" of the function f. The n^{th} order term contains the n successive application, from the right, of the operator $\{\cdots, H\}$ which can be written as the operator to the n^{th} power:

$$f(\boldsymbol{q}, \boldsymbol{p}, t) = \left[f e^{t\{\cdot, H\}} \right]_{t=0} \quad (12)$$

Equation (13) is called Lie Transformation [7] [8]; this transformation is usually used in quantum and classic mechanics to describe the dynamics of physical systems.

5. Lie Algebra

The Lie Algebra (LA) is a non-commutative algebra where the product is defined as:

$$\{f, g\} \equiv \sum_{i=1}^{n} \left((\partial f/\partial q_i)(\partial g/\partial p_i) - (\partial f/\partial p_i)(\partial g/\partial q_i) \right) \quad (13)$$

Let f, g, and h three functions of (q, p), then the following properties are fulfilled:

- Anti-symmetry: $\{g, h\} = -\{h, g\}$ and consequently $\{f, f\} = 0$
- Linearity: if a and b are constants, then $\{f, ag + bh\} = a\{f, g\} + b\{f, h\}$
- Product: $\{f, gh\} = \{f, h\}g + \{f, g\}h$
- Jacoby's Identity: $\{f, \{g, h\}\} = \{g, \{h, f\}\} = \{h, \{f, g\}\} = 0$

From this algebra, it is possible to show that there exists a sub-algebra called "Lie Differential Operator Algebra" defined as: $:f:g = \{f, g\}$. It is easy to show that this is a well-behaved algebra. The commutator between two operators is defined as the anti-symmetric operator: $[:f:, :g:] = :f::g: - :g::f:$

From this definition it is easy to prove that:

1) $[:f: + :g:, :h:] = [:f:, :h:] + [:g:, :h:]$

2) $[:f:,:g:+:h:] = [:f:,:g:] + [:f:,:h:]$

3) $a[:f:,:g:] = [a:f:,:h:] + [:f:,a:h:]$

4) $[:f:,:f:] = 0$

5) $[:h:,[:f:,:g:]] + [:f:,[:g:,:h:]] + [:g:,[:h:,:f:]] = 0$

The following theorem concerns to the well-behaved character of the LT [7]:

Theorem: Let two analytical functions f and g of (q, p), the function $G = G(\boldsymbol{q}, \boldsymbol{p}, t)$ can be written as $G = e^{(t:f:)}g = e^{(t\{f,\cdot\})}g$, where G is a real analytical function of (q, p) when the parameter t is sufficiently small.

6. Lie Group

As it is known, a group is a set of elements that fulfill the following requirements [9]:

1) Let a and b two elements of the group, then $a \otimes b$ belongs to the group.

2) The multiplication $\otimes$ is associative: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$.

3) The group contains the element identity e such that: $a \otimes e = e \otimes a$.

4) Each element a of the group has an inverse such that: $a \otimes a^{-1} = a^{-1} \otimes a = e$.

We shall restrict ourselves to continuous groups where the number of parameters is finite. Each of these parameters can vary continuously. Particularly, we are interested in the canonical transformation that forms a Lie Group.

$$Q_i = Q_i(\boldsymbol{q}, \boldsymbol{p}), \quad P_i = P_i(\boldsymbol{q}, \boldsymbol{p}) \tag{14}$$

In this particular case, this transformation connects two different coordinate systems of finite dimensionality. As known, the CT leaves invariant the Hamiltonian of the system, a fundamental requirement to leave unchanged the physical system. The CT can be written as a column vector of dimensionality n. The Lie group is isomorphic to a sub-group GL(n, R).

7. Symplectic Group

The isomorphism between the Lie group and GL(n, R) allows to characterize the CT by using the Symplectic Group (SG). The isomorphism between them allows to obtain a 2n-dimensional matrix. In order to obtain this matrix it is necessary to define the following vectors:

$$\zeta = (P_1, \cdots, P_n, Q_1, \cdots, Q_n), \quad \eta = (p_1, \cdots, p_n, q_1, \cdots, q_n) \tag{15}$$

where Q_i and P_i are the end conditions, and q_i and p_i are the initial conditions.

With these definitions, the symplectic matrix can be written as [6] [7] [10] [11] [12] [13] [14]:

$$M_{ij} = \partial\zeta_i / \partial\eta_j \tag{16}$$

It is easy to prove that this matrix fulfills the symplectic conditions:

$$\boldsymbol{M}^t \boldsymbol{J} \boldsymbol{M} = \boldsymbol{J} \quad \text{where} \quad \boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{17}$$

where 1 and −1 are block matrices of $n \times n$ and the determinant of M is:

$$\det(\boldsymbol{M}) = 1 \tag{18}$$

The infinitesimal transformation can be written as

$$\zeta = \eta + \delta\eta \tag{19}$$

and, after the substitution of the former equation, it is possible to obtain

$$M = 1 + \partial\delta\eta/\partial\eta \tag{20}$$

and applying the symplectic conditions it is possible to obtain the first order symplectic matrix.

8. Symplectic Maps Applied to Optical Systems

The transformation that takes from the initial condition to the end condition in phase space can be expressed formally as a Transference Map or Symplectic Map. In the same way that the Lie transformations are used in classical mechanics, these can be used in optical physics by using the following theorem [7]:

Theorem (Dragt-Finn): Let assume that M is any simplectic map which maps the phase space into itself, *i.e.* if $h = 0$, then $z = 0$. In this case M can be factored as product of Lie transformations:

$$M = \cdots e^{:f4:} e^{:f3:} e^{:f2:} \tag{21}$$

where f_n are polynomial homogeneous of n-degree in the variables (q, p). Furthermore, this map is symplectic for any set of polynomials. Additionally, if the product is cut at any stage of the expansion, the result is still a yimplectic map.

Each symplectic map whose existence is contemplated by the last theorem, can be used to represent optical systems. For example, lenses can be represented by Lie transformations whose exponent is polynomial of fourth-order [6] [11] [12] [13] [14].

The former definition of the symplectic matrix fulfills several properties [7] [12] [13]: suppose that M is a transformation matrix, or a product of transformations, and be g a function in the phase space, then:

As a corollary of this result it can be demonstrated that for two functions in phase space f and g the following relationship is satisfied:

$$\boldsymbol{M}\left[f(\eta), g(\eta)\right] = \left[\boldsymbol{M}f(\eta)\right]\left[\boldsymbol{M}g(\eta)\right]$$

where $\boldsymbol{M}$ is isomorphic respect to the ordinary multiplication of functions. By using the former result, it is possible to show that $\boldsymbol{M}\{f, g\} = \{\boldsymbol{M}f, \boldsymbol{M}g\}$ which means that $\boldsymbol{M}$ is isomorphic respect to PB.

The following theorem relates the SM to LT [8]:

Theorem: If M is a LT associated with an analytical function f, then:

$$\zeta_i = \boldsymbol{M}\eta_i = e^{(:f:)}\eta_i \tag{22}$$

is a convergent series, *i.e.* a symplectic map.

The proof of the previous theorem is difficult because the convergence of the series in the general form has never been demonstrated, neither using symbolic programming nor numerical methods.

An optical system is generally symmetric around the optical axis (z axis). This means that the mapping must contain only polynomial of even order, *i.e.* no odds terms are allowed to be present in the polynomials, because these terms are related to an asymmetric behavior of the light beam around the optical axis. Then:

$$\boldsymbol{M} = \cdots e^{:f6:} e^{:f4:} e^{:f2:} \tag{23}$$

The first term from the right (where appears the 2nd-order polynomial) is the Lie transformation of the paraxial optics; the next term (corresponding to a 4th-order polynomial) is related to aberration and image deformation. High order terms are related to small corrections, produced by the optical system (lenses, apertures bandwidth cutoff, etc.), that have to be applied to the image to recover the object.

The order of the operators is arbitrary; this means that the same mapping can be stated in ascending order:

$$\boldsymbol{M} = e^{:f2:} e^{:f4:} e^{:f6:} \cdots \tag{24}$$

The election of the order of the operators in the symplectic matrix $\boldsymbol{M}$ (ascending or descending) changes the direction of the light beam [6] [7] [12]. From the mathematical point of view, to choose ascending or descending order in the polynomial is equivalent to transpose the symplectic matrix $\boldsymbol{M}$.

As an example of a symplectic map let us consider a light source who is emitting rays into a medium characterized by a constant refraction index. The symplectic matrix for a ray traveling in this medium from z_i to z_f is given by [6]:

$$M = e^{-\int_{z_i}^{z_f} :H^{opt}:\mathrm{dz}} = e^{l:\sqrt{n^2 - p^{i2}}} \tag{25}$$

where H^{opt} is the Hamiltonian of the optical system, while l is the difference $l = z_f - z_i$.

Expanding the Hamiltonian in a power series in $\boldsymbol{q}$ and $\boldsymbol{p}$ it is possible to obtain:

$$\begin{gathered}\left[:\sqrt{n^2 - p^{i2}}:\right]^m \boldsymbol{p}^i = 0 \;\; \therefore m = 1,2,3,\cdots \\ :\sqrt{n^2 - p^{i2}}:\boldsymbol{q}^i = \boldsymbol{p}^i \Big/ \sqrt{n^2 - p^{i2}} \\ \left[:\sqrt{n^2 - p^{i2}}:\right]^m \boldsymbol{q}^i = 0 \;\; \therefore m = 2,3,\cdots\end{gathered} \tag{26}$$

The last result is obtained by expanding the square root in a power series and retaining only the lower order terms corresponding to a small deviations respect to the optical axis.

Therefore:

$$\boldsymbol{p}^f = \boldsymbol{M}\boldsymbol{p}^i = \boldsymbol{p}^i, \;\; \boldsymbol{q}^f = \boldsymbol{M}\boldsymbol{q}^i = \boldsymbol{q}^i + l\boldsymbol{p}^i \Big/ \sqrt{n^2 - p^{i2}} \tag{27}$$

Expanding the Hamiltonian in a Taylor series, the symplectic map has the form (Equation (23)):

$$\boldsymbol{M} = \cdots e^{l1/16n^5\left[:\left(p^i\right)^2:\right]^3} e^{l1/8n^3\left[:\left(p^i\right)^2:\right]^2} e^{l1/2n\left[:\left(p^i\right)^2:\right]} \tag{28}$$

Retaining only the first exponential, the final coordinates and momentum have the form:

$$\boldsymbol{p}^f = e^{l1/2n\left[:\left(p^i\right)^2:\right]}\boldsymbol{p}^i, \quad \boldsymbol{q}^f = e^{l1/2n\left[:\left(p^i\right)^2:\right]}\boldsymbol{q}^i = \boldsymbol{q}^i + \left(1/n\right)\boldsymbol{p}^i \tag{29}$$

9. Frequency-Dependent Symplectic Map

We know that the Optical Hamiltonian is:

$$\boldsymbol{H} = -\sqrt{n\left(\boldsymbol{q}, z\right)^2 - \boldsymbol{p}^2} \tag{30}$$

If the medium only depends on the frequency, the refraction index is:

$$n = n\left(\omega\right) \tag{31}$$

Nevertheless, the refraction index depends on

$$n\left(\omega\right) = \sqrt{\epsilon\mu} \tag{32}$$

In the case that we have a isotropic and non-linear medium, we have modeled the electrical permittivity as:

$$\epsilon = a - \left(\omega_m/\omega\right)^2 + \left(\omega_m/\omega\right)^4 \tag{33}$$

We assumed that medium have a single resonance frequency. The "a" is a constant and "ω" is the signal frequency. The refraction index behavior is:

$$n\left(\omega\right) = \sqrt{\mu\left(a - \left(\omega_m/\omega\right)^2 + \left(\omega_m/\omega\right)^4\right)} \tag{34}$$

since is constant we can obtain a new refraction index

$$n_\mu\left(\omega\right) = n\left(\omega\right)/\sqrt{\mu} = \sqrt{a - \left(\omega_m/\omega\right)^2 + \left(\omega_m/\omega\right)^4} \tag{35}$$

And the new Hamiltonian has the following form:

$$H = -\sqrt{a - \left(\omega_m/\omega\right)^2 + \left(\omega_m/\omega\right)^4 - \boldsymbol{p}^2} \tag{36}$$

With this Hamiltonian Equations (27) become:

$$\boldsymbol{p}^f = \boldsymbol{M}\boldsymbol{p}^i = \boldsymbol{p}^i, \quad \boldsymbol{q}^f = \boldsymbol{M}\boldsymbol{q}^i = \boldsymbol{q}^i + l\left(\boldsymbol{p}^i \Big/ \sqrt{a - \left(\omega_m/\omega\right)^2 + \left(\omega_m/\omega\right)^4 - \left(\boldsymbol{p}^i\right)^2}\right) \tag{37}$$

The coordinate presents a resonance when the medium frequency is equal to the signal frequency. In this frequency the ray has the maximum inclination with respect to the optical axis, that is, the $tg\left(\theta\right)$ is maximum is when $\omega = \omega_m$. When $\omega \ll 1$ we can expand the coordinate in a Taylor series:

$$\boldsymbol{q}^f - \boldsymbol{q}^i \cong \left(\omega^2/\omega_m^2\right) + \left(1/2\right)\left(\omega^4/\omega_m^4\right) \tag{38}$$

And if $\omega \gg 1$, the coordinate have the following form:

$$\boldsymbol{q}^f - \boldsymbol{q}^i \cong \left(1\Big/\sqrt{n^2 - \left(p^i\right)^2}\right) + \left(1/2\right)\left(\omega_m^2 \Big/ \left(\left(n^2 - \left(p^i\right)^2\right)^{3/2}\omega^2\right)\right) \tag{39}$$

The most interesting case is when because, from of a certain frequency the saturated medium behaves like vacuum (1st. term of the last series).

10. Conclusion

We have reviewed how a Lie transformation associated with an analytical function produces a Symplectic Map; this can be written as a product of Lie transformations. This formalism can be applied easily to complicated optical systems. The Lie group was used to construct the symplectic map which allows to obtain the equations that govern the behavior of a light beam passing through a system of thin lenses. When the medium is non-linear and isotropic and the signal depends on the frequency, this approach also allows to determine the ray propagation. In the case that the signal frequency is equal to the characteristic material frequency, the inclination angle is maxim: resonance point. By low signal frequency respect to material frequency, the inclination is quadratic; but by high signal frequency, the medium is saturated and its respond is the same, that it to say, it is like medium not depending to frequency with a refraction index equal one (vacuum).

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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